Journal of
Mathematical
Inequalities
Volume 11, Number 1 (2017), 167–179 doi:10.7153/jmi-11-17

NEW BOUNDS ON ZAGREB INDICES

ZHONGZHU LIU, QINGHUA MA AND YIZHI CHEN

(*Communicated by T. Burić*)

Abstract. The Zagreb indices are among the oldest and the most famous topological molecular structure-descriptors. The first Zagreb index is equal to the sum of the squares of the degrees of the vertices, and the second Zagreb index is equal to the sum of the products of the degrees of pairs of adjacent vertices of the respective graph. In this paper, we characterize the extremal graphs with maximal, second-maximal, third-maximal, fourth-maximal and minimal, second-minimal, third-minimal Zagreb indices among all Eulerian graphs, and then we give the tight conditions on the Zagreb indices of a graph for the existence of a spanning eulerian subgraph, dominating circuits, spanning circuits, Hamiltonian paths and cycles, respectively.

1. Introduction

Let $G=(V(G),E(G))$ be a simple graph with $n=|V(G)|$ vertices and $m=|E(G)|$ edges. The first Zagreb index $M_1(G)$ and the second Zagreb index $M_2(G)$ defined as follows:

$$M_1(G)=\sum_{u\in V(G)} d_G^2(u) \text{ and } M_2(G)=\sum_{uv\in E(G)} d_G(u)d_G(v),$$

where $d_G(v_1),d_G(v_2),\cdots,d_G(v_n)$ are degrees of vertices $v_1,\cdots,v_n$, respectively, while $d_G(v_i)d_G(v_j)$ represents weight associated to the edge v_iv_j. The Zagreb indices were classical topological indices introduced by I. Gutman and N. Trinajstić [21], which examined the dependence of total π-electron energy on molecular structure and elaborated in [20]. They are later separately as topological indices in QSPR/QSAR [27] and reflecting the extent of branching of the molecular carbon-atom skeleton [4, 30]. Main properties of Zagreb indices were summarized in [5, 6, 7, 22, 23, 26, 27, 34, 31]. In particular, Deng [16] gave a unified approach to determine extremal values of Zagreb indices for trees, unicyclic graphs, and bicyclic graphs, respectively. Other recent results on ordinary Zagreb indices can be found in [5, 6, 7, 19, 22, 23, 26, 31] and the references cited therein.

An Eulerian circuit in an undirected graph is a closed trail that uses each edge exactly once. If such a circuit exists, the graph is called Eulerian. According to the result of Euler [2], a graph is Eulerian if and only if it is connected and all its vertices have even degrees. Denoted by $\mathbb{G}_n$ the set of all Eulerian graphs with $n\geqslant 3$ vertices. In

Mathematics subject classification (2010): 05C07, 05C45.
Keywords and phrases: Zagreb index, Eulerian graph, supereulerian graph, dominating circuit, spanning circuit, Hamiltonian.

section 3, we use the techniques from [18] to determine the extremal graphs with maximal, second-maximal, third-maximal, fourth-maximal and minimal, second-minimal Zagreb indices among all Eulerian graphs.

If a graph is supereulerian if it has a spanning eulerian subgraph. Motivated by the Chinese Postman Problem, Boesch et al. [8] proposed the supereulerian graph problem: determine when a graph has a spanning eulerian subgraph. Pulleyblank [29] showed that such a decision problem, even when restricted to planar graphs, is NP-complete. The literatures on supereulerian graph can be found in [9, 12, 13, 14, 25]. In section 4, we give the tight conditions on Zagreb indices of a graph for the existence of supereulerian subgraphs, and then give the bounds of complement graphs on Zagreb indices for the existence of supereulerian subgraphs.

A spanning circuit of a graph G, is a circuit that contains all vertices of G, while a dominating circuit of G, is a circuit such that every edge of G is incident with at least one vertex of the circuit. A. Benhocine et al. [3] proposed the condition that the graph contains a spanning circuit and dominating circuit, respectively. In section 5, we propose the tight condition on Zagreb indices of a graph for the existence of spanning circuits or dominating circuits, and the bounds are in terms of number of vertices and edges.

