INEQUALITIES INVOLVING NORM AND ESSENTIAL NORM OF WEIGHTED COMPOSITION OPERATORS

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Abstract. We characterize the boundedness and compactness of the weighted composition operator acting from the weighted Bergman space \( A^p(\sigma) \) to the Zygmund-type space \( Z_\nu \), where \( \sigma \) is an admissible weight and \( \nu \) is a normal weight. Some upper and lower bounds for the norm and essential norm of the operator are also given.

1. Introduction and preliminaries

Let \( \mathbb{D} \) be the open unit disk in the complex plane \( \mathbb{C} \), \( H(\mathbb{D}) \) the space of all holomorphic functions on \( \mathbb{D} \) and \( H^\infty \) the Banach space of bounded analytic functions on \( \mathbb{D} \). Let \( \psi \in H(\mathbb{D}) \) and \( \phi \) be a holomorphic self-map of \( \mathbb{D} \). Then the weighted composition operator \( W_{\psi, \phi} \) is a linear operator on \( H(\mathbb{D}) \) defined by \( W_{\psi, \phi} f = \psi \cdot f \circ \phi \) for \( f \in H(\mathbb{D}) \). It is of interest to provide function-theoretic characterizations involving symbols \( \psi \) and \( \phi \) for the boundedness and compactness of \( W_{\psi, \phi} \) acting between different function spaces. Recently, several authors have studied these type of operators on different spaces of holomorphic functions, see for example, [2]–[34] and the related references therein.

Let \( \sigma : [0, 1) \to [0, \infty) \) be a non-increasing continuous function. We extend it on \( \mathbb{D} \) by \( \sigma(z) = \sigma(|z|), z \in \mathbb{D} \) and call it a weight or a weight function. Throughout this paper, we assume that a weight \( \sigma \) will also satisfy the following properties:

1. \( \sigma(r)(1 - r)^{-1}\gamma \) is non-decreasing for some \( \gamma > 0 \);
2. \( \lim_{r \to 1^-} \sigma(r) = 0 \).

Such a weight function is called an admissible weight. For \( 0 < p < \infty \) and \( \sigma \) an admissible weight, we denote by \( A^p(\sigma) \) the weighted Bergman space consisting of holomorphic functions \( f \) on \( \mathbb{D} \) such that

\[
||f||_{A^p(\sigma)}^p = \int_{\mathbb{D}} |f(z)|^p \sigma(z) dA(z) < \infty,
\]


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where \( dA(z) = \frac{1}{\pi} dx dy = \frac{1}{\pi} r dr d\theta \) stands for the normalized area measure in \( \mathbb{D} \). Recall that a weight \( v \) is normal if there exist positive numbers \( \eta \) and \( \tau \), \( 0 < \eta < \tau \) and \( \delta \in [0, 1) \) such that

\[
\frac{v(r)}{(1-r)^\eta} \text{ is decreasing on } [\delta, 1) \text{ and } \lim_{r \to 1} \frac{v(r)}{(1-r)^\eta} = 0;
\]

\[
\frac{v(r)}{(1-r)^\tau} \text{ is increasing on } [\delta, 1) \text{ and } \lim_{r \to 1} \frac{v(r)}{(1-r)^\tau} = \infty.
\]

It is well known that classical weights \( \sigma_\alpha(z) = (1-|z|^2)^\alpha \), \( \alpha > -1 \) are normal weights.

The following lemma is folklore and can be proved as Lemma 2.1 in [6] or Lemma 1 in [18], we omit the details.

**Lemma 1.1.** Assume that \( p > 0 \), \( k \in \mathbb{N}_0 \) and \( \sigma \) is an admissible weight. Then, there is a positive constant \( C \) such that

\[
|f^{(k)}(z)| \leq C \frac{\|f\|_{\mathcal{A}^p(\sigma)}}{\sigma^{1/p}(z)(1-|z|^2)^{k+2/p}}
\]

for every \( z \in \mathbb{D} \) and \( f \in \mathcal{A}^p(\sigma) \).

**Lemma 1.2.** Let \( \gamma > 0 \) and \( \sigma \) be an admissible weight. Then

\[
f_{\gamma, \lambda}(z) = \frac{(1-|\lambda|^2)^{2(1+\gamma)/p}}{\sigma^{1/p}(\lambda)(1-\lambda z)^{2(2+\gamma)/p}} \quad (\lambda \in \mathbb{D})
\]

is in \( \mathcal{A}^p(\sigma) \). Moreover, \( \sup_{\lambda \in \mathbb{D}} \|f_{\gamma, \lambda}\|_{\mathcal{A}^p(\sigma)} \lesssim 1 \).

