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INEQUALITIES RELATED TO HERON MEANS FOR POSITIVE OPERATORS

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Abstract. For positive operators A and B on a Hilbert space, their Heron mean is defined by $H_{r,\mu}(A,B) = rA\sharp_\mu B + (1-r)A\nabla_\mu B$, where $A\sharp_\mu B$ and $A\nabla_\mu B$ are the μ-geometric and μ-arithmetic means of A and B, respectively. Recently, the relationship $H_{r,\mu}(A,B) \leqslant A!_\mu B$ has been shown under some condition on μ and r. In this paper, we improve the condition and show more sufficient conditions for the relationship.

1. Introduction

Throughout the paper, we use uppercase letters for invertible positive operators on a Hilbert space and lowercase letters for real numbers. The following notation will be used:

- $A \geqslant B$ $(A > B)$ denotes that $A - B$ is a positive (invertible positive) operator.
- $A \geqslant 0$ $(A > 0)$ denotes that A is a positive (invertible positive) operator.

For $A, B > 0$ and $0 \leqslant \mu \leqslant 1$, the μ-arithmetic, μ-geometric, and μ-harmonic means of A and B are defined, respectively, by

$$A\nabla_\mu B = (1-\mu)A + \mu B,$$
$$A\sharp_\mu B = A^{1/2}(A^{-1/2}BA^{-1/2})^\mu A^{1/2},$$
$$A!_\mu B = ((1-\mu)A^{-1} + \mu B^{-1})^{-1}.$$

In the case $\mu = \frac{1}{2}$, we will omit the μ-value in them. For example, $A\nabla B$ denotes $A\nabla_{\frac{1}{2}}B$. As in the case of scalars, the following relationship is well-known:

$$A!_\mu B \leqslant A\sharp_\mu B \leqslant A\nabla_\mu B$$

for $A, B > 0$ and $0 \leqslant \mu \leqslant 1$. There are several means that interpolate between the basic three means. One of them is the Heron mean defined by

$$H_{r,\mu}(A,B) = rA\sharp_\mu B + (1-r)A\nabla_\mu B$$

for $r \in \mathbb{R}$ and $0 \leqslant \mu \leqslant 1$ (see [1]). It interpolates between the arithmetic mean and the geometric mean.

The following has been recently proved in [2].

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THEOREM 1. *Let* $A, B > 0$ *and* $r \in \mathbb{R}$.

1. *If* $0 \leqslant \mu \leqslant 1$ *and* $r \leqslant 1$, *then* $H_{r,\mu}(A,B) \geqslant A!_\mu B$.
2. *If* $\frac{1}{2} \leqslant \mu < 1$ *and* $r \geqslant r_\mu$, *then* $H_{r,\mu}(A,B) \leqslant A!_\mu B$, *where* $r_\mu = \frac{2(2-\mu)}{3(1-\mu)}$.

In this paper, we will improve the second result of the theorem and give other conditions on r, μ, A, and B for the relationship $H_{r,\mu}(A,B) \leqslant A!_\mu B$.

2. Improved result

LEMMA 2. *For* $0 < \mu < 1$ *and* $t > 0$, *define* $f_\mu(t)$ *by*

$$f_\mu(t) = \begin{cases} \frac{t-(1-\mu)t^{1+\mu}-\mu t^\mu}{(t-1)^2}, & t > 0,\ t \neq 1, \\ \frac{-\mu(1-\mu)}{2}, & t = 1. \end{cases}$$

Then for all $0 < \mu < 1$ *and* $t > 0$,

$$f_\mu(t) \geqslant \frac{-\mu(1-\mu)}{4\mu(1-\mu)+1}.$$

Proof. Since $(1-\mu)t^{1+\mu} + \mu t^\mu \geqslant t^{(1-\mu)(1+\mu)} t^{\mu^2} = t$ by Young inequality, $f_\mu(t) \leqslant 0$ for all $t > 0$. Moreover, it is easy to show that

$$\begin{aligned} &\lim_{t\to 0+} f_\mu(t) = \lim_{t\to +\infty} f_\mu(t) = 0, \\ &\lim_{t\to 1} f_\mu(t) = f_\mu(1), \\ &(t-1)^3 f'_\mu(t) = -1 - t + t^{\mu-1}\left\{((1-\mu)t+\mu)^2 + t\right\}, \qquad (1) \\ &f'_\mu(1) = \frac{\mu(1-\mu)(1-2\mu)}{6}. \end{aligned}$$

