Journal of Mathematical Inequalities
Volume 11, Number 1 (2017), 181–191 doi:10.7153/jmi-11-18

THE NATURAL APPROACH OF TRIGONOMETRIC INEQUALITIES — PADÉ APPROXIMANT

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(*Communicated by J. Pečarić*)

Abstract. The aim of this work is to provide simple proofs of some remarkable trigonometric inequalities: Jordan inequality, Kober inequality, Becker-Stark inequality, Wu-Srivastava inequality. The proofs are based on Padé approximant method. We also obtain rational refinements for these inequalities.

1. Introduction

The starting point of this paper is the following famous inequalities and their improvements.

The classical Jordan inequality (see [3]) states that

$$\frac{2}{\pi} \leqslant \frac{\sin x}{x} < 1 \quad \text{for} \quad 0 < x < \frac{\pi}{2}. \tag{1.1}$$

This inequality was improved in [1] as

$$\frac{2}{\pi}x + \frac{1}{\pi^3}\left(\pi^2 - 4x^2\right) \leqslant \sin x \quad \text{for} \quad x \in \left[0, \frac{\pi}{2}\right]. \tag{1.2}$$

Furthermore, $\frac{1}{\pi^3}$ is the best constant in (1.2) in the sense that it cannot be replaced by a larger constant.

In [6], R. Klen, M. Visuri and M. Vourinen established an upper bound for the function $\frac{\sin x}{x}$:

$$\frac{\sin x}{x} \leqslant 1 - \frac{2x^2}{3\pi^2}, \quad 0 < x < \frac{\pi}{2}. \tag{1.3}$$

In fact, the best possible inequality of this type is the following:

$$\frac{\sin x}{x} \leqslant 1 - \frac{x^2}{3\pi}, \quad 0 < x < \frac{\pi}{2}. \tag{1.4}$$

Mathematics subject classification (2010): 41A21, 26D05, 26D15, 33B10.
Keywords and phrases: Trigonometric inequalities, Padé approximant, rational refinement.

The following inequality is due to Kober [7]:

$$1-\frac{2x}{\pi}\leqslant \cos x\leqslant 1-\frac{x^2}{\pi},\quad \text{for every}\quad 0\leqslant x\leqslant \frac{\pi}{2}. \tag{1.5}$$

The left - hand side inequalities in (1.1) and (1.5) are equivalent to each other via the transformation $x\to\frac{\pi}{2}-x$.

Becker and Stark [2] obtained the following two-sided rational approximation for $\frac{\tan x}{x}$:

$$\frac{8}{\pi^2-4x^2}<\frac{\tan x}{x}<\frac{\pi^2}{\pi^2-4x^2}\quad \text{for all}\quad x\in\left(0,\frac{\pi}{2}\right). \tag{1.6}$$

Furthermore, 8 and π^2 are the best constants in (1.6).

Wu and Srivastava [15] proved the following inequality

$$\left(\frac{x}{\sin x}\right)^2+\frac{x}{\tan x}>2\quad \text{for every}\quad 0<x<\frac{\pi}{2}.$$

C. Mortici [9] obtained the following refinement

$$\left(\frac{x}{\sin x}\right)^2+\frac{x}{\tan x}>2+\frac{2x^4}{45}\quad \text{for every}\quad 0<x<\frac{\pi}{2}.$$

These inequalities have attracted the interest of many mathematicians in the recent past. We refer to [1]–[16] and closely related references therein.They were proved using the variation of some functions and their derivatives. Some of recent improvements are important through their symmetric form, but these inequalities have also practical applications, because they provide bounds for given functions. Some of these results were also obtained using Taylor's expansions of trigonometric functions.

It is known that a Padé approximant is the "best" approximation of a function by a rational function of given order. The rational approximation is also good for series with alternation terms and poor polynomial convergence.That is why in our paper we will use Padé approximant method for solving and improving these inequalities.