A Hamiltonian path (resp. cycle) is a path (resp. cycle) in a graph that visits each vertex exactly once. the problem of finding a Hamiltonian cycle is NP-complete. Fiedler and Nikiforo [17] gave tight conditions on spectral radius of a graph for the existence of Hamiltonian paths and cycles, and later B. Zhou [35] gave the similar result on the signless Laplacian spectral radius. Motivated by the results above, in section 6, we give the tight conditions on the Zagreb indices of a graph for the existence of Hamiltonian paths and cycles.

2. Preliminaries

Let $\mathbb{G}_n$ be the set of Eulerian graphs with $n \geqslant 3$ vertices. Let even-degree subgraph be a subgraph of G that all its degrees are positive even integers. Denoted by K_n the complete graph with n vertices. If n is odd, then $K_n \in \mathbb{G}_n$. If n is even, the cocktail-party graph $CP_n \in \mathbb{G}_n$ with n vertices, where CP_n obtained by deleting $n/2$ independent edges from K_n. Denoted by $K_{1,n-1}$ the start with n vertices, and by C_n the cycle with $n \geqslant 3$ vertices. Let $B(p,q)$ be the bicyclic graph obtained by identified a vertex of cycle C_p and a vertex of cycle C_q, and $B(r,s,t)$ the tricycle graphs obtained by identified a vertex of degree two of $B(r,s)$ with a vertex of cycle C_t, where $r+s \geqslant 7$ and two vertices with degree 4 are nonadjacent.

Let $\delta(G)$ and $\Delta(G)$ be the minimum degree and maximum degree of G, respectively. The edge connectivity is the size of a smallest edge cut, and a graph is called k-edge-connected if its edge connectivity is k or greater. The Petersen graph is an undirected graph with 10 vertices and 15 edges, and a pancyclic graph is a graph that contains cycles of all possible lengths from three up to the number of vertices in the graph.

LEMMA 2.1. [11] *Let G be a simple graph with $n \geqslant 3$ vertices and m edges. Then*

$$M_1(G) \leqslant m\left(\frac{2m}{n-1}+n-2\right) \tag{1}$$

with equality if and only if $G = S_n$ or K_n.

LEMMA 2.2. [25] *Let G be a simple graph with $n \geqslant 3$ vertices and m edges. Then*

$$M_2(G) \leqslant \frac{n}{2}(2m-n+1)^{\frac{3}{2}}. \tag{2}$$

LEMMA 2.3. [24] *Let G be a simple graph with $n \geqslant 3$ vertices and m edges. Then*

$$M_1(G) \geqslant \frac{4m^2}{n} \text{ and } M_2(G) \geqslant \frac{4m^3}{n^2}.$$

3. Eulerian graph with extremal Zagreb index

LEMMA 3.1. [1] *A connected graph G is Eulerian if and only if its edge set can be decomposed into cycles.*

THEOREM 3.1. *Let $G \in \mathbb{G}_n$ with $n \geqslant 4$.*

(a) If n is odd, then $M_1(G) \leqslant n(n-1)^2$ and $M_2(G) \leqslant \frac{1}{2}n(n-1)^3$, with both equalities if and only if $G = K_n$.

(b) If n is even, then $M_1(G) \leqslant n(n-2)^2$ and $M_2(G)_2 \leqslant \frac{1}{2}n(n-2)^3$, with equalities if and only if $G = CP_n$.

Proof. Case (a) is obviously, since K_n has maximal first and second Zagreb indices among all simple graphs in $\mathbb{G}_n$, it is easily to compute that $M_1(G) = n(n-1)^2$ and $M_2(K_n) = \frac{1}{2}n(n-1)^3$.

(b) If n is even, then $d_G(u) \leqslant n-2$ for $u \in V(G)$, $d_G(u)d_G(v) \leqslant (n-2)^2$ for every edge $uv \in E(G)$, and $|E(G)| \leqslant \frac{1}{2}n(n-2)$. Thus $M_1(G) \leqslant n(n-2)^2$ and $M_2(G) \leqslant \frac{1}{2}n(n-2)^3$, with both equalities if and only if each vertex of G has degree $n-2$. Note that CP_n is exactly one graph in $\mathbb{G}_n$ such that every vertex has degree $n-2$. Then the results hold. □

A even-degree subgraph is a subgraph of given graph of which each vertex has even degree. It is obviously that a induced connected even subgraph is Eulerian. If n is odd, then every graph $G \in \mathbb{G}_n$ can be obtained from K_n by deleting edges of some connected even subgraphs. If n is even, then every graph $G \in \mathbb{G}_n$ can be obtained from CP_n by deleting edges of some connected even subgraphs.

