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VOLUME EXTREMALS OF GENERAL L_p–CENTROID BODIES

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Abstract. Ludwig first discovered general L_p-centroid bodies in 2005 and Haberl and Schuster determined the extremals of their volume in 2009. In this paper, we extend the Haberl-Schuster result to all quermassintegrals and also obtain the extremals of the dual quermassintegrals of the polars of general L_p-centroid bodies.

1. Introduction

The classical Brunn-Minkowski theory was extended to the L_p-Brunn-Minkowski theory by Lutwak ([20, 21]). In 1997, Lutwak and Zhang ([29]) introduced the notion of L_p-centroid bodies and established an affine isoperimetric inequality for their polars which is an L_p version of the well-known Blaschke-Santaló inequality for origin-symmetric bodies. More recently, Lutwak, Yang and Zhang ([22]) proved the stronger L_p-Busemann-Petty centroid inequality which is equivalent to the L_p-Petty projection inequality. In the past 20 years, the L_p-Brunn-Minkowski theory expanded into an impressive body of results by Lutwak, Yang and Zhang, and many others (see e.g., [1, 2, 7, 8, 16, 22, 23, 24, 25, 26, 27, 28, 29, 40, 50, 51, 52, 53] and the two good books [6, 34]).

In 2005, Ludwig [14] discovered the general L_p-centroid bodies (she actually defined general L_p-moment bodies which are a dilation of the general L_p-centroid bodies). More recently, Haberl and Schuster [11] showed that the general L_p-centroid body is an L_p-Minkowski combination of the asymmetric L_p-centroid bodies. Moreover, they obtained a general version of the L_p-Busemann-Petty centroid inequality which is equivalent to the general L_p-Petty projection inequality. They also determined the extremals of volume of the family of general L_p-centroid bodies.

In this paper, we extend the Haberl-Schuster result on the volume extremals of the family of general L_p-centroid bodies to quermassintegrals. Moreover, the extremals of dual quermassintegrals of the polars of general L_p-centroid bodies are obtained. These results belong to the new and rapidly evolving asymmetric L_p-Brunn-Minkowski theory that has its origins in the work of Ludwig, Haberl and Schuster (see [9, 10, 11, 12, 14, 15]) and was further developed in [4, 5, 13, 31, 32, 33, 35, 36, 37, 43, 44, 46, 47, 48, 49].

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Let $\mathscr{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) in Euclidean space $\mathbb{R}^n$. For the set of convex bodies containing the origin in their interiors, we write $\mathscr{K}_o^n$. Let $\mathscr{S}_o^n$ denote the set of star bodies (about the origin) in $\mathbb{R}^n$. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$, denote by $V(K)$ the n-dimensional volume of a body K and for the standard unit ball B in $\mathbb{R}^n$, write $\omega_n = V(B)$.

Classical centroid bodies were attributed by Blaschke to Dupin (see [6, 34]), and their definition was extended by Petty. For a compact set K, the centroid body, ΓK, of K is the origin-symmetric convex body whose support function is given by (see [6])

$$h_{\Gamma K}(u) = \frac{1}{V(K)} \int_K |\, u \cdot x \,|\, dx \tag{1.1}$$

for all $u \in S^{n-1}$.

Centroid bodies have proven to be very important in Brunn-Minkowski theory. In the recent 30 years, classical centroid bodies have attracted increased attention (see, e.g., [18, 19, 30, 54] or the books [6, 34]).

In 1997, Lutwak and Zhang [29] introduced the notion of L_p-centroid bodies. For each compact star-shaped (about the origin) K in $\mathbb{R}^n$ and real $p \geqslant 1$, the L_p-centroid body, $\Gamma_p K$, of K is the origin-symmetric convex body whose support function is defined by

$$\begin{aligned} h^p_{\Gamma_p K}(u) &= \frac{1}{c_{n,p} V(K)} \int_K |\, u \cdot x \,|^p \, dx \\ &= \frac{1}{c_{n,p}(n+p) V(K)} \int_{S^{n-1}} |\, u \cdot v \,|^p \, \rho_K(v)^{n+p} dv \end{aligned} \tag{1.2}$$

for all $u \in S^{n-1}$. Here

$$c_{n,p} = \omega_{n+p} / \omega_2 \omega_n \omega_{p-1} \tag{1.3}$$

and dv is the standard spherical Lebesgue measure on S^{n-1}. The normalization above is chosen so that for the standard unit ball B in $\mathbb{R}^n$, we have $\Gamma_p B = B$. For the case $p = 1$, by (1.1) and (1.2) we see that $\Gamma_1 K$ is the classical centroid body ΓK under the normalization of definition (1.2) chosen such that $\Gamma B = B$ (rather than the classical $c_{n,1}^{-1} B$).

Regarding investigations of L_p-centroid bodies, we refer to [1, 2, 3, 22, 41, 42, 45] and the books [6, 34]. In particular, Lutwak, Yang and Zhang [22] established the following L_p-Busemann-Petty centroid inequality (for another proof see [1]): *If $K \in \mathscr{S}_o^n$, $p \geqslant 1$, then*

$$V(\Gamma_p K) \geqslant V(K), \tag{1.4}$$

with equality if and only if K is an ellipsoid centered at the origin.