**Proof.** Proof is an easy consequence of Lemma 2.4 in [2]. \( \square \)

The next lemma can be found in [13].

**Lemma 1.3.** Let \( 1 < p < \infty \) and \( \sigma \) be an admissible weight. If a bounded sequence \( \{f_k\}_{k \in \mathbb{N}} \) in \( \mathcal{A}^p(\sigma) \) converges to 0 uniformly on compact subsets of \( \mathbb{D} \), then \( \{f_k\}_{k \in \mathbb{N}} \) also converges to 0 weakly in \( \mathcal{A}^p(\sigma) \).

The following functions play an important role in the rest of the paper.

For \( \gamma > 0 \), \( n \in \mathbb{N}_0 \) and \( \lambda \in \mathbb{D} \), consider the family of function.

\[
f_{\gamma, n, \lambda}(z) = \frac{(1-|\lambda|^2)^{2(1+\gamma)/p+n}}{\sigma^{1/p}(\lambda)(1-\lambda z)^{2(2+\gamma)/p+n}}.
\]

(1)

Using Lemma 1.2 it is easy to show that \( f_{\gamma, n, \lambda} \in \mathcal{A}^p(\sigma) \). Moreover,

\[
f_{\gamma, n, \lambda}(\lambda) = \frac{1}{\sigma^{1/p}(\lambda)(1-|\lambda|^2)^{2/p}}.
\]

(2)
Also an easy calculation yields the following equalities:

\[
f'_{γ,n,λ}(z) = \left(\frac{4 + 2γ}{p} + n\right) \frac{1 - |λ|^2}{σ^{1/p}(λ)(1 - |λ|^2)} 2^{(1+γ)/p+n},
\]

\[
f'_{γ,n,λ}(λ) = \left(\frac{4 + 2γ}{p} + n\right) \frac{1 - |λ|^2}{σ^{1/p}(λ)(1 - |λ|^2)} 2^{1+2/p},
\]

\[
f''_{γ,n,λ}(z) = \left(\frac{4 + 2γ}{p} + n\right) \left(\frac{4 + 2γ}{p} + n + 1\right) \frac{1 - |λ|^2}{σ^{1/p}(λ)(1 - |λ|^2)} 2^{(1+γ)/p+n+1},
\]

\[
f''_{γ,n,λ}(λ) = \left(\frac{4 + 2γ}{p} + n\right) \left(\frac{4 + 2γ}{p} + n + 1\right) \frac{1 - |λ|^2}{σ^{1/p}(λ)(1 - |λ|^2)} 2^{1+2/p}.
\]

For a normal weight \(ν\), the Zenrugd-type space \(X_ν\) on \(D\) is the space of all holomorphic functions \(f\) on \(D\) such that

\[
\sup_{z \in D} ν(z)|f''(z)| < \infty.
\]

For \(ν(z) = 1 - |z|^2\) is obtained the (standard) Zygmund space \(X\), which was defined in [5]. The space \(X_ν\) is a Banach space with the norm

\[
\|f\|_{X_ν} = |f(0)| + |f'(0)| + \sup_{z \in D} ν(z)|f''(z)|.
\]

Zygmund-type spaces and operators on them have attracted a considerable attention recently. For some operators from or to the Zygmund-type spaces on the unit disk, see, for example, [3, 4, 5, 9, 7, 8, 12, 23, 27, 32, 34]. Zygmund-type spaces on the unit ball and operators from or to them are studied, for example, in [10, 11, 17, 31, 33], while some results in the setting of the upper half-plane can be found, for example, in [15, 21, 22]. For some generalizations of Zygmund-type spaces and operators on them, see, for example, [21, 20, 24, 25, 22, 26].

In this paper, we characterize the boundedness and compactness of weighted composition operators acting from weighted Bergman spaces \(A^p(σ)\) with admissible weights to Zygmund-type spaces \(X_ν\). We also give some upper and lower bounds for the norm and essential norm of the operators. For some results, in this direction, see, for example, [13, 14, 16, 30] and the references therein.

The next criterion for compactness follows by standard arguments similar to those outlined in the Proposition 3.11 in [1].

**Lemma 1.4.** Let \(σ\) be an admissible weight, \(ν\) a normal weight and \(W_{ϕ,φ} : A^p(σ) → X_ν\) is bounded. Then \(W_{ϕ,φ} : A^p(σ) → X_ν\) is compact if and only if for any bounded sequence \(\{f_n\}_{n \in ℕ}\) in \(A^p(σ)\) which converges to zero uniformly on compact subsets of \(D\), we have \(\|W_{ϕ,φ}f_n\|_{X_ν} → 0\) as \(n → ∞\).

