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INEQUALITIES INVOLVING NORM AND ESSENTIAL NORM OF WEIGHTED COMPOSITION OPERATORS

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Abstract. We characterize the boundedness and compactness of the weighted composition operator acting from the weighted Bergman space $\mathscr{A}^p(\sigma)$ to the Zygmund-type space $\mathscr{Z}_\nu$, where σ is an admissible weight and ν is a normal weight. Some upper and lower bounds for the norm and essential norm of the operator are also given.

1. Introduction and preliminaries

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, $H(\mathbb{D})$ the space of all holomorphic functions on $\mathbb{D}$ and H^∞ the Banach space of bounded analytic functions on $\mathbb{D}$. Let $\psi \in H(\mathbb{D})$ and φ be a holomorphic self-map of $\mathbb{D}$. Then the *weighted composition operator* $W_{\psi,\varphi}$ is a linear operator on $H(\mathbb{D})$ defined by $W_{\psi,\varphi}f = \psi \cdot f \circ \varphi$ for $f \in H(\mathbb{D})$. It is of interest to provide function-theoretic characterizations involving symbols ψ and φ for the boundedness and compactness of $W_{\psi,\varphi}$ acting between different function spaces. Recently, several authors have studied these type of operators on different spaces of holomorphic functions, see for example, [2]–[34] and the related references therein.

Let $\sigma : [0,1) \to [0,\infty)$ be a non-increasing continuous function. We extend it on $\mathbb{D}$ by $\sigma(z) = \sigma(|z|)$, $z \in \mathbb{D}$ and call it a *weight* or a *weight function*. Throughout this paper, we assume that a weight σ will also satisfy the following properties:

(1) $\sigma(r)(1-r)^{-(1+\gamma)}$ is non-decreasing for some $\gamma > 0$;

(2) $\lim_{r\to 1^-} \sigma(r) = 0$.

Such a weight function is called an *admissible weight*. For $0 < p < \infty$ and σ an admissible weight, we denote by $\mathscr{A}^p(\sigma)$ the weighted Bergman space consisting of holomorphic functions f on $\mathbb{D}$ such that

$$||f||^p_{\mathscr{A}^p(\sigma)} = \int_{\mathbb{D}} |f(z)|^p \sigma(z)dA(z) < \infty,$$

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where $dA(z) = \frac{1}{\pi} dxdy = \frac{1}{\pi} rdrd\theta$ stands for the normalized area measure in $\mathbb{D}$. Recall that a weight ν is normal if there exist positive numbers η and τ, $0 < \eta < \tau$ and $\delta \in [0,1)$ such that

$$\frac{\nu(r)}{(1-r)^{\eta}} \text{ is decreasing on } [\delta,1) \text{ and } \lim_{r\to 1} \frac{\nu(r)}{(1-r)^{\eta}} = 0;$$

$$\frac{\nu(r)}{(1-r)^{\tau}} \text{ is increasing on } [\delta,1) \text{ and } \lim_{r\to 1} \frac{\nu(r)}{(1-r)^{\tau}} = \infty.$$

It is well known that classical weights $\sigma_\alpha(z) = (1-|z|^2)^\alpha, \quad \alpha > -1$ are normal weights.

The following lemma is folklore and can be proved as Lemma 2.1 in [6] or Lemma 1 in [18], we omit the details.

LEMMA 1.1. *Assume that* $p > 0$, $k \in \mathbb{N}_0$ *and* σ *is an admissible weight. Then, there is a positive constant* C *such that*

$$|f^{(k)}(z)| \leqslant C \frac{\|f\|_{\mathscr{A}^p(\sigma)}}{\sigma^{1/p}(z)(1-|z|^2)^{k+2/p}}$$

for every $z \in \mathbb{D}$ *and* $f \in \mathscr{A}^p(\sigma)$.

LEMMA 1.2. *Let* $\gamma > 0$ *and* σ *be an admissible weight. Then*

$$f_{\gamma,\lambda}(z) = \frac{(1-|\lambda|^2)^{2(1+\gamma)/p}}{\sigma^{1/p}(\lambda)(1-\overline{\lambda}z)^{2(2+\gamma)/p}}, \quad (\lambda \in \mathbb{D})$$

is in $\mathscr{A}^p(\sigma)$. *Moreover,* $\sup_{\lambda\in\mathbb{D}} \|f_{\gamma,\lambda}\|_{\mathscr{A}^p(\sigma)} \lesssim 1$.

Proof. Proof is an easy consequence of Lemma 2.4 in [2]. □

The next lemma can be found in [13].

LEMMA 1.3. *Let* $1 < p < \infty$ *and* σ *be an admissible weight. If a bounded sequence* $\{f_k\}_{k\in\mathbb{N}}$ *in* $\mathscr{A}^p(\sigma)$ *converges to* 0 *uniformly on compact subsets of* $\mathbb{D}$, *then* $\{f_k\}_{k\in\mathbb{N}}$ *also converges to* 0 *weakly in* $\mathscr{A}^p(\sigma)$.

The following functions play an important role in the rest of the paper.

For $\gamma > 0$, $n \in \mathbb{N}_0$ and $\lambda \in \mathbb{D}$, consider the family of function.

$$f_{\gamma,n,\lambda}(z) = \frac{(1-|\lambda|^2)^{2(1+\gamma)/p+n}}{\sigma^{1/p}(\lambda)(1-\overline{\lambda}z)^{2(2+\gamma)/p+n}}. \tag{1}$$

Using Lemma 1.2 it is easy to show that $f_{\gamma,n,\lambda} \in \mathscr{A}^p(\sigma)$. Moreover,

$$f_{\gamma,n,\lambda}(\lambda) = \frac{1}{\sigma^{1/p}(\lambda)(1-|\lambda|^2)^{2/p}}. \tag{2}$$

Also an easy calculation yields the following equalities:

$$f'_{\gamma,n,\lambda}(z) = \left(\frac{4+2\gamma}{p}+n\right)\overline{\lambda}\,\frac{(1-|\lambda|^2)^{2(1+\gamma)/p+n}}{\sigma^{1/p}(\lambda)(1-\overline{\lambda}z)^{2(2+\gamma)/p+n+1}},$$

$$f'_{\gamma,n,\lambda}(\lambda) = \left(\frac{4+2\gamma}{p}+n\right)\frac{\overline{\lambda}}{\sigma^{1/p}(\lambda)(1-|\lambda|^2)^{1+2/p}}, \tag{3}$$

$$f''_{\gamma,n,\lambda}(z) = \left(\frac{4+2\gamma}{p}+n\right)\left(\frac{4+2\gamma}{p}+n+1\right)(\overline{\lambda})^2\frac{(1-|\lambda|^2)^{2(1+\gamma)/p+n}}{\sigma^{1/p}(\lambda)(1-\overline{\lambda}z)^{2(2+\gamma)/p+n+2}},$$

$$f''_{\gamma,n,\lambda}(\lambda) = \left(\frac{4+2\gamma}{p}+n\right)\left(\frac{4+2\gamma}{p}+n+1\right)\frac{(\overline{\lambda})^2}{\sigma^{1/p}(\lambda)(1-|\lambda|^2)^{2+2/p}}. \tag{4}$$

For a normal weight ν, the *Zygmund-type space* $\mathscr{Z}_\nu$ on $\mathbb{D}$ is the space of all holomorphic functions f on $\mathbb{D}$ such that

$$\sup_{z\in\mathbb{D}} \nu(z)|f''(z)| < \infty.$$

For $\nu(z) = 1-|z|^2$ is obtained the (standard) Zygmund space $\mathscr{Z}$, which was defined in [5]. The space $\mathscr{Z}_\nu$ is a Banach space with the norm

$$\|f\|_{\mathscr{Z}_\nu} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}} \nu(z)|f''(z)|.$$

Zygmund-type spaces and operators on them have attracted a considerable attention recently. For some operators from or to the Zygmund-type spaces on the unit disk, see, for example, [3, 4, 5, 9, 7, 8, 12, 23, 27, 32, 34]. Zygmund-type spaces on the unit ball and operators from or to them are studied, for example, in [10, 11, 17, 31, 33], while some results in the setting of the upper half-plane can be found, for example, in [15, 21, 22]. For some generalizations of Zygmund-type spaces and operators on them, see, for example, [21, 20, 24, 25, 22, 26].