In 2005, Ludwig [14] introduced a function $\varphi_\tau : \mathbb{R} \to [0, +\infty)$ by

$$\varphi_\tau(t) = |t| + \tau t \tag{1.5}$$

with a parameter $\tau \in [-1, 1]$. Using (1.5), Ludwig [14] (also see [5]) defined general L_p-centroid bodies as follows: For $K \in \mathscr{S}_o^n$, $p \geqslant 1$ and $\tau \in [-1, 1]$, the general L_p-

centroid body, $\Gamma_p^\tau K$, of K is the convex body whose support function is defined by

$$h^p_{\Gamma_p^\tau K}(u) = \frac{2}{c_{n,p}(\tau)V(K)}\int_K \varphi_\tau(u\cdot x)^p dx$$
$$= \frac{2}{c_{n,p}(\tau)(n+p)V(K)}\int_{S^{n-1}} \varphi_\tau(u\cdot v)^p \rho_K(v)^{n+p}dv, \tag{1.7}$$

where $c_{n,p}(\tau) = c_{n,p}[(1+\tau)^p + (1-\tau)^p]$. The normalization is chosen such that $\Gamma_p^\tau B = B$ for every $\tau \in [-1,1]$. Obviously, if $\tau = 0$ then $\Gamma_p^\tau K = \Gamma_p K$.

In 2009, Haberl and Schuster [11] (also see [5]) introduced the notion of asymmetric L_p-centroid bodies (they actually defined the asymmetric L_p-moment bodies which are a dilatation of the asymmetric L_p-centroid bodies) as follows: For $K \in \mathscr{S}_o^n$, $p \geqslant 1$, the asymmetric L_p-centroid body, $\Gamma_p^+ K$, of K is the convex body whose support function is defined by

$$h^p_{\Gamma_p^+ K}(u) = \frac{2}{c_{n,p}V(K)}\int_K (u\cdot x)_+^p dx$$
$$= \frac{2}{c_{n,p}(n+p)V(K)}\int_{S^{n-1}} (u\cdot v)_+^p \rho_K(v)^{n+p}dv, \tag{1.8}$$

where $(u\cdot x)_+ = \max\{u\cdot x, 0\}$. From (1.3) and (1.8), we see that $\Gamma_p^+ B = B$. In [11] Haberl and Schuster also defined

$$\Gamma_p^- K = \Gamma_p^+(-K). \tag{1.9}$$

From the definitions of $\Gamma_p^\pm K$ and (1.7), Haberl and Schuster [11] deduced that for $K \in \mathscr{S}_o^n$, $p \geqslant 1$ and $\tau \in [-1,1]$,

$$\Gamma_p^\tau K = f_1(\tau)\cdot \Gamma_p^+ K +_p f_2(\tau)\cdot \Gamma_p^- K, \tag{1.10}$$

where "$+_p$" denotes the L_p-Minkowski combination of convex bodies, and

$$f_1(\tau) = \frac{(1+\tau)^p}{(1+\tau)^p + (1-\tau)^p}, \qquad f_2(\tau) = \frac{(1-\tau)^p}{(1+\tau)^p + (1-\tau)^p}. \tag{1.11}$$

Setting $\tau = 0$ in (1.10) and combining it with (1.11), we see that

$$\Gamma_p K = \frac{1}{2}\cdot \Gamma_p^+ K +_p \frac{1}{2}\cdot \Gamma_p^- K. \tag{1.12}$$

If $\tau = \pm 1$ in (1.10) then, by (1.11), $\Gamma_p^{+1} K = \Gamma_p^+ K$, $\Gamma_p^{-1} K = \Gamma_p^- K$.

For general L_p-centroid bodies, Haberl and Schuster [11] proved the following general L_p-Busemann-Petty centroid inequality.

THEOREM 1.A. *If $K \in \mathscr{S}_o^n$, $p > 1$, then for every $\tau \in [-1,1]$,*

$$V(\Gamma_p^\tau K) \geqslant V(K), \tag{1.13}$$

with equality if and only if K is an ellipsoid centered at the origin.

Note that if $\tau = 0$ in Theorem 1.A, then inequality (1.13) is just inequality (1.4).

Moreoever, Haberl and Schuster [11] determined the following extremals of volume of the general L_p-centroid bodies:

THEOREM 1.B. *If* $K \in \mathscr{S}_o^n$, $p > 1$, $\tau \in [-1,1]$, *then*

$$V(\Gamma_p K) \geqslant V(\Gamma_p^{\tau} K) \geqslant V(\Gamma_p^{\pm} K). \tag{1.14}$$

If K *is not origin-symmetric and* p *is not an odd integer, then there is equality in the left inequality if and only if* $\tau = 0$ *and equality in the right inequality if and only if* $\tau = \pm 1$.

One goal of this work is to extend inequality (1.14) from volume to quermassintegrals and obtain the extremals of quermassintegrals of general L_p-centroid bodies.

THEOREM 1.1. *If* $K \in \mathscr{S}_o^n$, $p > 1$, $\tau \in [-1,1]$, $i = 0,1,\cdots,n-1$, *then*

$$W_i(\Gamma_p K) \geqslant W_i(\Gamma_p^{\tau} K) \geqslant W_i(\Gamma_p^{\pm} K). \tag{1.15}$$

If K *is not origin-symmetric and* p *is not an odd integer, then there is equality in the left inequality if and only if* $\tau = 0$ *and equality in the right inequality if and only if* $\tau = \pm 1$.

Here $W_i(Q)$ denotes the ith quermassintegral of $Q \in \mathscr{K}_o^n$.

Obviously, if $i = 0$, then Theorem 1.1 becomes Theorem 1.B by (2.7).

Another aim of this paper is to determine the extremals of the dual quermassintegrals of polars $\Gamma_p^{\tau,*} K$ of general L_p-centroid bodies $\Gamma_p^{\tau} K$.