Consider $t_\mu \neq 1$ at which f_μ attains its minimum value. Then $t = t_\mu$ satisfies

$$t^\mu = \frac{t(t+1)}{((1-\mu)t+\mu)^2 + t} \qquad (2)$$

by (1). Using the equation above, we can express $f_\mu(t_\mu)$ by

$$\begin{aligned} f_\mu(t_\mu) &= \frac{-\mu(1-\mu)t_\mu}{((1-\mu)t_\mu+\mu)^2 + t_\mu}, \\ &= \frac{-\mu(1-\mu)}{((1-\mu)t_\mu+\mu)^2/t_\mu + 1}. \end{aligned}$$

A direct computation shows that the map $t \mapsto ((1-\mu)t+\mu)^2/t$ attains its minimum value at $t=\mu/(1-\mu)$. Thus

$$f_\mu(t) \geqslant \frac{-\mu(1-\mu)}{4\mu(1-\mu)+1}$$

for all $t>0$ (note that the right hand side of the inequality is larger than or equal to $f_\mu(1)$). $\square$

The following improves the second result of Theorem 1.

THEOREM 3. *For* $0<\mu<1$, *let* $r'_\mu = 1+\frac{1}{4\mu(1-\mu)}$. *If* $r \geqslant r'_\mu$, *then*

$$H_{r,\mu}(A,B) \leqslant A!_\mu B$$

for all $A,B>0$.

Proof. Assume that the following holds for all $t>0$ and $r \geqslant r'_\mu$:

$$rt^\mu + (1-r)((1-\mu)+\mu t) \leqslant \left((1-\mu)+\mu t^{-1}\right)^{-1}. \tag{3}$$

Then for any $X>0$ we have

$$rX^\mu + (1-r)((1-\mu)I+\mu X) \leqslant \left((1-\mu)I+\mu X^{-1}\right)^{-1}$$

by the operator monotonicity of continuous functions, where I is the identity operator. Replacing X by $A^{-1/2}BA^{-1/2}$, we have

$$r(A^{-1/2}BA^{-1/2})^\mu + (1-r)\left((1-\mu)I+\mu A^{-1/2}BA^{-1/2}\right) \leqslant ((1-\mu)I+\mu A^{1/2}B^{-1}A^{1/2})^{-1}$$

and thereby the desired inequality $H_{r,\mu}(A,B) \leqslant A!_\mu B$.

Now we prove (3). A simple algebra shows that (3) can be written as

$$r(1-\mu+\mu t-t^\mu) \geqslant \mu(1-\mu)\frac{(t-1)^2}{(1-\mu)t+\mu}$$

which in terms of $f_\mu(t)$ defined in Lemma 2 is

$$r\left(\mu(1-\mu)+f_\mu(t)\right) \geqslant \mu(1-\mu). \tag{4}$$

By Lemma 2,

$$\begin{aligned}\mu(1-\mu)+f_\mu(t) &\geqslant \mu(1-\mu)-\frac{\mu(1-\mu)}{4\mu(1-\mu)+1}\\ &= \mu(1-\mu)\cdot\frac{4\mu(1-\mu)}{4\mu(1-\mu)+1}\\ &= \frac{\mu(1-\mu)}{r'_\mu}\end{aligned}$$

for all $t>0$. Thus if $r \geqslant r'_\mu$, then we have (4). $\square$

Note that Theorem 3 improves the second result of Theorem 1, since

1. $2 \leqslant r'_\mu \leqslant r_\mu$ for $\frac{1}{2} \leqslant \mu \leqslant 1$,

2. r'_μ is defined for all $\mu \in (0,1)$ and symmetric about $\mu = \frac{1}{2}$.