In this paper, we characterize the boundedness and compactness of weighted composition operators acting from weighted Bergman spaces $\mathscr{A}^p(\sigma)$ with admissible weights to Zygmund-type spaces $\mathscr{Z}_\nu$. We also give some upper and lower bounds for the norm and essential norm of the operators. For some results, in this direction, see, for example, [13, 14, 16, 30] and the references therein.

The next criterion for compactness follows by standard arguments similar to those outlined in the Proposition 3.11 in [1].

LEMMA 1.4. *Let σ be an admissible weight, ν a normal weight and $W_{\psi,\varphi}$: $\mathscr{A}^p(\sigma) \to \mathscr{Z}_\nu$ is bounded. Then $W_{\psi,\varphi} : \mathscr{A}^p(\sigma) \to \mathscr{Z}_\nu$ is compact if and only if for any bounded sequence $\{f_n\}_{n\in\mathbb{N}}$ in $\mathscr{A}^p(\sigma)$ which converges to zero uniformly on compact subsets of $\mathbb{D}$, we have $\|W_{\psi,\varphi}f_n\|_{\mathscr{Z}_\nu} \to 0$ as $n\to\infty$.*

Throughout this paper constants are denoted by C, they are positive and not necessarily the same at each occurrence. The notations $A \lesssim B$ means that A is less than or equal to a constant times B and $D \gtrsim E$, means that D is greater than or equal to a constant times E.

2. Boundedness and compactness of $W_{\psi,\varphi} : \mathscr{A}^p(\sigma) \to \mathscr{Z}_\nu$.

THEOREM 2.1. *Let $p>0$, σ an admissible weight, ν a normal weight, $\psi \in H(\mathbb{D})$ and φ be a holomorphic self-map of $\mathbb{D}$. Then $W_{\psi,\varphi} : \mathscr{A}^p(\sigma) \to \mathscr{Z}_\nu$ is bounded if and only if the following conditions are satisfied:*

(1) $M_1 = \sup_{z\in\mathbb{D}} \dfrac{\nu(z)|\psi''(z)|}{\sigma^{1/p}(\varphi(z))(1-|\varphi(z)|^2)^{2/p}} < \infty.$

(2) $M_2 = \sup_{z\in\mathbb{D}} \dfrac{\nu(z)|2\psi'(z)\varphi'(z)+\psi(z)\varphi''(z)|}{\sigma^{1/p}(\varphi(z))(1-|\varphi(z)|^2)^{1+2/p}} < \infty.$

(3) $M_3 = \sup_{z\in\mathbb{D}} \dfrac{\nu(z)|\psi(z)||\varphi'(z)|^2}{\sigma^{1/p}(\varphi(z))(1-|\varphi(z)|^2)^{2+2/p}} < \infty.$

Moreover, the following relation hold

$$\begin{aligned} M_1+M_2+M_3 &\lesssim \|W_{\psi,\varphi}\|_{\mathscr{A}^p(\sigma)\to\mathscr{Z}_\nu} \\ &\lesssim \frac{|\psi(0)|+|\psi'(0)|}{\sigma^{1/p}(\varphi(0))(1-|\varphi(0)|^2)^{2/p}} + \frac{|\psi(0)\varphi'(0)|}{\sigma^{1/p}(\varphi(0))(1-|\varphi(0)|^2)^{1+2/p}} \\ &\quad + M_1+M_2+M_3. \end{aligned} \tag{5}$$

Proof. First suppose that $W_{\psi,\varphi} : \mathscr{A}^p(\sigma) \to \mathscr{Z}_\nu$ is bounded. Then

$$\|W_{\psi,\varphi} f\|_{\mathscr{Z}_\nu} \leqslant \|W_{\psi,\varphi}\|_{\mathscr{A}^p(\sigma)\to\mathscr{Z}_\nu} \|f\|_{\mathscr{A}^p(\sigma)} \tag{6}$$

for every $f \in \mathscr{A}^p(\sigma)$. By taking $f(z)=1$ in (6) we have that

$$\sup_{z\in\mathbb{D}} \nu(z)|\psi''(z)| \leqslant \|W_{\psi,\varphi}\|_{\mathscr{A}^p(\sigma)\to\mathscr{Z}_\nu}. \tag{7}$$

Also by taking $f(z)=z$ in (6), using (7) and the fact that $|\varphi(z)|<1$, we see that

$$\sup_{z\in\mathbb{D}} \nu(z)|2\psi'(z)\varphi'(z)+\psi(z)\varphi''(z)| \lesssim \|W_{\psi,\varphi}\|_{\mathscr{A}^p(\sigma)\to\mathscr{Z}_\nu}. \tag{8}$$

Again by taking $f(z)=z^2/2$ in (6), using (7), (8) and the fact that $|\varphi(z)|<1$, we have that

$$\sup_{z\in\mathbb{D}} \nu(z)|\psi(z)||\varphi'(z)|^2 \lesssim \|W_{\psi,\varphi}\|_{\mathscr{A}^p(\sigma)\to\mathscr{Z}_\nu}. \tag{9}$$

Next we consider the following family of functions

$$f_\lambda(z) = f_{\gamma,0,\varphi(\lambda)}(z) - 2f_{\gamma,1,\varphi(\lambda)}(z) + f_{\gamma,2,\varphi(\lambda)}(z),$$

where $f_{\gamma,i,\varphi(\lambda)}$, $i=0,1,2$ are defined in (1). Then

$$f_\lambda(z) = \left[1 - \frac{2(1-|\varphi(\lambda)|^2)}{(1-\overline{\varphi(\lambda)}z)} + \frac{(1-|\varphi(\lambda)|^2)^2}{(1-\overline{\varphi(\lambda)}z)^2}\right] f_{\gamma,0,\varphi(\lambda)}(z) = \tau_{\varphi(\lambda)}(z) f_{\gamma,0,\varphi(\lambda)}(z),$$

where

$$\tau_{\varphi(\lambda)}(z)=1-\frac{2(1-|\varphi(\lambda)|^2)}{(1-\overline{\varphi(\lambda)}z)}+\frac{(1-|\varphi(\lambda)|^2)^2}{(1-\overline{\varphi(\lambda)}z)^2}.$$