THEOREM 1.2. *If* $K \in \mathscr{S}_o^n$, $p \geqslant 1$, $\tau \in [-1,1]$, *real* $i \neq n$, *then for* $i < n$ *or* $i > n+p$,

$$\widetilde{W}_i(\Gamma_p^* K) \leqslant \widetilde{W}_i(\Gamma_p^{\tau,*} K) \leqslant \widetilde{W}_i(\Gamma_p^{\pm,*} K); \tag{1.16}$$

for $n < i < n+p$,

$$\widetilde{W}_i(\Gamma_p^* K) \geqslant \widetilde{W}_i(\Gamma_p^{\tau,*} K) \geqslant \widetilde{W}_i(\Gamma_p^{\pm,*} K). \tag{1.17}$$

If K *is not origin-symmetric and* p *is not an odd integer, then there is equality in the left inequality of (1.16) (or (1.17)) if and only if* $\tau = 0$ *and equality in the right inequality of (1.16) (or (1.17)) if and only if* $\tau = \pm 1$. *For* $i = n+p$, *(1.16) (or (1.17)) becomes an equality.*

Here $\widetilde{W}_i(Q)$ denotes the ith dual quermassintegral of $Q \in \mathscr{S}_o^n$.

For $i = 0$ in Theorem 1.2, by (1.16) and (2.10) we have that

COROLLARY 1.1. *If* $K \in \mathscr{S}_o^n$, $p \geqslant 1$, $\tau \in [-1,1]$, *then*

$$V(\Gamma_p^* K) \leqslant V(\Gamma_p^{\tau,*} K) \leqslant V(\Gamma_p^{\pm,*} K). \tag{1.18}$$

If K *is not origin-symmetric and* p *is not an odd integer, then there is equality in the left inequality if and only if* $\tau = 0$ *and equality in the right inequality if and only if* $\tau = \pm 1$.

The proofs of Theorems 1.1 and 1.2 are given in Section 4. In Section 3, we collect some properties of general L_p-centroid bodies.

2. Background material

If $K\in\mathscr{K}^n$, then its support function, $h_K=h(K,\cdot):\mathbb{R}^n\longrightarrow(-\infty,+\infty)$, is defined by (see [6])

$$h(K,x)=\max\{x\cdot y: y\in K\},\quad x\in\mathbb{R}^n,$$

where $x\cdot y$ denotes the standard inner product of x and y. From the definition of support functions, we easily deduce that for $K\in\mathscr{K}^n$ and $c>0$, $h(cK,\cdot)=ch(K,\cdot)$; for $K,L\in\mathscr{K}^n$, $h(K,\cdot)=h(L,\cdot)$ if and only if $K=L$. And if $\phi\in GL(n)$ then (see [6, 34])

$$h_{\phi K}(u)=h_K(\phi^t u) \tag{2.1}$$

for all $u\in S^{n-1}$, where $GL(n)$ denotes the group of general (nonsingular) linear transformations and ϕ^t denotes the transpose of ϕ.

If K is a compact star-shaped (about the origin) set in $\mathbb{R}^n$, its radial function, $\rho_K=\rho(K,\cdot):\mathbb{R}^n\backslash\{0\}\longrightarrow[0,+\infty)$, is defined by (see [34])

$$\rho(K,x)=\max\{\lambda\geqslant 0:\lambda x\in K\},\quad x\in\mathbb{R}^n\backslash\{0\}.$$

If ρ_K is positive and continuous, K will be called a star body (about the origin). Two star bodies K and L are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u\in S^{n-1}$.

If E is a nonempty set in $\mathbb{R}^n$, the polar set of E, E^*, is defined by (see [6, 34])

$$E^*=\{x:x\cdot y\leqslant 1, y\in E\},\quad x\in\mathbb{R}^n.$$

From the above definitions, we see that if $K\in\mathscr{K}_o^n$, then (see [6, 34])

$$h_{K^*}=\frac{1}{\rho_K},\quad \rho_{K^*}=\frac{1}{h_K}. \tag{2.2}$$

For $K,L\in\mathscr{K}_o^n$, $p\geqslant 1$ and $\lambda,\mu\geqslant 0$ (not both zero), the L_p-Minkowski combination, $\lambda\cdot K+_p\mu\cdot L\in\mathscr{K}_o^n$, of K and L is defined by (see [20])

$$h(\lambda\cdot K+_p\mu\cdot L,\cdot)^p=\lambda h(K,\cdot)^p+\mu h(L,\cdot)^p. \tag{2.3}$$

Note that $\lambda\cdot K=\lambda^{1/p}K$.

For $K,L\in\mathscr{S}_o^n$, $p\geqslant 1$ and $\lambda,\mu\geqslant 0$ (not both zero), the L_p-harmonic radial combination, $\lambda\star K+_{-p}\mu\star L\in\mathscr{S}_o^n$, of K and L is defined by (see [21])

$$\rho(\lambda\star K+_{-p}\mu\star L,\cdot)^{-p}=\lambda\rho(K,\cdot)^{-p}+\mu\rho(L,\cdot)^{-p}. \tag{2.4}$$

Note that $\lambda\star K=\lambda^{-1/p}K$.