In the proof of Lemma 2, we simply used (2). We can analyze it more and produce another condition for $H_{r,\mu}(A,B) \leqslant A!_\mu B$.

LEMMA 4. *Let $f_\mu(t)$ be the function defined in Lemma 2. If $(2\mu-1)(t-1) < 0$, then*

$$f_\mu(t) > f_\mu(1) = \frac{-\mu(1-\mu)}{2}.$$

Proof. Letting $s = t^{-1}$ in (1), we have

$$t^{-\mu-1}(t-1)^3 f'_\mu(t) = (1-\mu+\mu s)^2 + s - s^\mu(s+1). \tag{5}$$

Putting the right hand side by $g_\mu(s)$, we will show that $g_\mu(s) > 0$ for s and μ with $(2\mu-1)(s-1) > 0$. A direct computation shows

$$\begin{aligned} g'_\mu(s) &= 2(1-\mu+\mu s)\mu + 1 - \mu s^{\mu-1} - (1+\mu)s^\mu, \\ g''_\mu(s)/\mu &= 2\mu + (1-\mu)s^{\mu-2} - (1+\mu)s^{\mu-1} \\ &= (1+\mu)\left(\frac{2\mu}{1+\mu} + \frac{1-\mu}{1+\mu}s^{\mu-2} - s^{\mu-1}\right). \end{aligned}$$

By Young inequality,

$$\frac{2\mu}{1+\mu} + \frac{1-\mu}{1+\mu}s^{\mu-2} \geqslant s^{\frac{(1-\mu)(\mu-2)}{1+\mu}}.$$

Moreover, since

$$\frac{(1-\mu)(\mu-2)}{1+\mu} - (\mu-1) = \frac{1-\mu}{1+\mu} \cdot (2\mu-1),$$

if $(2\mu-1)(s-1) > 0$, then $s^{(1-\mu)(\mu-2)/(1+\mu)} > s^{\mu-1}$ and thereby $g''_\mu(s) > 0$. Therefore, since $g'_\mu(1) = g_\mu(1) = 0$, $g_\mu(s) > 0$ for s and μ with $(2\mu-1)(s-1) > 0$. By (5), we have $(t-1)f'_\mu(t) > 0$ for t and μ with $(2\mu-1)(t-1) < 0$. That is, if $\mu < \frac{1}{2}$ and $t > 1$, then $f'_\mu(t) > 0$ and if $\mu > \frac{1}{2}$ and $t < 1$, then $f'_\mu(t) < 0$. Consequently, $f_\mu(t) > f_\mu(1)$ for t and μ with $(2\mu-1)(t-1) < 0$. □

We give another proof of [3, Corollary 2.7].

THEOREM 5. *If $r \geqslant 2$, then $H_{r,\mu}(A,B) \leqslant A!_\mu B$ for all $\mu \in (0,1)$ and $A,B > 0$ with $(2\mu-1)(B-A) < 0$.*

Proof. We have

$$r\left(\mu(1-\mu) + f_\mu(t)\right) \geqslant 2\left(\mu(1-\mu) - \frac{\mu(1-\mu)}{2}\right) = \mu(1-\mu)$$

by Lemma 4 and thus (4) is satisfied. By the argument used in the proof of Theorem 3, we have the desired result. □

A more delicate computation results in another sufficient condition for $H_{r,\mu}(A,B) \leqslant A!_\mu B$. By Lemma 4, we know that if $\mu < \frac{1}{2}$, $f'_\mu(t) > 0$ for $t > 1$ and if $\mu > \frac{1}{2}$, $f'_\mu(t) < 0$ for $t < 1$. We can expand the result as follows.