Then $\tau_{\varphi(\lambda)}\in H^\infty$ as

$$\sup_{z\in\mathbb{D}}|\tau_{\varphi(\lambda)}(z)|\leqslant\sup_{z\in\mathbb{D}}\left[1+\frac{2(1-|\varphi(\lambda)|^2)}{(1-|\varphi(\lambda)||z|)}+\frac{(1-|\varphi(z)|^2)^2}{(1-|\varphi(\lambda)||z|)^2}\right]\leqslant 9.$$

Therefore, $f_\lambda\in\mathscr{A}^p(\sigma)$ and $\sup_{\lambda\in\mathbb{D}}\|f_\lambda\|_{\mathscr{A}^p(\sigma)}\lesssim 1$. Moreover, using (2), (3) and (4), we have that

$$f_\lambda(\varphi(\lambda))=0,\ \ f'_\lambda(\varphi(\lambda))=0\ \text{ and }\ f''_\lambda(\varphi(\lambda))=\frac{2(\overline{\varphi(\lambda)})^2}{\sigma^{1/p}(\varphi(\lambda))(1-|\varphi(\lambda)|^2)^{2+2/p}}.$$

Thus

$$\begin{aligned}||W_{\psi,\varphi}||_{\mathscr{A}^p(\sigma)\to\mathscr{Z}_\nu}\gtrsim\ &||W_{\psi,\varphi}f_\lambda||_{Z_\nu}\\ \geqslant\ &\nu(\lambda)|\psi''(\lambda)f_\lambda(\varphi(\lambda))+(2\psi'(\lambda)\varphi'(\lambda)\\ &+\psi(\lambda)\varphi''(\lambda))f'_\lambda(\varphi(\lambda))+\psi(\lambda)(\varphi'(\lambda))^2f''_\lambda(\varphi(\lambda))|\\ \geqslant\ &2\frac{\nu(\lambda)|\psi(\lambda)||\varphi'(\lambda)|^2|\varphi(\lambda)|^2}{\sigma^{1/p}(\varphi(\lambda))(1-|\varphi(\lambda)|^2)^{2+2/p}}.\end{aligned}\tag{10}$$

Therefore, we have that

$$\sup_{\lambda\in\mathbb{D}}\frac{\nu(\lambda)|\psi(\lambda)||\varphi'(\lambda)|^2|\varphi(\lambda)|^2}{\sigma^{1/p}(\varphi(\lambda))(1-|\varphi(\lambda)|^2)^{2+2/p}}\lesssim||W_{\psi,\varphi}||_{\mathscr{A}^p(\sigma)\to\mathscr{Z}_\nu}.$$

Thus for fixed $\delta\in(0,1)$, we have that

$$\sup_{|\varphi(\lambda)|>\delta}\frac{\nu(\lambda)|\psi(\lambda)||\varphi'(\lambda)|^2}{\sigma^{1/p}(\varphi(\lambda))(1-|\varphi(\lambda)|^2)^{2+2/p}}\lesssim||W_{\psi,\varphi}||_{\mathscr{A}^p(\sigma)\to\mathscr{Z}_\nu}.\tag{11}$$

Since σ is non-increasing, so by using (9), we have that

$$\sup_{|\varphi(\lambda)|\leqslant\delta}\frac{\nu(\lambda)|\psi(\lambda)||\varphi'(\lambda)|^2}{\sigma^{1/p}(\varphi(\lambda))(1-|\varphi(\lambda)|^2)^{2+2/p}}\leqslant\frac{1}{\sigma^{1/p}(\delta)(1-\delta^2)^{2+2/p}}||W_{\psi,\varphi}||_{\mathscr{A}^p(\sigma)\to\mathscr{Z}_\nu}.\tag{12}$$

Hence from (11) and (12), we have that

$$\sup_{\lambda\in\mathbb{D}}\frac{\nu(\lambda)|\psi(\lambda)||\varphi'(\lambda)|^2}{\sigma^{1/p}(\varphi(\lambda))(1-|\varphi(\lambda)|^2)^{2+2/p}}\lesssim||W_{\psi,\varphi}||_{\mathscr{A}^p(\sigma)\to\mathscr{Z}_\nu}.\tag{13}$$

Again, let $\lambda=\varphi(\zeta)$ and consider the family of function

$$g_\lambda(z)=\left(\frac{8+4\gamma}{p}+4\right)f_{\gamma,0,\lambda}(z)-\left(\frac{16+8\gamma}{p}+6\right)f_{\gamma,1,\lambda}(z)+\left(\frac{8+4\gamma}{p}+2\right)f_{\gamma,2,\lambda}(z).$$

Then

$$g_\lambda(z) = \left[\left(\frac{8+4\gamma}{p}+4\right) - \left(\frac{16+8\gamma}{p}+6\right)\frac{1-|\lambda|^2}{1-\overline{\lambda}z} + \left(\frac{8+4\gamma}{p}+2\right)\frac{(1-|\lambda|^2)^2}{(1-\overline{\lambda}z)^2}\right] f_{\gamma,0,\lambda}(z)$$
$$= \rho_\lambda(z) f_{\gamma,0,\lambda}(z).$$

Since

$$\sup_{z\in\mathbb{D}}|\rho_\lambda(z)| = \sup_{z\in\mathbb{D}}\left|\left(\frac{8+4\gamma}{p}+4\right) - \left(\frac{16+8\gamma}{p}+6\right)\frac{1-|\lambda|^2}{1-\overline{\lambda}z} + \left(\frac{8+4\gamma}{p}+2\right)\frac{(1-|\lambda|^2)^2}{(1-\overline{\lambda}z)^2}\right| \leqslant 36\left(\frac{2+\gamma}{p}\right)+24,$$

so $\rho_\lambda \in H^\infty$. Therefore, we have that $g_\lambda \in \mathscr{A}^p(\sigma)$ and $\sup_{\lambda\in\mathbb{D}}\|g_\lambda\|_{\mathscr{A}^p(\sigma)} \lesssim 1$. Moreover,

$$g'_\lambda(z) = \left(\frac{8+4\gamma}{p}+4\right) f'_{\gamma,0,\lambda}(z) - \left(\frac{16+8\gamma}{p}+6\right) f'_{\gamma,1,\lambda}(z) + \left(\frac{8+4\gamma}{p}+2\right) f'_{\gamma,2,\lambda}(z)$$

$$g''_\lambda(z) = \left(\frac{8+4\gamma}{p}+4\right) f''_{\gamma,0,\lambda}(z) - \left(\frac{16+8\gamma}{p}+6\right) f''_{\gamma,1,\lambda}(z) + \left(\frac{8+4\gamma}{p}+2\right) f''_{\gamma,2,\lambda}(z).$$

Therefore, by using (2), (3) and (4), we have that

$$g_\lambda(\lambda) = 0, \;\; g''_\lambda(\lambda) = 0 \;\text{ and }\; g'_\lambda(\lambda) = \frac{-2\overline{\lambda}}{\sigma^{1/p}(\lambda)(1-|\varphi(\lambda)|^2)^{1+2/p}}$$

and so

$$\begin{aligned} ||W_{\psi,\varphi}||_{\mathscr{A}^p(\sigma)\to\mathscr{Z}_\nu} &\gtrsim ||W_{\psi,\varphi}g_\lambda||_{\mathscr{Z}_\nu} \\ &\geqslant \nu(\zeta)|\psi''(\zeta)g_\lambda(\varphi(\zeta)) + (2\psi'(\zeta)\varphi'(\zeta) \\ &\quad + \psi(\zeta)\varphi''(\zeta))g'_\lambda(\varphi(\zeta)) + \psi(\zeta)(\varphi'(\zeta))^2 g''_\lambda(\varphi(\zeta))| \\ &\geqslant 2\frac{\nu(\zeta)|2\psi'(\zeta)\varphi'(\zeta)+\psi(\zeta)\varphi''(\zeta)|}{\sigma^{1/p}(\varphi(\zeta))(1-|\varphi(\zeta)|^2)^{1+2/p}}|\varphi(\zeta)|. \end{aligned}$$