From (2.2), (2.3) and (2.4), we easily get that if $K,L\in\mathscr{K}_o^n$, $p\geqslant 1$, and $\lambda,\mu\geqslant 0$ (not both zero), then (see [21])

$$(\lambda\cdot K+_p\mu\cdot L)^*=\lambda\star K^*+_{-p}\mu\star L^*. \tag{2.5}$$

For $K \in \mathscr{K}^n$, $i = 0, 1, \cdots, n-1$, the quermassintegrals, $W_i(K)$, of K are defined by (see [6, 34])

$$W_i(K) = \frac{1}{n}\int_{S^{n-1}} h(K,u)dS_i(K,u), \tag{2.6}$$

where $S_i(K,\cdot)$ $(i = 0, 1, \cdots, n-1)$ denote the areas measures of order i of K, $S_0(K,\cdot)$ is just the surface area measure $S(K,\cdot)$ of K. From definition (2.6), we easily see that

$$W_0(K) = \frac{1}{n}\int_{S^{n-1}} h(K,u)dS(K,u) = V(K). \tag{2.7}$$

For the L_p-Minkowski combination, Lutwak [20] proved the following Brunn-Minkowski type inequality for quermassintegrals.

THEOREM 2.A. *If $K, L \in \mathscr{K}_o^n$, $p > 1$, $i = 0, 1, \cdots, n-1$, and $\lambda, \mu \geqslant 0$ (not both zero), then*

$$W_i(\lambda \cdot K +_p \mu \cdot L)^{\frac{p}{n-i}} \geqslant \lambda W_i(K)^{\frac{p}{n-i}} + \mu W_i(L)^{\frac{p}{n-i}}, \tag{2.8}$$

with equality if and only if K and L are dilates.

For $K \in \mathscr{S}_o^n$ and any real i, the dual quermassintegrals, $\widetilde{W}_i(K)$, of K are defined by (see [17])

$$\widetilde{W}_i(K) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n-i}du. \tag{2.9}$$

Obviously, if $i = 0$ in (2.9), then

$$\widetilde{W}_0(K) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^n du = V(K). \tag{2.10}$$

The L_p-dual mixed quermassintegrals were introduced by Wang and Leng (see [38]). For $K, L \in \mathscr{S}_o^n$, $p \geqslant 1$ and real $i \neq n$, the L_p-dual mixed quermassintegrals, $\widetilde{W}_{-p,i}(K,L)$, of K and L are defined by

$$\widetilde{W}_{-p,i}(K,L) = \frac{1}{n}\int_{S^{n-1}} \rho_K^{n+p-i}(u)\rho_L^{-p}(u)du. \tag{2.11}$$

From (2.11) and (2.9), we see that for each $K \in \mathscr{S}_o^n$ and $p \geqslant 1$,

$$\widetilde{W}_{-p,i}(K,K) = \widetilde{W}_i(K). \tag{2.12}$$

The Minkowski inequality for the L_p-dual mixed quermassintegrals states the following (see [38]).

THEOREM 2.B. *If $K, L \in \mathscr{S}_o^n$, $p \geqslant 1$ and real $i \neq n$, then for $i < n$ or $n < i < n+p$,*

$$\widetilde{W}_{-p,i}(K,L) \geqslant \widetilde{W}_i(K)^{\frac{n+p-i}{n-i}}\widetilde{W}_i(L)^{-\frac{p}{n-i}}; \tag{2.13}$$

for $i > n+p$,

$$\widetilde{W}_{-p,i}(K,L) \leqslant \widetilde{W}_i(K)^{\frac{n+p-i}{n-i}}\widetilde{W}_i(L)^{-\frac{p}{n-i}}. \tag{2.14}$$

Equality holds in each inequality if and only if K and L are dilates. For $i = n+p$, (2.13) (or (2.14)) becomes an equality.

Further, Wang and Leng [39] established the following Brunn-Minkowski type inequality for dual quermassintegrals.

THEOREM 2.C. *If $K,L\in\mathscr{S}_o^n$, $p\geqslant 1$, real $i\neq n$ and $\lambda,\mu\geqslant 0$ (not both zero), then for $i<n$ or $n<i<n+p$,*

$$\widetilde{W}_i(\lambda\star K+_{-p}\mu\star L)^{-\frac{p}{n-i}}\geqslant\lambda\widetilde{W}_i(K)^{-\frac{p}{n-i}}+\mu\widetilde{W}_i(L)^{-\frac{p}{n-i}};\tag{2.15}$$

for $i>n+p$,

$$\widetilde{W}_i(\lambda\star K+_{-p}\mu\star L)^{-\frac{p}{n-i}}\leqslant\lambda\widetilde{W}_i(K)^{-\frac{p}{n-i}}+\mu\widetilde{W}_i(L)^{-\frac{p}{n-i}}.\tag{2.16}$$

In each inequality, equality holds if and only if K and L are dilates. For $i=n+p$, (2.15) (or (2.16)) becomes an equality.

3. Some properties of general L_p-centroid bodies

THEOREM 3.1. *For $K\in\mathscr{S}_o^n$, $p\geqslant 1$ and $\tau\in[-1,1]$, if $\phi\in GL(n)$, then $\Gamma_p^\tau\phi K=\phi\Gamma_p^\tau K$.*

Proof. From definitions (1.7) and (2.1), we have that for $\phi\in GL(n)$ and all $u\in S^{n-1}$,

$$\begin{aligned}
h^p_{\Gamma_p^\tau\phi K}(u)&=\frac{2}{c_{n,p}(\tau)V(\phi K)}\int_{\phi K}\varphi_\tau(u\cdot x)^p dx\\
&=\frac{2}{c_{n,p}(\tau)|det\phi|V(K)}\int_K\varphi_\tau(u\cdot\phi y)^p|det\phi|dy\ \ (where\ \ y=\phi^{-1}x)\\
&=\frac{2}{c_{n,p}(\tau)V(K)}\int_K\varphi_\tau(\phi^t u\cdot y)^p dy\\
&=h^p_{\Gamma_p^\tau K}(\phi^t u)=h^p_{\phi\Gamma_p^\tau K}(u).
\end{aligned}$$