THEOREM 3.2. *(a) If n is odd, $n \geqslant 5$, then the graphs obtained from K_n by deleting the edges of a triangle, of a quadrangle, and of a pentagon have, respectively, second-maximal, third-maximal, and fourth-maximal first and second Zagreb indices in $\mathbb{G}_n$.*

(b) If n is even, $n \geqslant 6$, then the graphs obtained from CP_n by deleting the edges of a triangle, of a quadrangle, and of a pentagon have, respectively, seond-maximal, third-maximal, and fourth-maximal first and second Zagreb indices in $\mathbb{G}_n$.

Proof. (a) Let $G \in \mathbb{G}_n$ $(G \neq K_n)$, since n is odd, then $K_n - E(G) = H$ is a even-degree subgraph, which implies that $E(H)$ can be decomposed into cycles from Lemma 3.1. Thus G can be obtained from K_n by deleting edges of some cycles from H. Suppose that H contain k vertices with degree at least 2, where $3 \leqslant s,k \leqslant n$. Thus $M_1(G) \leqslant M_1(K_n - E(C_r))$. If $s = 3$, then H contains C_3 as its subgraph. If $s = 4$, then H contains C_4. If $s = 5$, then H contains $C_5, B(3,3)$ or K_5. Moreover,

$$\begin{aligned}
M_1(G) &= \sum_{u \in V(G)} (n-1-d_H(u))^2 \leqslant (n-s)(n-1)^2 + s(n-3)^2, \\
&\leqslant M_1(K_n - E(C_s)) \\
M_2(G) &= \sum_{uv \in E(G)} (n-1-d_H(u))\,(n-1-d_H(v)) \\
&\leqslant \left(\binom{n}{2} - s(n-s+1)\right)(n-1)^2 + s(n-s)(n-1)(n-3) \\
&\quad + s(n-3)^2 \leqslant M_2(K_n - E(C_s)).
\end{aligned}$$

If $H \neq C_3, C_4$, then $s \geqslant 5$, and by equalities above, we have

$$\begin{aligned}
M_1(G) &\leqslant (n-5)(n-1)^2 + 5(n-3)^2 = M_1(K_n - E(C_5)) \\
M_2(G) &\leqslant \left(\binom{n}{2} - 5(n-4)\right)(n-1)^2 + 5(n-5)(n-1)(n-3) + 5(n-3)^2 \\
&= M_2(K_n - E(C_5)),
\end{aligned}$$

with equality if and only if $G = K_n - E(C_5)$, and by directly calculation, $M_i(K_n - E(C_3)) \geqslant M_i(K_n - E(C_4)) \geqslant M_i(K_n - E(C_5))$ for $i = 1,2$. If $s \geqslant 6$, then $|E(H)| \geqslant 6$, thus $M_1(K_n - E(C_5)) > M_1(G)$ and $M_2(K_n - E(C_5)) > M_2(G)$. Then the result holds.

(b) The proof is similar to (a). □

THEOREM 3.3. [32, 33] *Let $G \in \mathbb{G}_n$ with $n \geqslant 3$. Then $M_1(G) \geqslant M_1(C_n)$ and $M_2(G) \geqslant M_2(C_n)$ with both equalities if and only if $G = C_n$.*

THEOREM 3.4. *Let $G \in \mathbb{G}_n$ with $n \geqslant 5$. If $G \neq C_n$, then $M_1(G) \geqslant 4n + 12$ and $M_2(G) \geqslant 4n + 20$, with both equalities if, and only if $G = B(p,q)$, where $3 \leqslant p \leqslant q \leqslant n-3$ and $p + q = n + 1$.*

Proof. Since $G \in \mathbb{G}_n$ and $\neq C_n$, G must contains a vertex w such that $d_G(w) \geqslant 4$, which implies that G contains 4 edges with weight at least 8, and $|E(G)| \geqslant n + 1$.