The Padé approximant [L/M] corresponds to the Taylor series. When it exists, the [L/M] Padé approximant to any power series $A(x)=\sum_{j=0}^{\infty}a_jx^j$ is unique. If $A(x)$ is a transcendental function, then the terms are given by the Taylor series about x_0, $a_n=\frac{1}{n!}A^{(n)}(x_0)$.

The coefficients are found by setting

$$A(x)=\frac{p_0+p_1x+\ldots+p_Lx^L}{1+q_1x+\ldots+q_Mx^M}.$$

These give the set of equations

$$\begin{cases} p_0 = a_0 \\ p_1 = a_0 q_1 + a_1 \\ p_2 = a_0 q_2 + a_1 q_1 + a_2 \\ \vdots \\ p_L = a_0 q_L + \ldots + a_{L-1} q_1 + a_L \\ 0 = a_{L-M+1} q_M + \ldots + a_L q_1 + a_{L+1} \\ 0 = a_L q_M + \ldots + a_{L+M-1} q_1 + a_{L+M}. \end{cases}$$

For example, we consider the Taylor series for sin:

$$\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + O\left(x^7\right)$$

and its associate polynomial:

$$x - \frac{1}{6}x^3 + \frac{1}{120}x^5.$$

The Padé approximant

$$\sin_{[3/3]}(x) = \frac{p_0 + p_1 x + p_2 x^2 + p_3 x^3}{1 + q_1 x + q_2 x^2 + q_3 x^3}$$

satisfies

$$\left(x - \frac{x^3}{6} + \frac{x^5}{120}\right)\left(1 + q_1 x + q_2 x^2 + q_3 x^3\right) = p_0 + p_1 + p_2 x^2 + p_3 x^3.$$

We find

$$p_0 = 0, \quad p_1 = 1, \quad p_2 = q_1 = 0, \quad p_3 = -\frac{7}{60}, \quad q_2 = \frac{1}{20}, \quad q_3 = 0.$$

Therefore

$$\sin_{[3/3]}(x) = \frac{x - \frac{7}{60}x^3}{1 + \frac{1}{20}x^2} = \frac{60x - 7x^3}{60 + 3x^2}.$$

Here are the first order versions of a few trigonometric functions which we will use:

Function	Padé Approximant	Associate Taylor Polynomials
$\sin x$	$\sin_{[5/2]}(x) = \dfrac{2520x - 360x^3 + 11x^5}{2520 + 60x^2}$	$x - \dfrac{x^3}{6} + \dfrac{x^5}{120} - \dfrac{x^7}{5040}$
$\cos x$	$\cos_{[4/2]}(x) = \dfrac{120 - 56x^2 + 3x^4}{120 + 4x^2}$	$1 - \dfrac{x^2}{2} + \dfrac{x^4}{24} - \dfrac{x^6}{720}$
$\cos x$	$\cos_{[4/4]}(x) = \dfrac{1080 - 480x^2 + 17x^4}{1080 + 60x^2 + 2x^4}$	$1 - \dfrac{x^2}{2} + \dfrac{x^4}{24} - \dfrac{x^6}{720}$

2. Main results

In order to attain our aim, we will prove our main results, rational refinements of the inequalities mentioned in the first section.

THEOREM 2.1. *The following inequalities hold*

(*i*) $$\frac{-7x^2+60}{3x^2+60} < \frac{\sin x}{x} < \frac{11x^4-360x^2+2520}{60x^2+2520}$$

for every $x \in \left(0, \frac{\pi}{2}\right)$*;*

(*ii*) $$\frac{2}{\pi} + \frac{\pi^2-4x^2}{\pi^3} < \frac{-7x^2+60}{3x^2+60}$$

for every $x \in (0, 1.4163)$*;*

(*iii*) $$\frac{11x^4-360x^2+2520}{60x^2+2520} < 1 - \frac{x^2}{3\pi}$$

for every $x \in \left(0, \frac{\pi}{2}\right)$.