Thus we have

$$\sup_{\zeta\in\mathbb{D}}\frac{\nu(\zeta)|2\psi'(\zeta)\varphi'(\zeta)+\psi(\zeta)\varphi''(\zeta)||\varphi(\zeta)|}{\sigma^{1/p}(\varphi(\zeta))(1-|\varphi(\zeta)|^2)^{1+1/p}} \lesssim ||W_{\psi,\varphi}||_{\mathscr{A}^p(\sigma)\to\mathscr{Z}_\nu}.$$

Thus there exist some $\delta_1 \in (0,1)$, such that

$$\sup_{|\varphi(\zeta)|>\delta_1}\frac{\nu(\zeta)|2\psi'(\zeta)\varphi'(\zeta)+\psi(\zeta)\varphi''(\zeta)|}{\sigma^{1/p}(\varphi(\zeta))(1-|\varphi(\zeta)|^2)^{1+1/p}} \lesssim ||W_{\psi,\varphi}||_{\mathscr{A}^p(\sigma)\to\mathscr{Z}_\nu}. \tag{14}$$

Since σ is monotonically increasing, so by (8), we have that

$$\sup_{|\varphi(\zeta)|\leqslant\delta_1} \frac{\nu(\zeta)|2\psi'(\zeta)\varphi'(\zeta)+\psi(\zeta)\varphi''(\zeta)|}{\sigma^{1/p}(\varphi(\zeta))(1-|\varphi(\zeta)|^2)^{1+1/p}} \leqslant \frac{1}{\sigma(\delta_1)(1-\delta_1^2)^{1+2/p}}||W_{\psi,\varphi}||_{\mathscr{A}^p(\sigma)\to\mathscr{Z}_\nu}. \tag{15}$$

Combining (14) and (15) we have that

$$\sup_{\zeta\in\mathbb{D}} \frac{\nu(\zeta)|2\psi'(\zeta)\varphi'(\zeta)+\psi(\zeta)\varphi''(\zeta)|}{\sigma^{1/p}(\varphi(\zeta))(1-|\varphi(\zeta)|^2)^{1+1/p}} \lesssim ||W_{\psi,\varphi}||_{\mathscr{A}^p(\sigma)\to\mathscr{Z}_\nu}. \tag{16}$$

Again, let $\lambda=\varphi(\zeta)$ and consider the family of functions

$$\begin{aligned} h_\lambda(z) &= 2\left(\frac{2+\gamma}{p}+1\right)\left(\frac{4+2\gamma}{p}+1\right)f_{\gamma,0,\lambda}(z)\\ &\quad -8\left(\frac{2+\gamma}{p}\right)\left(\frac{2+\gamma}{p}+1\right)f_{\gamma,1,\lambda}(z)+2\left(\frac{2+\gamma}{p}\right)\left(\frac{4+2\gamma}{p}+1\right)f_{\gamma,2,\lambda}(z). \end{aligned}$$

Then

$$\begin{aligned} h_\lambda(z) &= \left[2\left(\frac{2+\gamma}{p}+1\right)\left(\frac{4+2\gamma}{p}+1\right)-8\left(\frac{2+\gamma}{p}\right)\left(\frac{2+\gamma}{p}+1\right)\frac{1-|\lambda|^2}{1-\overline{\lambda}z}\right.\\ &\quad \left.+2\left(\frac{2+\gamma}{p}\right)\left(\frac{4+2\gamma}{p}+1\right)\frac{(1-|\lambda|^2)^2}{(1-\overline{\lambda}z)^2}\right]f_{\gamma,0,\lambda}(z)\\ &= g_1(z)f_{\gamma,0,\lambda}(z). \end{aligned}$$

Proceeding as above we can show that $g_1\in H^\infty$, $h_\lambda\in\mathscr{A}^p(\sigma)$ and $\sup_{\lambda\in\mathbb{D}}\|h_\lambda\|_{\mathscr{A}^p(\sigma)}\lesssim 1$ and

$$h'_\lambda(\lambda)=0, h''_\lambda(\lambda)=0 \ \text{ and } \ h_\lambda(\lambda)=\frac{2}{\sigma^{1/p}(\lambda)(1-|\lambda|^2)^{2/p}}.$$

Therefore,

$$||W_{\psi,\varphi}||_{\mathscr{A}^p(\sigma)\to\mathscr{Z}_\nu} \gtrsim ||W_{\psi,\varphi}h_\lambda||_{\mathscr{Z}_\nu} \geqslant \frac{\nu(\zeta)|\psi''(\zeta)|}{\sigma^{1/p}(\varphi(\zeta))(1-|\varphi(\zeta)|^2)^{2/p}}.$$

Taking the supermum over $\zeta\in\mathbb{D}$, we have that

$$\sup_{\zeta\in\mathbb{D}} \frac{\nu(\zeta)|\psi''(\zeta)|}{\sigma^{1/p}(\varphi(\zeta))(1-|\varphi(\zeta)|^2)^{2/p}} \lesssim ||W_{\psi,\varphi}||_{\mathscr{A}^p(\sigma)\to\mathscr{Z}_\nu}. \tag{17}$$

From (13), (16) and (17) we have that

$$M_1+M_2+M_3 \lesssim \|W_{\psi,\varphi}\|_{\mathscr{A}^p(\sigma)\to\mathscr{Z}_\nu}. \tag{18}$$

Conversely, suppose that the conditions (1)–(3) hold. Then

$$\begin{aligned}\nu(z)|(W_{\psi,\varphi}f)''(z)| &= \nu(z)|\psi''(z)f(\varphi(z)) + (2\psi'(z)\varphi'(z)\\ &\quad + \psi(z)\varphi''(z))f'(\varphi(z)) + \psi(z)(\varphi'(z))^2 f''(\varphi(z))|\\ &\lesssim \Bigg(\frac{\nu(z)|\psi''(z)|}{\sigma^{1/p}(\varphi(z))(1-|\varphi(z)|^2)^{2/p}} + \frac{\nu(z)|2\psi'(z)\varphi'(z)+\psi(z)\varphi''(z)|}{\sigma^{1/p}(\varphi(z))(1-|\varphi(z)|^2)^{1+2/p}}\\ &\quad + \frac{\nu(z)|\psi(z)||\varphi'(z)|^2}{\sigma^{1/p}(\varphi(z))(1-|\varphi(z)|^2)^{2+2/p}}\Bigg)\|f\|_{\mathscr{A}^p(\sigma)}.\end{aligned}$$

Taking the supremum over $z\in\mathbb{D}$, we get

$$\sup_{z\in\mathbb{D}} \nu(z)|(W_{\psi,\varphi}f)''(z)| \lesssim (M_1+M_2+M_3)\|f\|_{\mathscr{A}^p(\sigma)}.$$