Thus, $d_G^2(v) \geqslant 4$ for every vertex v and $d_G(u)d_G(v) \geqslant 4$ for every edge uv. Therefore,

$$\begin{aligned} M_1(G) &= \sum_{u\neq w, u\in V(G)} d_G^2(u) + d_G^2(w) \geqslant 4(n-1)+16 = 4n+12 = M_1(B(p,q)), \\ M_2(G) &= \sum_{uv\in E(G), u,v\neq w} d_G(u)d_G(v) + \sum_{uw\in E(G)} d_G(u)d_G(w) \\ &\geqslant 4(|E(G)|-4) + 4\sum_{uw\in E(G)} d_G(u) \\ &\geqslant 4(n-3) + 4\times 8 = 4n+20 = M_2(B(p,q)). \end{aligned}$$

Then the result holds. □

THEOREM 3.5. *Let $G \in \mathbb{G}_n$ with $n \geqslant 8$. If $G \neq C_n$, $B(p,q)$, where $p+q = n+1$, then*

$$M_1(G) \geqslant M_1(B(r,s,t)), \ \textit{and} \ M_2(G) \geqslant M_2(B(r,s,t)),$$

with both equalities hold if and only if $G = B(r,s,t)$, where $r+s+t = n+2$ and $s > 3$.

Proof. If $G \neq C_n, B(p,q)$, then G contain a vertex u with $d_G(u) \geqslant 6$ or two vertices v,w with $d_G(v), d_G(w) \geqslant 4$. Let $|E(G)| = m$.

Case 1. G contains a vertex u such that $d_G(u) \geqslant 6$. Then G contain at least 6 edges with weight more than 12. We have

$$\begin{aligned} M_1(G) &= d_G^2(u) + \sum_{v\neq u, v\in V(G)} d_G^2(v) \geqslant 36 + 4(n-1) = 4n+32, \\ M_2(G) &= \sum_{uv} d_G(u)d_G(v) + \sum_{wv\in E(G), w,v\neq u} d_G(w)d_G(v) \\ &\geqslant 12\times 6 + 4(m-6) = 4m+48. \end{aligned}$$

Case 2. G contain two vertices v,w with degrees at least 4.

Case 2.1. If v adjacent to w, then G contain at least 6 edges connected to v or w with weight at least 8. We have

$$\begin{aligned} M_1(G) &\geqslant 32 + 4(n-2) = 4n+24, \\ M_2(G) &\geqslant 8\times 6 + 16 + 4(m-7) = 4m+36. \end{aligned}$$

Case 2.2. If v nonadjacent to w, then G contain at least 8 edges with weight at least 8, thus

$$\begin{aligned} M_1(G) &\geqslant 32 + 4(n-2) = 4n+24, \\ M_2(G) &\geqslant 8\times 8 + 4(m-8) = 4m+32. \end{aligned}$$

By directly calculation, we have $M_1(G) \geqslant 4n+24$ and $M_2(G) \geqslant 4m+32$, with both equalities hold if, and only if $G = B(r,s,t)$, then the result holds. □

4. Existence of supereulerian subgraphs

LEMMA 4.1. [3] *Let G be a 2-edge-connected simple graph with $n \geqslant 3$ vertices. If*

$$d_G(u) + d_G(v) \geqslant \frac{2n+3}{3},$$

whenever $uv \notin E(G)$, then G is supereulerian.

LEMMA 4.2. *Let G be a simple graph with $n \geqslant 3$ vertices and m edges. If*

$$m \geqslant \frac{n(3n-11)}{6} + 4,$$

then G is supereulerian.