REMARK 2.1.
1. We notice that both sides of inequality (i) are positives for $x \in \left(0, \frac{\pi}{2}\right)$.
2. We also mention that our rational inequalities (i), (ii) and (iii) improve the results (1.2) and (1.3).

Proof. (i) We introduce the function

$$r(x) = \left(3x^2+60\right)\sin x - 60x + 7x^3.$$

Easy computation yields

$$r'(x) = 6x\sin x + \left(3x^2+60\right)\cos x - 60 + 21x^2,$$

$$r^{(2)}(x) = -54\sin x + 120x\cos x - 3x^2\sin x + 42x,$$

$$r^{(3)}(x) = -42\cos x - 18x\sin x - 3x^2\cos x + 42,$$

$$r^{(4)}(x) = 24\sin x - 24x\cos x + 3x^2\sin x,$$

$$r^{(5)}(x) = 30x\sin x + 3x^2\cos x.$$

Evidently $r^{(5)} > 0$ on $\left(0, \frac{\pi}{2}\right)$. Then $r^{(4)}$ is strictly increasing on $\left(0, \frac{\pi}{2}\right)$. As $r^{(4)}(0) = 0$, we get $r^{(4)} > 0$ on $\left(0, \frac{\pi}{2}\right)$. Continuing the algorithm, finally we obtain $r(x) > 0$ for all $x \in \left(0, \frac{\pi}{2}\right)$.

Let

$$s(x) = \left(60x^2 + 2520\right)\sin x - 11x^5 + 360x^3 - 2520x.$$

Then

$$s'(x) = 120x\sin x + \left(60x^2 + 2520\right)\cos x - 55x^4 + 1080x^2 - 2520,$$

$$s^{(2)}(x) = -2400\sin x + 240x\cos x - 60x^2\sin x - 220x^3 + 2160x,$$

$$s^{(3)}(x) = -2160\cos x - 360x\sin x - 60x^2\cos x - 660x^2 + 2160,$$

$$s^{(4)}(x) = 1800\sin x - 480x\cos x + 60x^2\sin x - 1320x,$$

$$s^{(5)}(x) = 1320\cos x + 600x\sin x + 60x^2\cos x - 1320,$$

$$s^{(6)}(x) = -720\sin x + 720x\cos x - 60x^2\sin x,$$

$$s^{(7)}(x) = -840x\sin x - 60x^2\cos x.$$

The function $s^{(7)} < 0$ for all $x \in \left(0, \frac{\pi}{2}\right)$, therefore $s^{(6)}$ is strictly decreasing on $\left(0, \frac{\pi}{2}\right)$. As $s^{(6)}(0) = 0$, we get $s^{(6)} < 0$ on $\left(0, \frac{\pi}{2}\right)$. Using the same arguments, finally we have $s(x) < 0$ for every $x \in \left(0, \frac{\pi}{2}\right)$.

(ii) The difference

$$E(x) = \frac{-7x^2 + 60}{3x^2 + 60} - \left(\frac{2}{\pi} + \frac{\pi^2 - 4x^2}{\pi^3}\right)$$

has the equivalent form

$$E(x) = \frac{12x^4 + \left(240 - 7\pi^3 - 9\pi^2\right)x^2 + 60\pi^3 - 180\pi^2}{\pi^3\left(3x^2 + 60\right)}.$$

The polynomial function

$$P(x) = 12x^4 + \left(240 - 7\pi^3 - 9\pi^2\right)x^2 + 60\pi^3 - 180\pi^2$$

has the real roots

$$x_1 \approx -1.8663, \quad x_2 \approx -1.4163, \quad x_3 \approx 1.4163 \quad \text{and} \quad x_4 \approx 1.8663.$$

Therefore $P(x) > 0$ for all $x \in (0, 1.4163)$.