Further, we have

$$\|W_{\psi,\varphi}\|_{\mathscr{A}^p(\sigma)\to\mathscr{Z}_\nu} \lesssim M_1+M_2+M_3, \tag{19}$$

$$|W_{\psi,\varphi}f(0)| = |\psi(0)||f(\varphi(0))| \lesssim \frac{|\psi(0)|\|f\|_{\mathscr{A}^p(\sigma)}}{\sigma^{1/p}(\varphi(0))(1-|\varphi(0)|^2)^{2/p}} \tag{20}$$

and

$$\begin{aligned}&|(W_{\psi,\varphi}f)'(0)|\\ &= |\psi'(0)f(\varphi(0)) + \psi(0)\varphi'(0)f'(\varphi(0))|\\ &\lesssim \left(\frac{|\psi'(0)|}{\sigma^{1/p}(\varphi(0))(1-|\varphi(0)|^2)^{2/p}} + \frac{|\psi(0)\varphi'(0)|}{\sigma^{1/p}(\varphi(0))(1-|\varphi(0)|^2)^{1+2/p}}\right)\|f\|_{\mathscr{A}^p(\sigma)}.\end{aligned} \tag{21}$$

Combining (19), (20) and (21), we have that

$$\begin{aligned}\|W_{\psi,\varphi}\|_{\mathscr{A}^p(\sigma)\to\mathscr{Z}_\nu} &\lesssim \frac{|\psi(0)|+|\psi'(0)|}{\sigma^{1/p}(\varphi(0))(1-|\varphi(0)|^2)^{2/p}}\\ &\quad + \frac{|\psi(0)\varphi'(0)|}{\sigma^{1/p}(\varphi(0))(1-|\varphi(0)|^2)^{1+2/p}} + M_1+M_2+M_3.\end{aligned} \tag{22}$$

From (18) and (22), (6) holds. □

THEOREM 2.2. *Let $p>0$, σ an admissible weight, ν a normal weight, $\psi\in H(\mathbb{D})$ and φ be a holomorphic self map of $\mathbb{D}$ such that $\|\varphi\|_\infty<1$. Let $W_{\psi,\varphi}:\mathscr{A}^p(\sigma)\to\mathscr{Z}_\nu$ is bounded. Then following conditions are equivalent:*

(1) *$W_{\psi,\varphi}:\mathscr{A}^p(\sigma)\to\mathscr{Z}_\nu$ is bounded.*

(2) *$W_{\psi,\varphi}:\mathscr{A}^p(\sigma)\to\mathscr{Z}_\nu$ is compact.*

(3) $N_1 = \sup_{z\in\mathbb{D}} \nu(z)|\psi''(z)| < \infty,$

$N_2 = \sup_{z\in\mathbb{D}} \nu(z)|2\psi'(z)\varphi'(z)+\psi(z)\varphi''(z)| < \infty,$

$N_3 = \sup_{z\in\mathbb{D}} \nu(z)|\psi(z)||\varphi'(z)|^2 < \infty.$

Proof. $(2) \Rightarrow (1)$ is trivially true. To complete the theorem we only need to prove that $(1) \Rightarrow (3)$ and $(3) \Rightarrow (2)$.

$(1) \Rightarrow (3)$ By taking $f(z) = 1, f(z) = z$ and $f(z) = z^2/2$, respectively in $\|W_{\psi,\varphi}f\|_{\mathscr{Z}_\nu} \leqslant \|W_{\psi,\varphi}\|\|f\|_{A^p(\sigma)}$. We see that all the conditions in (3) hold.

$(3) \Rightarrow (2)$. Suppose that conditions in (3) hold. Let $\{f_j\}$ be a bounded sequence in $\mathscr{A}^p(\sigma)$ such that $\sup_j \|f_j\|_{A^p(\sigma)} \leqslant M$ and $f_j \to 0$ uniformly on compact subsets of $\mathbb{D}$. From this and by the Cauchy inequality we have that $f_j' \to 0$ and $f_j'' \to 0$ uniformly on compact subsets of $\mathbb{D}$. Thus for fixed $z \in \mathbb{D}$, we have that

$$\begin{aligned}
\nu(z)|(W_{\psi,\varphi}f_j)''(z)| &\leqslant \nu(z)|\psi''(z)||f_j(\varphi(z))| + \nu(z)|2\psi'(z)\varphi'(z) + \psi(z)\varphi''(z)||f_j'(\varphi(z))| \\
&\quad + \nu(z)|\psi(z)||\varphi'(z)|^2|f_j''(\varphi(z))| \\
&\leqslant \nu(z)|\psi''(z)| \max_{|\zeta|\leqslant\|\varphi\|_\infty} |f_j(\zeta)| + \nu(z)|2\psi'(z)\varphi'(z) \\
&\quad + \psi(z)\varphi''(z)| \max_{|\zeta|\leqslant\|\varphi\|_\infty} |f_j'(\zeta)| + \nu(z)|\psi(z)||\varphi'(z)|^2 \max_{|\zeta|\leqslant\|\varphi\|_\infty} |f_j''(\zeta)|.
\end{aligned}$$

Thus

$$\begin{aligned}
\sup_{z\in\mathbb{D}} \nu(z)|(W_{\psi,\varphi}f_j)''(z)| &\leqslant N_1 \max_{|\zeta|\leqslant\|\varphi\|_\infty} |f_j(\zeta)| + N_2 \max_{|\zeta|\leqslant\|\varphi\|_\infty} |f_j'(\zeta)| \\
&\quad + N_3 \max_{|\zeta|\leqslant\|\varphi\|_\infty} |f_j''(\zeta)| \to 0 \text{ as } j \to \infty
\end{aligned}$$

and so $\lim_{j\to\infty} \sup_{z\in\mathbb{D}} \nu(z)|(W_{\psi,\varphi}f_j)''(z)| = 0$. Moreover, $|\psi(0)||f_j(\varphi(0))| \to 0$ and $|\psi'(0)f_j(\varphi(0)) + \psi(0)\varphi'(0)f_j(\varphi(0))| \to 0$ as $j \to \infty$. Thus

$$\lim_{j\to\infty} \|W_{\psi,\varphi}f_j\|_{\mathscr{Z}_\nu} = 0.$$

Hence by Lemma 1.4, $W_{\psi,\varphi} : \mathscr{A}^p(\sigma) \to Z_\nu$ is compact. □

3. Essential norm of $W_{\psi,\varphi} : \mathscr{A}^p(\sigma) \to \mathscr{Z}_\nu$.

In this section, we give some upper and lower bounds for the essential norm of the operators.

Recall that if X and Y are two Banach spaces, then the essential norm $\|T\|_{e,X\to Y}$ of a bounded linear operator $T : X \to Y$ is defined as

$$\|T\|_{e,X\to Y} = \inf\{\|T - K\| : K \text{ is compact from } X \text{ to } Y\},$$

where $\|T\|$ denote the usual operator norm. Clearly T is compact if and only if $\|T\|_{e,X\to Y} = 0$.