This gives the desired result. □

THEOREM 3.2. *If $K\in\mathscr{S}_o^n$, $p\geqslant 1$ and $\tau\in[-1,1]$, then*

$$\Gamma_p^\tau(-K)=-\Gamma_p^\tau K=\Gamma_p^{-\tau}K.\tag{3.1}$$

Proof. From definition (1.8), we have

$$h^p_{-\Gamma_p^+K}(u)=h^p_{\Gamma_p^+K}(-u)=\frac{2}{c_{n,p}V(K)}\int_K(-u\cdot x)_+^p dx=h^p_{\Gamma_p^-K}(u)$$

for all $u\in S^{n-1}$. This together with (1.9) yields

$$\Gamma_p^-K=\Gamma_p^+(-K)=-\Gamma_p^+K.\tag{3.2}$$

Similarly,

$$\Gamma_p^+K=\Gamma_p^-(-K)=-\Gamma_p^-K.\tag{3.3}$$

Hence, by (3.2), (3.3) and (1.10) we obtain

$$\Gamma_p^{\tau}(-K) = -\Gamma_p^{\tau}K. \tag{3.4}$$

In addition, by (1.11) we have that

$$f_1(\tau) + f_2(\tau) = 1, \tag{3.5}$$

$$f_1(-\tau) = f_2(\tau), \quad f_2(-\tau) = f_1(\tau). \tag{3.6}$$

This together with (3.2), (3.3) and (1.10) yields

$$\begin{aligned} \Gamma_p^{-\tau}K &= f_1(-\tau)\cdot\Gamma_p^{+}K +_p f_2(-\tau)\cdot\Gamma_p^{-}K \\ &= f_2(\tau)\cdot\Gamma_p^{-}(-K) +_p f_1(\tau)\cdot\Gamma_p^{+}(-K) = \Gamma_p^{\tau}(-K). \end{aligned} \tag{3.7}$$

Obviously, (3.4) and (3.7) yield (3.1). □

THEOREM 3.3. *If* $K \in \mathscr{S}_o^n$, $p \geqslant 1$, $\tau \in [-1,1]$ *and* $\tau \neq 0$, *then*

$$\Gamma_p^{\tau}K = \Gamma_p^{-\tau}K \quad \Longleftrightarrow \quad \Gamma_p^{+}K = \Gamma_p^{-}K. \tag{3.8}$$

Proof. By (1.10), for $K \in \mathscr{S}_o^n$, $p \geqslant 1$ and $\tau \in [-1,1]$, we have that for all $u \in S^{n-1}$,

$$h^p_{\Gamma_p^{\tau}K}(u) = f_1(\tau)h^p_{\Gamma_p^{+}K}(u) + f_2(\tau)h^p_{\Gamma_p^{-}K}(u). \tag{3.9}$$

On the other hand, by (1.10) and (3.6), we also have

$$\Gamma_p^{-\tau}K = f_2(\tau)\cdot\Gamma_p^{+}K +_p f_1(\tau)\cdot\Gamma_p^{-}K.$$

Thus, we get for all $u \in S^{n-1}$,

$$h^p_{\Gamma_p^{-\tau}K}(u) = f_2(\tau)h^p_{\Gamma_p^{+}K}(u) + f_1(\tau)h^p_{\Gamma_p^{-}K}(u). \tag{3.10}$$

Hence, by (3.5), (3.9) and (3.10), if $\Gamma_p^{+}K = \Gamma_p^{-}K$, then for all $u \in S^{n-1}$,

$$h^p_{\Gamma_p^{\tau}K}(u) = h^p_{\Gamma_p^{-\tau}K}(u).$$

This gives $\Gamma_p^{\tau}K = \Gamma_p^{-\tau}K$.

Conversely, if $\Gamma_p^{\tau}K = \Gamma_p^{-\tau}K$, then (3.9) and (3.10) yield that

$$[f_1(\tau) - f_2(\tau)]h^p_{\Gamma_p^{+}K}(u) = [f_1(\tau) - f_2(\tau)]h^p_{\Gamma_p^{-}K}(u),$$

for all $u \in S^{n-1}$. Since $f_1(\tau) - f_2(\tau) \neq 0$ when $\tau \neq 0$, we get $\Gamma_p^{+}K = \Gamma_p^{-}K$. □

In [11], Haberl and Schuster proved the following a fact.

THEOREM 3.A. *If $K \in \mathscr{S}_o^n$, $p \geqslant 1$ and p is not odd integer, then $\Gamma_p^+ K = \Gamma_p^- K$ if and only if K is origin-symmetric.*

From Theorem 3.3 and Theorem 3.A, we deduce that

COROLLARY 3.1. *If $K \in \mathscr{S}_o^n$, $p \geqslant 1$ and p is not odd integer, then for $\tau \in [-1,1]$ and $\tau \neq 0$, $\Gamma_p^\tau K = \Gamma_p^{-\tau} K$ if and only if K is origin-symmetric.*

THEOREM 3.4. *If $K \in \mathscr{S}_o^n$, $p \geqslant 1$, $\tau \in [-1,1]$ and $\tau \neq 0$, then*

$$\Gamma_p^\tau K +_p \Gamma_p^{-\tau} K = \Gamma_p^+ K +_p \Gamma_p^- K. \tag{3.11}$$