(iii) The difference

$$F(x) = \frac{11x^4 - 360x^2 + 2520}{60x^2 + 2520} - \frac{3\pi - x^2}{3\pi}$$

can be re - written as

$$F(x) = \frac{x^2\left((60 + 33\pi)x^2 + (2520 - 1260\pi)\right)}{3\pi\left(60x^2 + 2520\right)}.$$

The polynomial function

$$Q(x) = (60+33\pi)x^2 + (2520-1260\pi)$$

has the real roots

$$x_1 = -2\sqrt{\frac{105(\pi-2)}{20+11\pi}} \approx -2.9644 \quad \text{and} \quad x_2 = 2\sqrt{\frac{105(\pi-2)}{20+11\pi}} \approx 2.9644.$$

Hence $Q(x) < 0$ for all $x \in \left(0, \frac{\pi}{2}\right)$. □

Related to Kober's inequality, our result is

THEOREM 2.2. *The following inequalities hold:*

$$(i) \qquad \frac{17x^4 - 480x^2 + 1080}{2x^4 + 60x^2 + 1080} < \cos x < \frac{3x^4 - 56x^2 + 120}{4x^2 + 120},$$

for every $x \in \left(0, \frac{\pi}{2}\right)$*;*

$$(ii) \qquad 1 - \frac{2x}{\pi} < \frac{17x^4 - 480x^2 + 1080}{2x^4 + 60x^2 + 1080}$$

for every $x \in (0, 1.5689)$.

REMARK 2.2.

1. We notice that the polynomial function $t(x) = 17x^4 - 480x^2 + 1080$ has the real roots

$$x_1 = -\sqrt{\frac{240+6\sqrt{1090}}{17}} \approx -5.0764, \quad x_2 = -\sqrt{\frac{240-6\sqrt{1090}}{17}} \approx -1.5701,$$
$$x_3 = \sqrt{\frac{240-6\sqrt{1090}}{17}} \approx 1.5701 \quad \text{and} \quad x_4 = \sqrt{\frac{240+6\sqrt{1090}}{17}} \approx 5.0764,$$

therefore $t(x) > 0$ for all

$$x \in \left(0, \sqrt{\frac{240-6\sqrt{1090}}{17}}\right) = (0, 1.5701) \subset \left(0, \frac{\pi}{2}\right) = (0, 1.5707).$$

2. We also specify that our rational refinement for cos improves the Kober's inequality for all $x \in (0, 1.5689)$.

Proof. (i) Let

$$f(x) = \left(4x^2 + 120\right)\cos x - 3x^4 + 56x^2 - 120.$$

We get

$$f'(x) = 8x\cos x - \left(4x^2+120\right)\sin x - 12x^3 + 112x,$$

$$f^{(2)}(x) = \left(-4x^2-112\right)\cos x - 16x\sin x - 36x^2 + 112,$$

$$f^{(3)}(x) = -24x\cos x + \left(4x^2+96\right)\sin x - 72x,$$

$$f^{(4)}(x) = \left(4x^2+72\right)\cos x + 32x\sin x - 72,$$

$$f^{(5)}(x) = -4x^2\sin x - 40\sin x + 40x\cos x,$$

$$f^{(6)}(x) = -4x^2\cos x - 48x\sin x.$$

Evidently $f^{(6)} < 0$ on $\left(0,\frac{\pi}{2}\right)$. It follows that $f^{(5)}$ is strictly decreasing on $\left(0,\frac{\pi}{2}\right)$. As $f^{(5)}(0) = 0$, we get $f^{(5)} < 0$ on $\left(0,\frac{\pi}{2}\right)$. Using the same algorithm, we finally obtain $f(x) < 0$ for $x \in \left(0,\frac{\pi}{2}\right)$.