THEOREM 3.1. *Let $p > 1, \sigma$ an admissible weight, ν a normal weight, $\psi \in H(\mathbb{D})$ and φ be a holomorphic self map of $\mathbb{D}$ such that $\|\varphi\|_\infty = 1$. Let $W_{\psi,\varphi}$:*

$\mathscr{A}^p(\sigma) \to \mathscr{Z}_\nu$ is bounded. Then

$$\|W_{\psi,\varphi}\|_{e,\mathscr{A}^p(\sigma)\to\mathscr{Z}_\nu} \asymp \limsup_{|\varphi(z)|\to 1} \frac{\nu(z)|\psi''(z)|}{\sigma^{1/p}(\varphi(z))(1-|\varphi(z)|^2)^{2/p}} + \limsup_{|\varphi(z)|\to 1} \frac{\nu(z)|2\psi'(z)\varphi'(z)+\psi(z)\varphi''(z)|}{\sigma^{1/p}(\varphi(z))(1-|\varphi(z)|^2)^{1+2/p}} + \limsup_{|\varphi(z)|\to 1} \frac{\nu(z)|\psi(z)||\varphi'(z)|^2}{\sigma^{1/p}(\varphi(z))(1-|\varphi(z)|^2)^{2+2/p}}.$$

Proof. Lower Bound. Let $\{z_j\}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_j)| \to 1$ as $j \to \infty$. Consider the functions f_j, g_j and h_j defined, respectively as

$$f_j(z) = f_{\gamma,0,\varphi(z_j)}(z) - 2f_{\gamma,1,\varphi(z_j)}(z) + f_{\gamma,2,\varphi(z_j)}(z),$$

$$g_j(z) = \left(\frac{8+4\gamma}{p}+4\right) f_{\gamma,0,\varphi(z_j)}(z) - \left(\frac{16+8\gamma}{p}+6\right) f_{\gamma,1,\varphi(z_j)}(z) + \left(\frac{8+4\gamma}{p}+2\right) f_{\gamma,2,\varphi(z_j)}(z),$$

$$h_j(z) = 2\left(\frac{2+\gamma}{p}+1\right)\left(\frac{4+2\gamma}{p}+1\right) f_{\gamma,0,\varphi(z_j)}(z) - 8\left(\frac{2+\gamma}{p}\right)\left(\frac{2+\gamma}{p}+1\right) f_{\gamma,1,\varphi(z_j)}(z) + 2\left(\frac{2+\gamma}{p}\right)\left(\frac{4+2\gamma}{p}+1\right) f_{\gamma,2,\varphi(z_j)}(z),$$

where $f_{\gamma,0,\varphi(z_j)}$, $f_{\gamma,1,\varphi(z_j)}$ and $f_{\gamma,2,\varphi(z_j)}$ are defined as in (1). As in Theorem 2.1, we have that $\{f_j\}$, $\{g_j\}$ and $\{h_j\}$ are bounded in $\mathscr{A}^p(\sigma)$ and $\sup_j \|K_j\| \leqslant M$, where $K_j = f_j$ or g_j or h_j. Moreover $K_j \to 0$ as $j \to \infty$ uniformly on compact subsets of $\mathbb{D}$. Let $K : \mathscr{A}^p(\sigma) \to \mathscr{Z}_\nu$ is compact. Then by Lemma 1.3, $\|Kf_j\|_{\mathscr{Z}_\nu} \to 0$ as $j \to \infty$. As in Theorem 2.1,

$$f_j(\varphi(z_j)) = 0, f_j'(\varphi(z_j)) = 0 \text{ and } f_j''(\varphi(z_j)) = \frac{2(\overline{\varphi(z_j)})^2}{\sigma^{1/p}(\varphi(z_j))(1-|\varphi(z_j)|^2)^{2+2/p}}.$$

Therefore,

$$\begin{aligned}\|W_{\psi,\varphi}\|_{e,\mathscr{A}^p(\sigma)\to\mathscr{Z}_\nu} &\geqslant C \limsup_{j\to\infty} \|W_{\psi,\varphi}f_j - Kf_j\|_{Z_\nu} \\ &\geqslant C \limsup_{j\to\infty} \|W_{\psi,\varphi}f_j\|_{Z_\nu} - \limsup_{j\to\infty} \|Kf_j\|_{Z_\nu} \\ &\geqslant C \limsup_{j\to\infty} \frac{\nu(z)|\psi(z_j)||\varphi'(z_j)|^2}{\sigma^{1/p}(\varphi(z_j))(1-|\varphi(z_j)|^2)^{2+2/p}}. \end{aligned} \tag{23}$$

Proceeding as in Theorem 2.1, we have that

$$g_j(\varphi(z_j)) = 0, \; g_j''(\varphi(z_j)) = 0 \text{ and } g_j'(\varphi(z_j)) = \frac{-2\overline{\varphi(z_j)}}{\sigma^{1/p}(\varphi(z_j))(1-|\varphi(z_j)|^2)^{1+2/p}}$$

and so

$$\begin{aligned}\|W_{\psi,\varphi}\|_{e,\mathscr{A}^p(\sigma)\to\mathscr{Z}_\nu} &\geqslant C\limsup_{j\to\infty}\|W_{\psi,\varphi}g_j-Kg_j\|_{\mathscr{Z}_\nu}\\ &\geqslant C\limsup_{j\to\infty}\|W_{\psi,\varphi}g_j\|_{\mathscr{Z}_\nu}-\limsup_{j\to\infty}\|Kg_j\|_{\mathscr{Z}_\nu}\\ &\geqslant C\limsup_{j\to\infty}\frac{\nu(z)|2\psi'(z_j)\varphi'(z_j)+\psi(z_j)\varphi''(z_j)|}{\sigma^{1/p}(\varphi(z_j))(1-|\varphi(z_j)|^2)^{1+2/p}}. \qquad (24)\end{aligned}$$

Once again as in Theorem 2.1, we have that

$$h_j'(\varphi(z_j))=0,\ \ h_j''(\varphi(z_j))=0 \ \text{ and } \ h_j(\varphi(z_j))=\frac{2}{\sigma^{1/p}(\varphi(z_j))(1-|\varphi(z_j)|^2)^{2/p}}.$$

Therefore,

$$\begin{aligned}\|W_{\psi,\varphi}\|_{e,\mathscr{A}^p(\sigma)\to\mathscr{Z}_\nu} &\geqslant C\limsup_{j\to\infty}\|W_{\psi,\varphi}h_j-Kh_j\|_{\mathscr{Z}_\nu}\\ &\geqslant C\limsup_{j\to\infty}\|W_{\psi,\varphi}h_j\|_{\mathscr{Z}_\nu}-\limsup_{j\to\infty}\|Kh_j\|_{\mathscr{Z}_\nu}\\ &\geqslant C\limsup_{j\to\infty}\frac{\nu(z)|\psi(z_j)||\varphi'(z_j)|^2}{\sigma^{1/p}(\varphi(z_j))(1-|\varphi(z_j)|^2)^{2/p}}. \qquad (25)\end{aligned}$$

Combining (23), (24) and (25), we get

$$\begin{aligned}\|W_{\psi,\varphi}\|_{e,\mathscr{A}^p(\sigma)\to\mathscr{Z}_\nu} \gtrsim\ &\limsup_{|\varphi(z)|\to1}\frac{\nu(z)|\psi''(z)|}{\sigma^{1/p}(\varphi(z))(1-|\varphi(z)|^2)^{2/p}}\\ &+\limsup_{|\varphi(z)|\to1}\frac{\nu(z)|2\psi'(z)\varphi'(z)+\psi(z)\varphi''(z)|}{\sigma^{1/p}(\varphi(z))(1-|\varphi(z)|^2)^{1+2/p}}\\ &+\limsup_{|\varphi(z)|\to1}\frac{\nu(z)|\psi(z)||\varphi'(z)|^2}{\sigma^{1/p}(\varphi(z))(1-|\varphi(z)|^2)^{2+2/p}}. \qquad (26)\end{aligned}$$