LEMMA 6. *Let $f_\mu(t)$ be the function in Lemma 2. If $\mu < \frac{1}{2}$, $f'_\mu(t) > 0$ on $(\frac{\mu}{1-\mu}, 1)$ and if $\mu > \frac{1}{2}$, $f'_\mu(t) < 0$ on $(1, \frac{\mu}{1-\mu})$. Thus if $(2\mu - 1)((1-\mu)t - \mu) < 0$, then $f_\mu(t) > f_\mu(\frac{\mu}{1-\mu})$.*

Proof. From (5), it suffices to show that if $\mu > \frac{1}{2}$, then $g_\mu(s) < 0$ for $\frac{1-\mu}{\mu} < s < 1$ and if $\mu < \frac{1}{2}$, then $g_\mu(s) < 0$ for $1 < s < \frac{1-\mu}{\mu}$, where $g_\mu(s)$ is the right hand side of (5), *i.e.*,

$$g_\mu(s) = (1 - \mu + \mu s)^2 + s - s^\mu(s+1).$$

Case $\mu > \frac{1}{2}$ and $\frac{1-\mu}{\mu} < s < 1$: We will fix $0 < s < 1$, consider $g_\mu(s)$ as a function in μ, and show $h(\mu) = -g_\mu(s) > 0$ for $\mu \in I_s = (\frac{1}{1+s}, 1)$. A simple algebra shows

$$\begin{aligned} h'(\mu) &= (s+1)s^\mu \ln s - 2(s-1)(1-\mu+\mu s), \\ h''(\mu) &= (s+1)s^\mu (\ln s)^2 - 2(s-1)^2, \\ h'''(\mu) &= (s+1)s^\mu (\ln s)^3. \end{aligned}$$

Since $h'''(\mu) < 0$, h'' is a decreasing function. We will show that $h''(\frac{1}{1+s}) < 0$, *that is*,

$$(s+1)s^{1/(1+s)}(\ln s)^2 < 2(s-1)^2 \text{ for } 0 < s < 1,$$

or replacing s by t,

$$(t+1)t^{1/(t+1)}(\ln t)^2 < 2(t-1)^2 \text{ for } t > 1.$$

Taking the logarithm, $\varphi'(t) > 0$ will be shown for $t > 1$, where

$$\varphi(t) = \ln 2 + 2\ln(t-1) - \ln(t+1) - \frac{\ln t}{t+1} - 2\ln\ln t.$$

Since $\varphi'(t) = \frac{2}{t-1} - \frac{1}{t} + \frac{\ln t}{(t+1)^2} - \frac{2}{t\ln t}$,

$$\varphi'(t) > 0 \iff \frac{t+1}{t-1} + \frac{t\ln t}{(t+1)^2} > \frac{2}{\ln t}.$$

It is trivial to show $(t+1)\ln t > 2(t-1)$ for $t > 1$, which implies $\frac{t+1}{t-1} > \frac{2}{\ln t}$. Thus $\varphi'(t) > 0$ follows directly from the above relationship. Therefore $h''(\frac{1}{1+s}) < 0$ and thus $h''(\mu) < 0$ on I_s. Since $h'(\mu)$ is decreasing on the interval, if $h'(\frac{1}{1+s}) \leqslant 0$, $h(\mu)$

is also decreasing. In this case, since $h(1)=0$, $h(\mu)>0$ for $\mu\in I_s$. On the other hand, if $h'(\frac{1}{1+s})>0$, $h(\mu)>\min\{h(\frac{1}{1+s}),h(1)\}$ on I_s. Thus it suffices to show that

$$h(\frac{1}{1+s})=s^{1/(1+s)}(s+1)-s-\frac{4s^2}{(1+s)^2}$$

is positive for $0<s<1$. Since

$$\begin{aligned}h(\frac{1}{1+s})>0 &\iff s^{-s/(1+s)}>\frac{(1+s)^2+4s}{(1+s)^3}\\ &\iff -\frac{s\ln s}{1+s}+3\ln(1+s)-\ln(s^2+6s+1)>0\\ &\iff \psi(s)\equiv -s\ln s+3(1+s)\ln(1+s)-(1+s)\ln(s^2+6s+1)>0,\end{aligned}$$

we will show that $\psi(s)>0$ for $0<s<1$. A direct computation shows

$$\begin{aligned}\psi'(s) &= -\ln s+3\ln(1+s)-\ln(s^2+6s+1)+\frac{4(s-1)}{s^2+6s+1},\\ \psi''(s) &= \frac{-s^4+12s^3-22s^2+12s-1}{s(s+1)(s^2+6s+1)^2}.\end{aligned}$$