If we consider

$$g(x) = \left(2x^4+60x^2+1080\right)\cos x - 17x^4 + 480x^2 - 1080,$$

then we have

$$g'(x) = \left(8x^3+120x\right)\cos x - \left(2x^4+60x^2+1080\right)\sin x - 68x^3 + 960x,$$

$$g^{(2)}(x) = \left(-2x^4-36x^2-960\right)\cos x - \left(16x^3+240x\right)\sin x - 204x^2 + 960,$$

$$g^{(3)}(x) = \left(-24x^3-312x\right)\cos x + \left(2x^4-12x^2+720\right)\sin x - 408x,$$

$$g^{(4)}(x) = \left(2x^4-84x^2+408\right)\cos x + \left(32x^3+288x\right)\sin x - 408,$$

$$g^{(5)}(x) = \left(40x^3+120x\right)\cos x + \left(-2x^4+180x^2-120\right)\sin x,$$

$$g^{(6)}(x) = \left(-2x^4+300x^2\right)\cos x + \left(-48x^3+240x\right)\sin x.$$

The positivity of $g^{(6)}$ on $\left(0,\frac{\pi}{2}\right)$ yields that $g^{(5)}$ is strictly increasing on $\left(0,\frac{\pi}{2}\right)$. As $g^{(5)}(0) = 0$, we get $g^{(5)} > 0$ on $\left(0,\frac{\pi}{2}\right)$. Similar arguments lead us to the positivity of g on $\left(0,\frac{\pi}{2}\right)$.

(ii) The difference

$$G(x) = \frac{17x^4-480x^2+1080}{2x^4+60x^2+1080} - \frac{\pi-2x}{\pi}$$

becomes

$$G(x) = \frac{x\left(4x^4+15\pi x^3+120x^2-540\pi x+2160\right)}{\pi\left(2x^4+60x^2+1080\right)}.$$

The polynomial function

$$R(x) = 4x^4 + 15\pi x^3 + 120x^2 - 540\pi x + 2160$$

has the real roots

$$x_1 \approx 1.5689 \text{ and } x_2 \approx 3.2693.$$

Hence, we have $G(x) > 0$ for all $x \in (0, 1.5689)$. □

Using Padé approximation for the sine and cosine functions, we obtain rational refinement of Becker-Stark inequality as follows.

THEOREM 2.3. *The following inequalities hold:*

(*i*)

$$\frac{-28x^4 - 600x^2 + 7200}{9x^6 + 12x^4 - 3000x^2 + 7200} < \frac{\tan x}{x} < \frac{22x^8 - 60x^6 - 4680x^4 - 237600x^2 + 2721600}{1020x^6 + 14040x^4 - 1144800x^2 + 2721600}$$

for all $x \in (0, 1.5701)$;

(*ii*)

$$\frac{8}{\pi^2 - 4x^2} < \frac{-28x^4 - 600x^2 + 7200}{9x^6 + 12x^4 - 3000x^2 + 7200}$$

for all $x \in (0, 1.52305)$;

(*iii*)

$$\frac{22x^8 - 60x^6 - 4680x^4 - 237600x^2 + 2721600}{1020x^6 + 14040x^4 - 1144800x^2 + 2721600} < \frac{\pi^2}{\pi^2 - 4x^2}$$

for all $x \in (0, 1.5672)$.

Proof. (i) Since both inequalities (i) from Theorem 2.1 and respectively Theorem 2.2 contain only positive functions for all $x \in (0, 1.5701)$, we can multiply them and obtain

$$\frac{\left(-7x^6 + 60\right)\left(4x^2 + 120\right)}{(3x^2 + 60)(3x^4 - 56x^2 + 120)} < \frac{\tan x}{x} < \frac{\left(11x^4 - 360x^2 + 2520\right)\left(2x^4 + 60x^2 + 1080\right)}{(60x^2 + 2520)(17x^4 - 480x^2 + 1080)}$$

or, equivalently,

$$\frac{-28x^4 - 600x^2 + 7200}{9x^6 + 12x^4 - 3000x^2 + 7200} < \frac{\tan x}{x} < \frac{22x^8 - 60x^6 - 4680x^4 - 237600x^2 + 2721600}{1020x^6 + 14040x^4 - 1144800x^2 + 2721600}$$

for all $x \in \left(0, \sqrt{\frac{240 - 6\sqrt{1090}}{17}}\right) = (0, 1.5701)$.