Upper Bound. Let $\varphi_k(z)=\frac{k}{k+1}z$. Then $\|\varphi_k\|_\infty<1$. Let

$$L_kf(z)=C_{\varphi_k}f(z)=f\left(\frac{k}{k+1}z\right).$$

Then by Corollary 4 in [13], we have that $L_k:\mathscr{A}^p(\sigma)\to\mathscr{A}^p(\sigma)$ is compact. Since $W_{\psi,\varphi}:\mathscr{A}^p(\sigma)\to\mathscr{Z}_\nu$ is bounded, so $W_{\psi,\varphi}L_k:\mathscr{A}^p(\sigma)\to\mathscr{Z}_\nu$ is compact. Thus

$$\begin{aligned}\|W_{\psi,\varphi}\|_e &\leqslant \|W_{\psi,\varphi}-W_{\psi,\varphi}L_k\|\\ &\leqslant \sup_{\|f\|_{\mathscr{A}^p(\sigma)}\leqslant1}\|W_{\psi,\varphi}(I-L_k)f\|_{\mathscr{Z}_\nu}\\ &\leqslant \sup_{\|f\|_{\mathscr{A}^p(\sigma)}\leqslant1}\Big[|W_{\psi,\varphi}(I-L_k)f(0)|+|(W_{\psi,\varphi}(I-L_k)f)'(0)|\\ &\quad+\sup_{z\in\mathbb{D}}\nu(z)|(W_{\psi,\varphi}(I-L_k)f)''(z)|\Big], \qquad (27)\end{aligned}$$

where I is the identity operator on $\mathscr{A}^p(\sigma)$. For any $r \in (0,1)$, we can write

$$\sup_{z\in\mathbb{D}} \nu(z)|(W_{\psi,\varphi}(I-L_k)f)''(z)| = \sup_{|\varphi(z)|\leqslant r} \nu(z)|(W_{\psi,\varphi}(I-L_k)f)''(z)| + \sup_{|\varphi(z)|>r} \nu(z)|(W_{\psi,\varphi}(I-L_k)f)''(z)|. \tag{28}$$

Now

$$\begin{aligned}|(W_{\psi,\varphi}(I-L_k)f)''(z)| = \Bigg|\psi''(z)\Big\{f(\varphi(z)) - f\Big(\frac{k}{k+1}\varphi(z)\Big)\Big\} &+ \big(2\psi'(z)\varphi'(z) \\ + \psi(z)\varphi''(z)\big)\Big\{f'(\varphi(z)) - \frac{k}{k+1}f'\Big(\frac{k}{k+1}\varphi(z)\Big)\Big\} \\ + \psi(z)(\varphi'(z))^2\Big\{f''(\varphi(z)) - \frac{k^2}{(k+1)^2}f''\Big(\frac{k}{k+1}\varphi(z)\Big)\Big\}\Bigg|.\end{aligned} \tag{29}$$

Let $|\varphi(z)| \leqslant r$ and $w = \varphi(z)$. Denote the straight line segment from $kw/(k+1)$ to w by $[kw/(k+1), w]$. Then $[kw/(k+1), w] \subset D(0,r)$, where $D(0,r) = \{z : |z| \leqslant r\}$. Thus for $i \in \{0,1,2\}$, by Lemma 1.1 and the fact that σ is non-increasing, we have that

$$\begin{aligned}\left|f^{(i)}(w) - f^{(i)}\Big(\frac{k}{k+1}w\Big)\right| &= \left|\int_{[kw/(k+1),w]} f^{(i+1)}(\zeta)d\zeta\right| \\ &\leqslant \frac{|w|}{k+1}\sup_{\zeta\in D(0,r)}|f^{(i+1)}(\zeta)| \\ &\lesssim \frac{|w|}{k+1}\sup_{\zeta\in D(0,r)}\frac{\|f\|_{\mathscr{A}^p(\sigma)}}{\sigma^{1/p}(\zeta)\big(1-|\zeta|^2\big)^{i+1+2/p}} \\ &\lesssim \frac{|w|}{k+1}\frac{\|f\|_{\mathscr{A}^p(\sigma)}}{\sigma^{1/p}(r)\big(1-r^2\big)^{i+1+2/p}}.\end{aligned} \tag{30}$$

Using Lemma 1.1, (29) and (30), we have that

$$\sup_{\|f\|_{\mathscr{A}^p(\sigma)}\leqslant 1}\ \sup_{|\varphi(z)|\leqslant r} \nu(z)|(W_{\psi,\varphi}(I-L_k)f)''(z)| \to 0 \text{ as } k\to\infty. \tag{31}$$

Using (30) with $i = 0$, we have that

$$\begin{aligned}|W_{\psi,\varphi}(I-L_k)f(0)| &= \left|\psi(0)f(\varphi(0)) - \psi(0)f\Big(\frac{k}{k+1}\varphi(0)\Big)\right| \\ &\lesssim \frac{|\psi(0)\varphi(0)|}{k+1}\frac{\|f\|_{\mathscr{A}^p(\sigma)}}{\sigma^{1/p}(r)\big(1-r^2\big)^{1+2/p}}.\end{aligned} \tag{32}$$

On the other hand, by Lemma 1.1 and (30) with $i = 0$ and $i = 1$, we have that

$$\begin{aligned}&|(W_{\psi,\varphi}(I-L_k)f)'(0)| \\ &\leqslant |\psi'(0)|\left|f(\varphi(0)) - f\Big(\frac{k}{k+1}\varphi(0)\Big)\right| + |\psi(0)\varphi'(0)|\left|f'(\varphi(0)) - \frac{k}{k+1}f'\Big(\frac{k}{k+1}\varphi(0)\Big)\right|\end{aligned}$$

$$
\begin{aligned}
&\leqslant |\psi'(0)|\left|f(\varphi(0)) - f\left(\frac{k}{k+1}\varphi(0)\right)\right| + \frac{|\psi(0)\varphi'(0)|}{k+1}|f'(\varphi(0))| \\
&\quad + |\psi(0)\varphi'(0)|\frac{k}{k+1}\left|f'(\varphi(0)) - f'\left(\frac{k}{k+1}\varphi(0)\right)\right| \\
&\lesssim \left(\frac{|\psi'(0)\varphi(0)|}{k+1}\frac{1}{\sigma^{1/p}(r)\left(1-r^2\right)^{1+2/p}} + \frac{|\psi(0)\varphi'(0)|}{k+1}\frac{1}{\sigma^{1/p}(r)\left(1-r^2\right)^{1+2/p}}\right. \\
&\quad \left. + \frac{|\psi(0)\varphi(0)\varphi'(0)|}{k+1}\frac{k}{k+1}\frac{1}{\sigma^{1/p}(r)\left(1-r^2\right)^{2+2/p}}\right)\|f\|_{\mathscr{A}^p(\sigma)}. \qquad (33)
\end{aligned}
$$