Letting $\varphi(s)=-s^4+12s^3-22s^2+12s-1$, we can show that

$$\begin{aligned}\varphi'(s)/4 &= -s^3+9s^2-11s+3,\\ \varphi''(s)/4 &= -3s^2+19s-11\end{aligned}$$

and that $\varphi''(s)$ is an increasing function with $\varphi''(0)<0$ and $\varphi''(1)>0$. Since $\varphi'(0)>0$ and $\varphi'(1)=0$, $\varphi'(s_0)>0$ on $(0,s_0)$ and $\varphi'(s_0)<0$ on $(s_0,1)$ for some $s_0\in(0,1)$. Moreover, since $\varphi(0)<0$ and $\varphi(1)=0$, $\varphi(s)<0$ on $(0,s_1)$ and $\varphi(s)>0$ on $(s_1,1)$ for some $s_1\in(0,1)$. Thus $\psi''(s)<0$ on $(0,s_1)$ and $\psi''(s)>0$ on $(s_1,1)$. Since $\lim_{s\to 0+}\psi'(s)=+\infty$ and $\psi'(1)=0$, $\psi'(s)>0$ on $(0,s_2)$ and $\psi'(s)<0$ on $(s_2,1)$ for some $s_2\in(0,1)$. Finally, since $\psi(0)=\psi(1)=0$, we conclude $\psi(s)>0$ for $s\in(0,1)$.

Case $\mu<\frac{1}{2}$ and $1<s<\frac{1-\mu}{\mu}$: By the previous result, $g_\mu(s)<0$ for s and μ with $\mu>\frac{1}{2}$ and $\frac{1-\mu}{\mu}<s<1$. Replacing s by t^{-1},

$$\begin{aligned}g_\mu(s) &= g_\mu(t^{-1})\\ &= t^{-2}\left[((1-\mu)t+\mu)^2+t-t^{1-\mu}(t+1)\right]\\ &= t^{-2}g_{1-\mu}(t).\end{aligned}$$

Thus $g_{1-\mu}(t)<0$ for t and μ with $\mu>\frac{1}{2}$ and $1<t<\frac{\mu}{1-\mu}$. Replacing μ by $1-\mu$, $g_\mu(t)<0$ for all t and μ such that $\mu<\frac{1}{2}$ and $1<t<\frac{1-\mu}{\mu}$. $\square$

THEOREM 7. *For* $\mu \in (0,1)$, *let*

$$\tilde{r}_\mu = \frac{(2\mu-1)^2}{(2\mu-1)^2 - 2\mu^\mu(1-\mu)^{1-\mu}+1}.$$

If $r \geqslant \tilde{r}_\mu$ *,then* $H_{r,\mu}(A,B) \leqslant A!_\mu B$ *for* $A,B>0$ *with* $(2\mu-1)((1-\mu)B-\mu A)<0$.

Proof. Since

$$f_\mu\left(\frac{\mu}{1-\mu}\right) = -\frac{\mu(1-\mu)}{(2\mu-1)^2}\cdot\left(2\mu^\mu(1-\mu)^{1-\mu}-1\right),$$

we have

$$\begin{aligned}\mu(1-\mu)+f_\mu(t) &\geqslant \mu(1-\mu) - \frac{\mu(1-\mu)}{(2\mu-1)^2}\cdot\left(2\mu^\mu(1-\mu)^{1-\mu}-1\right)\\ &= \mu(1-\mu)\cdot\frac{(2\mu-1)^2-2\mu^\mu(1-\mu)^{1-\mu}+1}{(2\mu-1)^2}\\ &= \frac{\mu(1-\mu)}{\tilde{r}_\mu}\end{aligned}$$

by Lemma 6. Thus if $r \geqslant \tilde{r}_\mu$, then r satisfies (4). By the argument in the proof of Theorem 3, we have the desired result. □

REMARK 8. For r'_μ and $\tilde{r}_\mu$ defined in Theorem 3 and Theorem 7, respectively, a direct computation shows $\tilde{r}_\mu \leqslant r'_\mu$ for $0<\mu<1$.

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