(ii) The difference

$$H(x) = \frac{-28x^4 - 600x^2 + 7200}{9x^6 + 12x^4 - 3000x^2 + 7200} - \frac{8}{\pi^2 - 4x^2}$$

takes the form

$$H(x)=\frac{40x^6+\left(2304-28\pi^2\right)x^4-\left(600\pi^2+4800\right)x^2+7200\pi^2-57600}{(\pi^2-4x^2)(9x^6+12x^4-3000x^2+7200)}.$$

The polynomial function

$$S(x)=40x^6+\left(2304-28\pi^2\right)x^4-\left(600\pi^2+4800\right)x^2+7200\pi^2-57600$$

has the real roots

$$x_1\approx-1.61505,\quad x_2\approx-1.52305,\quad x_3\approx1.52305\quad\text{and}\quad x_4\approx1.61505,$$

so $S(x)>0$ for all $x\in(0,1.52305)$.

Therefore the inequality

$$\frac{-28x^4-600x^2+7200}{9x^6+12x^4-3000x^2+7200}<\frac{\tan x}{x}$$

is true for all $x\in(0,1.52305)$.

(iii) We consider the function

$$I(x)=\frac{22x^8-60x^6-4680x^4-237600x^2+2721600}{1020x^6+14040x^4-1144800x^2+2721600}-\frac{\pi^2}{\pi^2-4x^2}.$$

An easy computation leads to

$$I(x)=\frac{x^2\left(-88x^8+\left(22\pi^2+240\right)x^6-\left(1080\pi^2-18720\right)x^4-\left(18720\pi^2-950400\right)x^2-\left(10886400-907200\pi^2\right)\right)}{\left(\pi^2-4x^2\right)\left(1020x^6+14040x^4-1144800x^2+2721600\right)}.$$

Since the polynomial function

$$\begin{aligned}T(x)=&-88x^8+\left(22\pi^2+240\right)x^6-\left(1080\pi^2-18720\right)x^4\\&-\left(18720\pi^2-950400\right)x^2-\left(10886400-907200\pi^2\right)\end{aligned}$$

has the real roots

$$x_1\approx-4.8322,\quad x_2\approx-1.5672,\quad x_3\approx1.5672\quad\text{and}\quad x_4\approx4.8322,$$

it follows that $T(x)<0$ for all $x\in(0,1.5672)$. □

Using the results from the above theorems, we can state and prove the refinement of Mortici's improvement of Wu - Srivastava inequality.

THEOREM 2.4. *The following inequality*

$$\begin{aligned}&\left(\frac{x}{\sin x}\right)^2+\frac{x}{\tan x}\\&>\frac{11220x^{10}-205560x^8-14256000x^6+512179200x^4-3157056000x^2+13716864000}{242x^{12}-8580x^{10}+25560x^8-1080000x^6+103680000x^4-1578528000x^2+6858432000}\\&>2+\frac{2}{45}x^4>2\end{aligned}$$

is true for all $x\in\left(0,\sqrt{\frac{240-6\sqrt{1090}}{17}}\right)=(0,1.5701)$.

Proof. We remind that for every $x \in \left(0, \sqrt{\dfrac{240-6\sqrt{1090}}{17}}\right)$, all functions that appear in the previous theorems are positive.

Then

$$\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} > M(x),$$

where

$$M(x) = \left(\frac{60x^2+2520}{11x^4-360x^2+2520}\right)^2 + \frac{\left(60x^2+2520\right)\left(17x^4-480x^2+1080\right)}{\left(11x^4-360x^2+2520\right)\left(2x^4+60x^2+1080\right)}.$$

After some elementary transformations we get

$$M(x) = \frac{11220x^{10}-205560x^8-14256000x^6+512179200x^4-3157056000x^2+13716864000}{242x^{12}-8580x^{10}+25560x^8-1080000x^6+103680000x^4-1578528000x^2+6858432000}.$$