Proof. By (3.9) and (3.10), and using (3.5), we have that for any $u \in S^{n-1}$,

$$h^p(\Gamma_p^\tau K, u) + h^p(\Gamma_p^{-\tau} K, u) = h^p(\Gamma_p^+ K, u) + h^p(\Gamma_p^- K, u), \tag{3.12}$$

i.e.,

$$h^p(\Gamma_p^\tau K +_p \Gamma_p^{-\tau} K, u) = h^p(\Gamma_p^+ K +_p \Gamma_p^- K, u).$$

This proves (3.11). $\square$

From (3.12) and (1.12), we easily get

COROLLARY 3.2. *If $K \in \mathscr{S}_o^n$, $p \geqslant 1$, $\tau \in [-1,1]$, then*

$$\Gamma_p K = \frac{1}{2} \cdot \Gamma_p^\tau K +_p \frac{1}{2} \cdot \Gamma_p^{-\tau} K. \tag{3.13}$$

4. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. Using (1.10) and inequality (2.8), we have

$$W_i(\Gamma_p^\tau K)^{\frac{p}{n-i}} \geqslant f_1(\tau) W_i(\Gamma_p^+ K)^{\frac{p}{n-i}} + f_2(\tau) W_i(\Gamma_p^- K)^{\frac{p}{n-i}}.$$

This combine with (3.5), (3.2) and (3.3), yields

$$W_i(\Gamma_p^\tau K) \geqslant W_i(\Gamma_p^\pm K).$$

This is just the right inequality of (1.15).

From the condition of equality in (2.8), we see that equality holds in the right inequality of (1.15) for $p > 1$ if and only if $\Gamma_p^+ K$ and $\Gamma_p^- K$ are dilates. From this, let $\Gamma_p^+ K = c\Gamma_p^- K$ $(c > 0)$ and use $W_i(\Gamma_p^+ K) = W_i(\Gamma_p^- K)$, then $c = 1$, i.e. $\Gamma_p^+ K = \Gamma_p^- K$. Thus, by Theorem 3.A we see that if K is not origin-symmetric and p is not an odd integer, then equality holds in the right inequality of (1.15) if and only if $\tau = \pm 1$.

On the other hand, from (3.13) and inequality (2.8), we obtain

$$W_i(\Gamma_p K)^{\frac{p}{n-i}} \geqslant \frac{1}{2} W_i(\Gamma_p^\tau K)^{\frac{p}{n-i}} + \frac{1}{2} W_i(\Gamma_p^{-\tau} K)^{\frac{p}{n-i}},$$

this together with (3.1), yields

$$W_i(\Gamma_p K) \geqslant W_i(\Gamma_p^\tau K).$$

This gives the left inequality of (1.15). And equality holds in the left inequality of (1.15) for $p>1$ if and only if $\Gamma_p^\tau K$ and $\Gamma_p^{-\tau} K$ are dilates, this means $\Gamma_p^\tau K = \Gamma_p^{-\tau} K$. Hence, together with Corollary 3.1, we know that if K is not origin-symmetric and p is not an odd integer, then equality holds in the left inequality of (1.15) if and only if $\tau = 0$. □

Proof of Theorem 1.2. From (1.10) and (2.5), we have

$$\Gamma_p^{\tau,*} K = f_1(\tau) \star \Gamma_p^{+,*} K +_{-p} f_2(\tau) \star \Gamma_p^{-,*} K. \tag{4.1}$$

Thus, for $i<n$ or $n<i<n+p$, by (4.1) and inequality (2.15) we know that

$$\widetilde{W}_i(\Gamma_p^{\tau,*} K)^{-\frac{p}{n-i}} \geqslant f_1(\tau)\widetilde{W}_i(\Gamma_p^{+,*} K)^{-\frac{p}{n-i}} + f_2(\tau)\widetilde{W}_i(\Gamma_p^{-,*} K)^{-\frac{p}{n-i}}. \tag{4.2}$$

This together with (3.2) and (3.3), yields

$$\widetilde{W}_i(\Gamma_p^{\tau,*} K)^{-\frac{p}{n-i}} \geqslant \widetilde{W}_i(\Gamma_p^{\pm,*} K)^{-\frac{p}{n-i}}. \tag{4.3}$$

Hence, if $i<n$ then

$$\widetilde{W}_i(\Gamma_p^{\tau,*} K) \leqslant \widetilde{W}_i(\Gamma_p^{\pm,*} K). \tag{4.4}$$

Inequality (4.4) is just the right inequality of (1.16). If $n<i<n+p$, then by (4.3) we get

$$\widetilde{W}_i(\Gamma_p^{\tau,*} K) \geqslant \widetilde{W}_i(\Gamma_p^{\pm,*} K), \tag{4.5}$$

this gives the right inequality of (1.17).

For $i>n+p$, from (4.1) and inequality (2.16), we get that

$$\widetilde{W}_i(\Gamma_p^{\tau,*} K)^{-\frac{p}{n-i}} \leqslant \widetilde{W}_i(\Gamma_p^{\pm,*} K)^{-\frac{p}{n-i}},$$

this yields (4.4).

According to the equality conditions of (2.15) and (2.16), we know that equality hold in (4.4) and (4.5) if and only if $\Gamma_p^{+,*} K$ and $\Gamma_p^{-,*} K$ are dilates. From this, we may get $\Gamma_p^{+,*} K = \Gamma_p^{-,*} K$, i.e. $\Gamma_p^{+} K = \Gamma_p^{-} K$. Hence, from Theorem 3.A, we see that if K is not origin-symmetric and p is not an odd integer, then equality hold in the right inequalities of (1.16) and (1.17) if and only if $\tau = \pm 1$.

Now we prove the left inequalities of (1.16) and (1.17).