Throughout this paper constants are denoted by \(C\), they are positive and not necessarily the same at each occurrence. The notations \(A ≲ B\) means that \(A\) is less than or equal to a constant times \(B\) and \(D \gtrsim E\), means that \(D\) is greater than or equal to a constant times \(E\).
2. Boundedness and compactness of $W_{\psi, \phi} : \mathcal{A}^P(\sigma) \to \mathcal{L}_\nu$.

**Theorem 2.1.** Let $p > 0$, $\sigma$ an admissible weight, $\nu$ a normal weight, $\psi \in H(\mathbb{D})$ and $\phi$ be a holomorphic self-map of $\mathbb{D}$. Then $W_{\psi, \phi} : \mathcal{A}^P(\sigma) \to \mathcal{L}_\nu$ is bounded if and only if the following conditions are satisfied:

1. $M_1 = \sup_{z \in \mathbb{D}} \frac{v(z)\psi''(z)}{\sigma^{1/p}(\phi(z))(1 - |\phi(z)|^2)^{-1/2}} < \infty$.

2. $M_2 = \sup_{z \in \mathbb{D}} \frac{v(z)|2\psi'(z)\psi'(z) + \psi(z)\psi''(z)|}{\sigma^{1/p}(\phi(z))(1 - |\phi(z)|^2)^{1/2}} < \infty$.

3. $M_3 = \sup_{z \in \mathbb{D}} \frac{v(z)\psi(z)|\psi'(z)|^2}{\sigma^{1/p}(\phi(z))(1 - |\phi(z)|^2)^{1/2}} < \infty$.

Moreover, the following relation hold

$$M_1 + M_2 + M_3 \lesssim \|W_{\psi, \phi}\|_{\mathcal{A}^P(\sigma) \to \mathcal{L}_\nu} \lesssim \frac{|\psi(0)| + |\psi'(0)|}{\sigma^{1/p}(\phi(0))(1 - |\phi(0)|^2)^{2/2}} + \frac{|\psi(0)\phi'(0)|}{\sigma^{1/p}(\phi(0))(1 - |\phi(0)|^2)^{1/2}} + M_1 + M_2 + M_3.$$  \hfill (5)

**Proof.** First suppose that $W_{\psi, \phi} : \mathcal{A}^P(\sigma) \to \mathcal{L}_\nu$ is bounded. Then

$$\|W_{\psi, \phi}f\|_{\mathcal{L}_\nu} \leq \|W_{\psi, \phi}\|_{\mathcal{A}^P(\sigma) \to \mathcal{L}_\nu} \|f\|_{\mathcal{A}^P(\sigma)}$$ \hfill (6)

for every $f \in \mathcal{A}^P(\sigma)$. By taking $f(z) = 1$ in (6) we have that

$$\sup_{z \in \mathbb{D}} v(z)|\psi''(z)| \leq \|W_{\psi, \phi}\|_{\mathcal{A}^P(\sigma) \to \mathcal{L}_\nu}.$$ \hfill (7)

Also by taking $f(z) = z$ in (6), using (7) and the fact that $|\phi(z)| < 1$, we see that

$$\sup_{z \in \mathbb{D}} v(z)|2\psi'(z)\psi'(z) + \psi(z)\psi''(z)| \lesssim \|W_{\psi, \phi}\|_{\mathcal{A}^P(\sigma) \to \mathcal{L}_\nu}.$$ \hfill (8)

Again by taking $f(z) = z^2/2$ in (6), using (7), (8) and the fact that $|\phi(z)| < 1$, we have that

$$\sup_{z \in \mathbb{D}} v(z)|\psi(z)||\phi'(z)|^2 \lesssim \|W_{\psi, \phi}\|_{\mathcal{A}^P(\sigma) \to \mathcal{L}_\nu}.$$ \hfill (9)

Next we consider the following family of functions

$$f_{\lambda}(z) = f_{\gamma, 0, \phi(\lambda)}(z) - 2f_{\gamma, 1, \phi(\lambda)}(z) + f_{\gamma, 2, \phi(\lambda)}(z),$$

where $f_{\gamma, i, \phi(\lambda)}$, $i = 0, 1, 2$ are defined in (1). Then

$$f_{\lambda}(z) = \left[1 - \frac{2(1 - |\phi(\lambda)|^2)}{(1 - \phi(\lambda)^2)} + \frac{(1 - |\phi(\lambda)|^2)^2}{(1 - \phi(\lambda)^2)^2}\right]f_{\gamma, 0, \phi(\lambda)}(z) = \tau_{\phi(\lambda)}(z)f_{\gamma, 0, \phi(\lambda)}(z),$$
where
\[ \tau_{\phi(\lambda)}(z) = 1 - \frac{2(1 - |\phi(\lambda)|^2)}{(1 - \phi(\lambda)z)} + \frac{(1 - |\phi(\lambda)|^2)^2}{(1 - \phi(\lambda)z)^2}. \]