Proof. Suppose by contrary that there exist nonadjacent vertices u and v in G, such that $d_G(u) + d_G(v) < \frac{2n+3}{3}$. Note that G can be obtained from K_n by deleting h edges. Therefore,

$$\frac{4}{3}n - 4 = \binom{n}{2} - \left(\frac{n(3n-11)}{6} + 4\right) \geqslant \binom{n}{2} - m = h,$$

and

$$h \geqslant 2(n-1) - 1 - (d_G(u) + d_G(v)) > 2(n-1) - 1 - \frac{2n+3}{3} = \frac{4}{3}n - 4,$$

it is a contradiction. Thus $d_G(u) + d_G(v) \geqslant \frac{2n+3}{3}$ for $uv \notin E(G)$, and by Lemma 4.1, G is supereulerian. Then the result holds. □

LEMMA 4.3. [13] *Let G be a 3-edge-connected simple graph with n vertices. If n is large and if for every edge $uv \in E(G)$,*

$$d_G(u) + d_G(v) \geqslant \frac{n}{6} - 2,$$

then either G is a supereulerian graph, or G has the Pertesen graph as its reduction.

By the similar proof of Lemma 4.2, we have

LEMMA 4.4. *Let G be a simple graph with n vertices and m edges. If n is large and*

$$m \geqslant \frac{n(3n-14)}{6},$$

then then either G is a supereulerian graph, or G has the Pertesen graph as its reduction.

THEOREM 4.1. *Let G be a 2-edge-connected simple graph with $n \geqslant 3$ vertices. If*

$$M_1(G) \geqslant \frac{2m(3n^2-10n+15)}{3(n-1)} \tag{3}$$

or

$$M_2(G) \geqslant \frac{n}{2}\left(n^2-\frac{14}{3}n+9\right)^{\frac{3}{2}}, \tag{4}$$

then G is supereulerian.

Proof. Since G is a 2-edge-connected simple graph with $n \geqslant 3$ vertices,
(1) If $M_1(G) \geqslant \frac{2m(3n^2-10n+15)}{3(n-1)}$, then by Lemma 2.1, we have

$$m\left(\frac{2m}{n-1}+n-2\right) \geqslant \frac{2m(3n^2-10n+15)}{3(n-1)},$$

which implies that

$$m \geqslant \frac{n(3n-11)}{6}+4.$$

Then the result holds from Lemma 4.2.

(2) If $M_2(G) \geqslant \frac{n}{2}\left(n^2-\frac{14}{3}n+9\right)^{\frac{3}{2}}$, then by Lemma 2.2, we have

$$\frac{n}{2}(2m-n+1)^{\frac{3}{2}} \geqslant \frac{n}{2}\left(n^2-\frac{14}{3}n+9\right)^{\frac{3}{2}},$$

by the properties of the power function, we conclude that

$$m \geqslant \frac{n(3n-11)}{6}+4.$$

Then the result holds from Lemma 4.2. □

Similarly, we have

THEOREM 4.2. *Let G be a 3-edge-connected simple graph with $n \geqslant 5$ vertices and m edges. If n is large and*

$$M_1(G) \geqslant \frac{m(6n^2-23n+6)}{3(n-1)}$$

or

$$M_2(G) \geqslant \frac{n}{2}\left(n^2-\frac{17}{3}n+1\right)^{\frac{3}{2}},$$

then either G is a supereulerian graph, or G has the Pertesen graph as its reduction.

THEOREM 4.3. *Let G be a 2-edge-connected simple graph with $n \geqslant 3$ vertices. If*

$$M_1(\overline{G}) \leqslant \frac{64(n-3)^2}{9n} \tag{5}$$

or

$$M_2(\overline{G}) \leqslant \frac{256(n-3)^3}{27n^2}, \tag{6}$$

then G is supereulerian.

Proof. Since G be a 2-edge-connected simple graph with $n \geqslant 3$ vertices.

If $M_1(\overline{G}) \leqslant \frac{64(n-3)^2}{9n}$, then by Lemma 2.3, we have

$$\frac{64(n-3)^2}{9n} \geqslant \frac{4\left(\binom{n}{2}-m\right)^2}{n},$$

which implies that

$$m \geqslant \frac{n(3n-11)}{6}+4.$$

Then the result holds from Lemma 4.2.