The inequality

$$M(x) > 2 + \frac{2}{45}x^4$$

is equivalent to the following form

$$x^6\left(-\frac{484}{45}x^{10} + \frac{1144}{3}x^8 - 1620x^6 + 76380x^4 - 4864680x^2 + 58060800\right) > 0.$$

The last inequality is true for every $x \in \left(0, \sqrt{\dfrac{240-6\sqrt{1090}}{17}}\right)$, because the polynomial function

$$U(x) = -\frac{484}{45}x^{10} + \frac{1144}{3}x^8 - 1620x^6 + 76380x^4 - 4864680x^2 + 58060800$$

has the real roots

$$x_1 \approx -5.1532 \quad \text{and} \quad x_2 \approx 5.1532. \qquad \square$$

3. Final remarks

Using Padé approximation we obtain good rational refinements near the origin of some remarkable trigonometric inequalities. We are convinced that Padé approximation method is suitable to establish many other similar inequalities.

REFERENCES

[1] U. ABEL, D. CACCIA, *A sharpening of Jordan's inequality*, Amer. Math. Mon., **93** (7) (1986) 568–569.

[2] M. BECKER, E. L. STARK, *On a hierarchy of quolynomial inequalities for* $\tan x$, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. (602–633) (1978) 133–138.

[3] P. S. BULLEN, *A Dictionary of Inequalities*, Vol. 97 of Pitmann Monographs and Surveys in Pure and Applied Mathematics, Longman, Harlow, UK, (1998).

[4] L. DEBNATH, C. MORTICI, L. ZHU, *Refinements of Jordan-Steckin and Becker-Stark inequalities*, Results Math. **67**, (1–2), (2015) 207–215.

[5] W.-D. JIANG, Q.-M. LUO, F. QI, *Refinements and sharpening of some Huygens and Wilker Type inequalities*, Turk. J. Anal. Number Theory, **2** (4) (2014) 134–139.

[6] R. KLÉN, M. VISURI, M. VUORINEN, *On Jordan type inequalities for hyperbolic functions*, J. Inequal. Appl. **2010** (2010), Article ID 362548, 14 pages.

[7] H. KOBER, *Approximation by integral functions in the complex domain*, Trans. Am. Math. Soc. **56** (1994), 7–31.

[8] D. S. MITRINOVIĆ, *Analytic Inequalities*, Springer-Verlag, Berlin, (1970).

[9] C. MORTICI, *The natural approach of Wilker-Cusa-Huygens Inequalities*, Math. Inequal. Appl. **14** (3) (2011) 535–541.

[10] C. MORTICI, *A subtly analysis of Wilker inequality*, Appl. Math. Comput. **231** (2014) 516–520.

[11] C. MORTICI, H.-M. SRIVASTAVA, *Estimates for the arctangent function related to Shafer's inequality*, Colloq. Math. **136** (2) (2014) 263–270.

[12] F. QI, D.-W. NIU, B. N. GUO, *Refinements, generalizations, and applications of Jordan's inequality and related problems*, J. Inequal. Appl. **2009** (2009), Article ID 271923, 52 pages.

[13] J. SÁNDOR, *On new refinements of Kober's and Jordan's trigonometric inequalities*, Notes Number Theory Discrete Math. **19** (1) (2013) 73–83.

[14] Z. J. SUN, L. ZHU, *Simple proofs of the Cusa-Huygens-Type and Becker-Stark-Type inequalities*, J. Math. Inequal. **7** (4) (2013) 563–567.

[15] S.-H. WU, H. M. SRIVASTAVA, *A weighted and exponential generalization of Wilker's inequality and its applications*, Integr. Transf. Spec. Funct. **18** (7–8) (2007) 529–535.

[16] L. ZHU, J. HUA, *Sharpening the Becker-Stark inequalities*, J. Inequal. Appl. **2010** (2010), Article ID 931275, 4 pages.

(Received January 30, 2016)

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