Combining (32) and (33), we have that

$$
\sup_{\|f\|_{\mathscr{A}^p(\sigma)} \leqslant 1}\left[|W_{\psi,\varphi}(I-L_k)f(0)| + |(W_{\psi,\varphi}(I-L_k)f)'(0)|\right] \to 0 \qquad (34)
$$

as $k \to \infty$. The second term in the right hand side of (28) is dominated by

$$
\begin{aligned}
&\sup_{|\varphi(z)|>r} \nu(z)|\psi''(z)|\left\{|f(\varphi(z))| + \left|f\left(\frac{k}{k+1}\varphi(z)\right)\right|\right\} \\
&\quad + \sup_{|\varphi(z)|>r} \nu(z)\left|2\psi'(z)\varphi'(z) + \psi(z)\varphi''(z)\right|\left\{|f'(\varphi(z))| + \frac{k}{k+1}\left|f'\left(\frac{k}{k+1}\varphi(z)\right)\right|\right\} \\
&\quad + \sup_{|\varphi(z)|>r} \nu(z)|\psi(z)(\varphi'(z))^2|\left\{|f''(\varphi(z))| + \frac{k^2}{(k+1)^2}\left|f''\left(\frac{k}{k+1}\varphi(z)\right)\right|\right\},
\end{aligned}
$$

which is further dominated by a constant multiple of

$$
\begin{aligned}
&\sup_{|\varphi(z)|>r} \nu(z)|\psi''(z)|\left\{\frac{\|f\|_{\mathscr{A}^p(\sigma)}}{\sigma^{1/p}(\varphi(z))(1-|\varphi(z)|^2)^{2/p}} + \frac{\|f\|_{\mathscr{A}^p(\sigma)}}{\sigma^{1/p}(\frac{k}{k+1}\varphi(z))(1-\frac{k^2}{(k+1)^2}|\varphi(z)|^2)^{2/p}}\right\} \\
&\quad + \sup_{|\varphi(z)|>r} \nu(z)\left|2\psi'(z)\varphi'(z) + \psi(z)\varphi''(z)\right|\left\{\frac{\|f\|_{\mathscr{A}^p(\sigma)}}{\sigma^{1/p}(\varphi(z))(1-|\varphi(z)|^2)^{1+2/p}}\right. \\
&\quad \left. + \frac{k}{k+1}\frac{\|f\|_{\mathscr{A}^p(\sigma)}}{\sigma^{1/p}(\frac{k}{k+1}\varphi(z))(1-\frac{k^2}{(k+1)^2}|\varphi(z)|^2)^{1+2/p}}\right\} \\
&\quad + \sup_{|\varphi(z)|>r} \nu(z)|\psi(z)(\varphi'(z))^2|\left\{\frac{\|f\|_{\mathscr{A}^p(\sigma)}}{\sigma^{1/p}(\varphi(z))(1-|\varphi(z)|^2)^{2+2/p}}\right. \\
&\quad \left. + \frac{k^2}{(k+1)^2}\frac{\|f\|_{\mathscr{A}^p(\sigma)}}{\sigma^{1/p}(\frac{k}{k+1}\varphi(z))(1-\frac{k^2}{(k+1)^2}|\varphi(z)|^2)^{2+2/p}}\right\}, \qquad (35)
\end{aligned}
$$

Letting $k \to \infty$ in (35), we get

$$
\begin{aligned}
&\limsup_{k\to\infty}\sup_{\|f\|_{\mathscr{A}^p(\sigma)}\leqslant 1}\sup_{|\varphi(z)|>r} \nu(z)|(W_{\psi,\varphi}(I-L_k)f)''(z)| \\
&\qquad \lesssim \sup_{|\varphi(z)|>r}\frac{\nu(z)|\psi''(z)|}{\sigma^{1/p}(\varphi(z))(1-|\varphi(z)|^2)^{2/p}}
\end{aligned}
$$

$$+\sup_{|\varphi(z)|>r}\frac{\nu(z)|2\psi'(z)\varphi'(z)+\psi(z)\varphi''(z)|}{\sigma^{1/p}(\varphi(z))(1-|\varphi(z)|^2)^{1+2/p}}$$
$$+\sup_{|\varphi(z)|>r}\frac{\nu(z)|\psi(z)||\varphi'(z)|^2}{\sigma^{1/p}(\varphi(z))(1-|\varphi(z)|^2)^{2+2/p}}. \tag{36}$$

Using (28), (31), (34) and (36) in (27), we have that

$$\begin{aligned}\|W_{\psi,\varphi}\|_{e,\mathscr{A}^p(\sigma)\to\mathscr{Z}_\nu} \lesssim & \sup_{|\varphi(z)|>r}\frac{\nu(z)|\psi''(z)|}{\sigma^{1/p}(\varphi(z))(1-|\varphi(z)|^2)^{2/p}}\\ &+\sup_{|\varphi(z)|>r}\frac{\nu(z)|2\psi'(z)\varphi'(z)+\psi(z)\varphi''(z)|}{\sigma^{1/p}(\varphi(z))(1-|\varphi(z)|^2)^{1+2/p}}\\ &+\sup_{|\varphi(z)|>r}\frac{\nu(z)|\psi(z)||\varphi'(z)|^2}{\sigma^{1/p}(\varphi(z))(1-|\varphi(z)|^2)^{2+2/p}}.\end{aligned}$$

Finally, letting $r\to 1$, then we get

$$\begin{aligned}\|W_{\psi,\varphi}\|_{e,\mathscr{A}^p(\sigma)\to\mathscr{Z}_\nu} \lesssim & \limsup_{|\varphi(z)|\to 1}\frac{\nu(z)|\psi''(z)|}{\sigma^{1/p}(\varphi(z))(1-|\varphi(z)|^2)^{2/p}}\\ &+\limsup_{|\varphi(z)|\to 1}\frac{\nu(z)|2\psi'(z)\varphi'(z)+\psi(z)\varphi''(z)|}{\sigma^{1/p}(\varphi(z))(1-|\varphi(z)|^2)^{1+2/p}}\\ &+\limsup_{|\varphi(z)|\to 1}\frac{\nu(z)|\psi(z)||\varphi'(z)|^2}{\sigma^{1/p}(\varphi(z))(1-|\varphi(z)|^2)^{2+2/p}}.\end{aligned} \tag{37}$$

Combining (26) and (37), we get the desired result. □

COROLLARY 3.2. *Let $p>1$, σ an admissible weight, ν a normal weight, $\psi\in H(\mathbb{D})$ and φ be a holomorphic self map of $\mathbb{D}$, such that $\|\varphi\|_\infty = 1$. Let $W_{\psi,\varphi}:\mathscr{A}^p(\sigma)\to\mathscr{Z}_\nu$ is bounded. Then $W_{\psi,\varphi}:\mathscr{A}^p(\sigma)\to\mathscr{Z}_\nu$ is compact if and only if the following conditions are satisfied*

(1) $\displaystyle\limsup_{|\varphi(z)|\to 1}\frac{\nu(z)|\psi''(z)|}{\sigma^{1/p}(\varphi(z))(1-|\varphi(z)|^2)^{2/p}}=0.$

(2) $\displaystyle\limsup_{|\varphi(z)|\to 1}\frac{\nu(z)|2\psi'(z)\varphi'(z)+\psi(z)\varphi''(z)|}{\sigma^{1/p}(\varphi(z))(1-|\varphi(z)|^2)^{1+2/p}}=0.$

(3) $\displaystyle\limsup_{|\varphi(z)|\to 1}\frac{\nu(z)|\psi(z)||\varphi'(z)|^2}{\sigma^{1/p}(\varphi(z))(1-|\varphi(z)|^2)^{2+2/p}}=0.$

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