If $M_2(\overline{G}) \leqslant \frac{256(n-3)^3}{27n^2}$, then by Lemma 2.3, we have

$$\frac{256(n-3)^3}{27n^2} \geqslant \frac{4\left(\binom{n}{2}-m\right)^3}{n^2}$$

which implies that

$$m \geqslant \frac{n(3n-11)}{6}+4.$$

Then the result holds from Lemma 4.2. □

Similar to the proof of Theorem 4.3

THEOREM 4.4. *Let G be a 3-edge-connected simple graph with $n \geqslant 5$ vertices and m edges. If*

$$M_1(\overline{G}) \leqslant \frac{121n}{9} \quad \text{or} \quad M_2(\overline{G}) \leqslant \frac{1331n}{54},$$

then either G is a supereulerian graph, or G has the Pertesen graph as its reduction.

5. Existence of a dominating circuit or spanning circuit

LEMMA 5.1. [3] *Let G be a 2-edge-connected simple graph with $n \geqslant 3$ vertices, and $G \ncong K_{1,n-1}$. If $d_G(u)+d_G(v) \geqslant \frac{2n+1}{3}$ for every pair of nonadjacent vertices u and v, then G contains a dominating circuit.*

Similar to the proof of Lemma 4.2, we have

LEMMA 5.2. *Let G be a 2-edge-connected simple graph with $n \geqslant 3$ vertices and m edges, and $G \ncong K_{1,n-1}$. If*

$$m \geqslant \frac{3n^2 - 11n + 20}{6},$$

then G contains a dominating circuit.

LEMMA 5.3. [3] *Let G be a 2-edge-connected simple graph with $n \geqslant 3$ vertices. If $d_G(u) + d_G(v) \geqslant \frac{2n+3}{3}$ for every nonadjacent vertices u and v, then G contains an spanning circuit.*

Similar to the proof of Lemma 4.2, we have

LEMMA 5.4. *Let G be a 2-edge-connected simple graph with $n \geqslant 3$ vertices. If*

$$m \geqslant \frac{(3n-2)(n-3)}{6} + 3,$$

then G contains an spanning circuit.

LEMMA 5.5. [3] *Let G be a connected simple graph with $n \geqslant 6$ vertices and $\delta(G) \geqslant 2$. If $d_G(u) + d_G(v) \geqslant n-1$ for every nonadjacent vertices u and v, then G contains an spanning circuit.*

Similar to the proof of Lemma 4.2, we have

LEMMA 5.6. *Let G be a connected simple graph with $n \geqslant 6$ vertices, m edges and $\delta(G) \geqslant 2$. If*

$$m \geqslant \frac{1}{2}(n-2)(n-1) + 1,$$

then G contains an spanning circuit.

LEMMA 5.7. [3] *Let G be a hamilton graph with $n \geqslant 3$ vertices. If $d_G(u) + d_G(v) \geqslant \frac{n}{2}$ for every adjacent vertices u and v, then $L(G)$ is pancyclic.*

Similar to the proof of Lemma 4.2, we have

LEMMA 5.8. *Let G be a simple connected graph with $n \geqslant 3$ vertices. If*

$$m \geqslant \frac{(n-2)^2}{2},$$

then $L(G)$ is pancyclic.

Similar to the proof of Theorems 4.1 and 4.3, we obtain theorems following:

THEOREM 5.1. *Let G be a simple 2-edge-connected graph with $n \geqslant 3$ vertices and m edges, and $G \ncong K_{1,n-1}$. If*

$$M_1(G) \geqslant m\left(2n+\frac{4}{n-1}-\frac{14}{3}\right)$$

or

$$M_2(G) \geqslant \frac{\sqrt{3}n}{18}\left(3n^2-14n+23\right)^{\frac{3}{2}},$$

then G contains a dominating circuit.

THEOREM 5.2. *Let G be a simple 2-edge-connected graph with $n \geqslant 3$ vertices and m edges. If*

$$M_1(G) \geqslant \frac{m}{3(n-1)}(6n^2-20n+30)$$

or

$$M_2(G) \geqslant \frac{n}{2}\left(n^2-\frac{14}{3}n+17\right)^{\frac{3}{2}},$$

then G contains an spanning circuit.

THEOREM 5.3. *Let G be a connected simple graph with $n \geqslant 6$ and $\delta(G) \geqslant 2$. If*

$$M_1(\overline{G}) \leqslant \frac{4(n-2)^2}{n} \quad or \quad M_2(\overline{G}) \leqslant \frac{4(n-2)^3}{n^2},$$

then G contains a spanning circuit.