Using (2.9), we have that

$$\widetilde{W}_i(\Gamma_p^{\tau,*} K) = \frac{1}{n}\int_{S^{n-1}} \rho_{\Gamma_p^{\tau,*} K}^{n-i}(u)du = \frac{1}{n}\int_{S^{n-1}} (\rho_{\Gamma_p^{\tau,*} K}^{-p}(u))^{-\frac{n-i}{p}} du.$$

But (4.1) gives that

$$\rho_{\Gamma_p^{\tau,*} K}^{-p}(u) = f_1(\tau)\rho_{\Gamma_p^{+,*} K}^{-p}(u) + f_2(\tau)\rho_{\Gamma_p^{-,*} K}^{-p}(u), \tag{4.6}$$

for any $u \in S^{n-1}$. Together with (4.6), we calculate the derivative of the function $\widetilde{W}_i(\Gamma_p^{\tau,*}K)$ with respect to τ as follows: For every $\tau \in [-1,1]$,

$$\begin{aligned}\frac{\partial}{\partial\tau}(\widetilde{W}_i(\Gamma_p^{\tau,*}K)) &= -\frac{n-i}{pn}\int_{S^{n-1}}(\rho_{\Gamma_p^{\tau,*}K}^{-p}(u))^{-\frac{n+p-i}{p}}\frac{\partial}{\partial\tau}(\rho_{\Gamma_p^{\tau,*}K}^{-p}(u))du \\ &= \frac{n-i}{pn}f(\tau)\int_{S^{n-1}}\rho_{\Gamma_p^{\tau,*}K}^{n+p-i}(u)[\rho_{\Gamma_p^{+,*}K}^{-p}(u)-\rho_{\Gamma_p^{-,*}K}^{-p}(u)]du,\end{aligned}$$

where

$$f(\tau) = -f_1'(\tau) = f_2'(\tau) = -\frac{2(1-\tau^2)^{p-1}}{[(1+\tau)^p+(1-\tau)^p]^2} \leqslant 0, \quad \tau \in [-1,1].$$

This combined with (2.11), yields

$$\frac{\partial}{\partial\tau}(\widetilde{W}_i(\Gamma_p^{\tau,*}K)) = \frac{n-i}{p}f(\tau)[\widetilde{W}_{-p,i}(\Gamma_p^{\tau,*}K,\Gamma_p^{+,*}K) - \widetilde{W}_{-p,i}(\Gamma_p^{\tau,*}K,\Gamma_p^{-,*}K)].$$

Note that $p > 1$,

$$f(\tau) = 0 \quad \Longleftrightarrow \quad \tau = \pm 1.$$

Thus, when

$$\frac{\partial}{\partial\tau}(\widetilde{W}_i(\Gamma_p^{\tau,*}K)) = 0,$$

we get $\tau = \pm 1$ for $p > 1$ or

$$\widetilde{W}_{-p,i}(\Gamma_p^{\tau,*}K,\Gamma_p^{+,*}K) = \widetilde{W}_{-p,i}(\Gamma_p^{\tau,*}K,\Gamma_p^{-,*}K).$$

If $i < n$ or $i > n+p$, then by the right inequality of (1.16) we know that

$$\widetilde{W}_i(\Gamma_p^{\tau,*}K) \leqslant \widetilde{W}_i(\Gamma_p^{\pm,*}K),$$

this means $\widetilde{W}_i(\Gamma_p^{\tau,*}K)$ attains its maximum at $\tau = \pm 1$. Thus the points where the minimum of $\widetilde{W}_i(\Gamma_p^{\tau,*}K)$ is attained are contained in $(-1,1)$. If $\tau = \bar{\tau}$ is such a point, then

$$\frac{\partial}{\partial\tau}(\widetilde{W}_i(\Gamma_p^{\tau,*}K))|_{\tau=\bar{\tau}} = 0$$

or, equivalently,

$$\widetilde{W}_{-p,i}(\Gamma_p^{\bar{\tau},*}K,\Gamma_p^{+,*}K) = \widetilde{W}_{-p,i}(\Gamma_p^{\bar{\tau},*}K,\Gamma_p^{-,*}K). \tag{4.7}$$

Therefore, by (4.1), (2.11) and (2.12)

$$\begin{aligned}\widetilde{W}_i(\Gamma_p^{\bar{\tau},*}K) &= \widetilde{W}_{-p,i}(\Gamma_p^{\bar{\tau},*}K,\Gamma_p^{\bar{\tau},*}K) \\ &= \widetilde{W}_{-p,i}(\Gamma_p^{\bar{\tau},*}K, f_1(\bar{\tau})\star\Gamma_p^{+,*}K +_{-p} f_2(\bar{\tau})\star\Gamma_p^{-,*}K) \\ &= f_1(\bar{\tau})\widetilde{W}_{-p,i}(\Gamma_p^{\bar{\tau},*}K,\Gamma_p^{+,*}K) + f_2(\bar{\tau})\widetilde{W}_{-p,i}(\Gamma_p^{\bar{\tau},*}K,\Gamma_p^{-,*}K),\end{aligned}$$

which together with (4.7) and equality (2.11), yields

$$\begin{aligned}\widetilde{W}_i(\Gamma_p^{\bar{\tau},*}K) &= f_1(\bar{\tau})\widetilde{W}_{-p,i}(\Gamma_p^{\bar{\tau},*}K,\Gamma_p^{-,*}K) + f_2(\bar{\tau})\widetilde{W}_{-p,i}(\Gamma_p^{\bar{\tau},*}K,\Gamma_p^{+,*}K)\\ &= \widetilde{W}_{-p,i}(\Gamma_p^{\bar{\tau},*}K, f_1(\bar{\tau})\star\Gamma_p^{-,*}K +_{-p} f_2(\bar{\tau})\star(\Gamma_p^{+,*}K))\\ &= \widetilde{W}_{-p,i}(\Gamma_p^{\bar{\tau},*}K, f_2(-\bar{\tau})\star\Gamma_p^{-,*}K +_{-p} f_1(-\bar{\tau})\star(\Gamma_p^{+,*}K))\\ &= \widetilde{W}_{-p,i}(\Gamma_p^{\bar{\tau},*}K,\Gamma_p^{-\bar{\tau},*}K). \end{aligned} \tag{4.8}$$