THEOREM 5.4. *Let G be a hamilton graph with $n \geqslant 3$ vertices, and $\overline{G}$ the complement of G. If*

$$M_1(\overline{G}) \leqslant \frac{(3n-4)^2}{n} \quad or \quad M_2(\overline{G}) \leqslant \frac{(3n-4)^3}{2n^2},$$

then $L(G)$ is pancyclic.

6. Existence of Hamiltonian paths and cycles

LEMMA 6.1. [28] *Let G be a simple graph with n vertices. If*

$$d_G(u)+d_G(v) \geqslant n-1$$

for every nonadjacent vertices u and v, then G contains a Hamiltonian path.

LEMMA 6.2. [17] *Let G be a graph with n vertices and m edges. If*

$$m \geqslant \binom{n-1}{2}, \tag{7}$$

then G contains a Hamiltonian path unless $G = K_{n-1}+v$. If the equality (7) is strict, then G contains a Hamiltonian cycle unless $G = K_{n-1}+e$.

THEOREM 6.1. *Let G be a graph with n vertices and m edges. If*

$$(a)\ M_1(G) \geqslant 2(n-2), or \tag{8}$$

$$(b)\ M_2(G) \geqslant \frac{n}{2}(n^2-4n+3)^{\frac{3}{2}}, \tag{9}$$

then G contains a Hamiltonian path unless $G = K_{n-1}+v$. If the equalities (i) or (ii) is strict, then G contains a Hamiltonian cycle unless $G = K_{n-1}+e$.

Proof. (a) By Lemma 2.1,

$$M_1(G) \leqslant m\left(\frac{2m}{n-1}+n-2\right),$$

together with (8), implies that

$$m\left(\frac{2m}{n-1}+n-2\right) \geqslant 2m(n-2).$$

Hence, we obtain

$$m \geqslant \binom{n-1}{2}$$

with strict inequality if (8) is strict. Then the theorem holds.

(b) By Lemma 2.2,

$$M_1(G) \leqslant \frac{n}{2}(2m-n+1)^{\frac{3}{2}},$$

together with (9), implies that

$$\frac{n}{2}(2m-n+1)^{\frac{3}{2}} \geqslant \frac{n}{2}(n^2-4n+3)^{\frac{3}{2}}.$$

Hence, we obtain

$$m \geqslant \binom{n-1}{2}$$

with strict inequality if (9) is strict. Then the theorem holds. □

From the proof of Theorem 3 in [17], we have

THEOREM 6.2. *Let G be a graph with n vertices and $\overline{G}$ the complement of G. If*

$$M_1(\overline{G}) \leqslant n(n-1),$$

then G contains a Hamiltonian path unless $G = K_{n-1}+v$.

If

$$M_1(\overline{G}) \leqslant n(n-2),$$

then G contains a Hamiltonian cycle unless $G = K_{n-1}+e$.

Similar to the proof of Theorems 4.3, we have

THEOREM 6.3. *Let G be a connected simple graph with n vertices and m edges, and $\overline{G}$ the complement of G. If*

$$M_2(\overline{G}) \leqslant \frac{(n-1)^3}{n^2}, \tag{10}$$

then G contains a Hamiltonian path.

If

$$M_2(\overline{G}) \leqslant \frac{(n-2)^3}{n^2}, \tag{11}$$

then G contains a Hamiltonian cycle unless $G = K_{n-1} + e$.

Acknowledgements. This work was supported by NSF of Guangdong Province of China (No. 2015A030313574, 2014B020004027), the Outstanding Young Teacher Training Program in Guangdong Universities (No. YQ2015155) and the National Social Science Fund in China (No. 15BTJ024).

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(Received January 26, 2016)

Zhongzhu Liu
Department of Mathematics, Huizhou University
Guangdong, 516007, P. R. China
e-mail: zhongzhuliu@126.com

Qinghua Ma
Applied Mathematics Department
School of Finance
Guangdong University of Foreign Studies
510006, P. R. China
e-mail: mqh@mail.gdufs.edu.cn

Yizhi Chen
Department of Mathematics
Huizhou University
Guangdong, 516007, P. R. China
e-mail: yizhichen1980@126.com