Hence, for $i<n$, by (2.13) and (4.8) we have

$$\widetilde{W}_i(\Gamma_p^{\bar{\tau},*}K) \geqslant \widetilde{W}_i(\Gamma_p^{\bar{\tau},*}K)^{\frac{n+p-i}{n-i}}\widetilde{W}_i(\Gamma_p^{-\bar{\tau},*}K)^{-\frac{p}{n-i}},$$

i.e.

$$\widetilde{W}_i(\Gamma_p^{\bar{\tau},*}K) \leqslant \widetilde{W}_i(\Gamma_p^{-\bar{\tau},*}K); \tag{4.9}$$

for $i>n+p$, by (2.14) and (4.8) we obtain

$$\widetilde{W}_i(\Gamma_p^{\bar{\tau},*}K) \leqslant \widetilde{W}_i(\Gamma_p^{\bar{\tau},*}K)^{\frac{n+p-i}{n-i}}\widetilde{W}_i(\Gamma_p^{-\bar{\tau},*}K)^{-\frac{p}{n-i}},$$

this still yields (4.9).

According to the equality conditions of (2.13) and (2.14), we know that equality holds in (4.9) if and only if $\Gamma_p^{\bar{\tau},*}K$ and $\Gamma_p^{-\bar{\tau},*}K$ are dilates.

Similarly, we also get for $i<n$ or $i>n+p$,

$$\widetilde{W}_i(\Gamma_p^{-\bar{\tau},*}K) \leqslant \widetilde{W}_i(\Gamma_p^{\bar{\tau},*}K), \tag{4.10}$$

and with equality if and only if $\Gamma_p^{\bar{\tau},*}K$ and $\Gamma_p^{-\bar{\tau},*}K$ are dilates.

Thus (4.9) and (4.10) give that for $i<n$ or $i>n+p$,

$$\widetilde{W}_i(\Gamma_p^{\bar{\tau},*}K) = \widetilde{W}_i(\Gamma_p^{-\bar{\tau},*}K)$$

and $\Gamma_p^{\bar{\tau},*}K$ and $\Gamma_p^{-\bar{\tau},*}K$ are dilates. This together with (2.9) yields $\Gamma_p^{\bar{\tau},*}K = \Gamma_p^{-\bar{\tau},*}K$, i.e.,

$$\rho^{-p}(\Gamma_p^{\bar{\tau},*}K,u) = \rho^{-p}(\Gamma_p^{-\bar{\tau},*}K,u)$$

for any $u\in S^{n-1}$. Hence, by (4.6) we obtain

$$[f_1(\bar{\tau}) - f_2(\bar{\tau})][\rho^{-p}(\Gamma_p^{+,*}K,u) - \rho^{-p}(\Gamma_p^{-,*}K,u)] = 0.$$

From this, if $f_1(\bar{\tau}) - f_2(\bar{\tau}) = 0$, then we have $\bar{\tau}=0$. This means $\widetilde{W}_i(\Gamma_p^{\tau,*}K)$ attains its minimum at $\tau=0$, i.e., for $i<n$ or $i>n+p$,

$$\widetilde{W}_i(\Gamma_p^{\tau,*}K) \geqslant \widetilde{W}_i(\Gamma_p^{*}K).$$

So we get the left inequality of (1.16).

If $\rho^{-p}(\Gamma_p^{+,*}K,u)-\rho^{-p}(\Gamma_p^{-,*}K,u)=0$ for all $u\in S^{n-1}$, then $\Gamma_p^{+,*}K=\Gamma_p^{-,*}K$, i.e., $\Gamma_p^{+}K=\Gamma_p^{-}K$. Thus by Theorem 3.3 we know that if $\tau\neq 0$ then $\Gamma_p^{\tau}K=\Gamma_p^{-\tau}K$ for $\tau\in[-1,1]$. This together with (3.13), yields $\Gamma_p^{\tau}K=\Gamma_pK$, i.e., for $\tau\in[-1,1]$,

$$\widetilde{W}_i(\Gamma_p^{\tau,*}K)=\widetilde{W}_i(\Gamma_p^{*}K).$$

But by Theorem 3.A, we see that $\Gamma_p^{+}K=\Gamma_p^{-}K$ if and only if K is origin-symmetric. This means that if K is not origin-symmetric then equality holds in the left inequality of (1.16) if and only if $\tau=0$.

If $n<i<n+p$, then by the right inequality of (1.17) we know that

$$\widetilde{W}_i(\Gamma_p^{\tau,*}K)\geqslant\widetilde{W}_i(\Gamma_p^{\pm,*}K),$$

this means $\widetilde{W}_i(\Gamma_p^{\tau,*}K)$ attains its minimum at $\tau=\pm1$. Thus the points where the maximum of $\widetilde{W}_i(\Gamma_p^{\tau,*}K)$ is attained are contained in $(-1,1)$. Similar to the above proof of the cases $i<n$ or $i>n+p$, we can obtain the left inequality of (1.17). □

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