Then \( \tau_{\phi(\lambda)} \in H^\infty \) as
\[
\sup_{\lambda \in \mathbb{D}} |\tau_{\phi(\lambda)}(z)| \leq \sup_{\lambda \in \mathbb{D}} \left[ 1 + \frac{2(1 - |\phi(\lambda)|^2)}{(1 - |\phi(\lambda)||z|)} + \frac{(1 - |\phi(z)|^2)^2}{(1 - |\phi(\lambda)||z|)^2} \right] \leq 9.
\]

Therefore, \( f_\lambda \in \mathcal{A}^p(\sigma) \) and \( \sup_{\lambda \in \mathbb{D}} \|f_\lambda\|_{\mathcal{A}^p(\sigma)} \lesssim 1. \) Moreover, using \( (2), (3) \) and \( (4) \), we have
\[
f_\lambda(\phi(\lambda)) = 0, \quad f_\lambda'(\phi(\lambda)) = 0 \quad \text{and} \quad f_\lambda''(\phi(\lambda)) = \frac{2(\phi(\lambda))^2}{\sigma^{1/p}(\phi(\lambda))(1 - |\phi(\lambda)|^2)^{2+2/p}}.
\]

Thus
\[
\|W_{\psi, \phi}\|_{\mathcal{A}^p(\sigma) \rightarrow \mathcal{Z}_v} \gtrsim \|W_{\psi, \phi} f_\lambda\|_{\mathcal{Z}_v} \\
\geq v(\lambda) |\psi''(\lambda) f_\lambda(\phi(\lambda)) + (2\psi'(\lambda) \phi'(\lambda) + \psi(\lambda) \phi''(\lambda)) f_\lambda'(\phi(\lambda)) + \psi(\lambda) (\phi'(\lambda))^2 f_\lambda''(\phi(\lambda))| \\
\geq 2 \frac{v(\lambda) |\psi(\lambda)||\phi'(\lambda)|^2 |\phi(\lambda)|^2}{\sigma^{1/p}(\phi(\lambda))(1 - |\phi(\lambda)|^2)^{2+2/p}}.
\]

Therefore, we have that
\[
\sup_{\lambda \in \mathbb{D}} \frac{v(\lambda) |\psi(\lambda)||\phi'(\lambda)|^2 |\phi(\lambda)|^2}{\sigma^{1/p}(\phi(\lambda))(1 - |\phi(\lambda)|^2)^{2+2/p}} \lesssim \|W_{\psi, \phi}\|_{\mathcal{A}^p(\sigma) \rightarrow \mathcal{Z}_v}.
\]

Thus for fixed \( \delta \in (0, 1) \), we have that
\[
\sup_{|\phi(\lambda)| > \delta} \frac{v(\lambda) |\psi(\lambda)||\phi'(\lambda)|^2}{\sigma^{1/p}(\phi(\lambda))(1 - |\phi(\lambda)|^2)^{2+2/p}} \lesssim \|W_{\psi, \phi}\|_{\mathcal{A}^p(\sigma) \rightarrow \mathcal{Z}_v}.
\]

Since \( \sigma \) is non-increasing, so by using \( (9) \), we have that
\[
\sup_{|\phi(\lambda)| \leq \delta} \frac{v(\lambda) |\psi(\lambda)||\phi'(\lambda)|^2}{\sigma^{1/p}(\phi(\lambda))(1 - |\phi(\lambda)|^2)^{2+2/p}} \leq \frac{1}{\sigma^{1/p}(\delta)(1 - \delta^2)^{2+2/p}} \|W_{\psi, \phi}\|_{\mathcal{A}^p(\sigma) \rightarrow \mathcal{Z}_v}.
\]

Hence from \( (11) \) and \( (12) \), we have that
\[
\sup_{\lambda \in \mathbb{D}} \frac{v(\lambda) |\psi(\lambda)||\phi'(\lambda)|^2}{\sigma^{1/p}(\phi(\lambda))(1 - |\phi(\lambda)|^2)^{2+2/p}} \lesssim \|W_{\psi, \phi}\|_{\mathcal{A}^p(\sigma) \rightarrow \mathcal{Z}_v}.
\]

Again, let \( \lambda = \phi(\zeta) \) and consider the family of function
\[
g_\lambda(z) = \left( \frac{8 + 4\gamma}{p} + 4 \right) f_{\gamma,0,\lambda}(z) - \left( \frac{16 + 8\gamma}{p} + 6 \right) f_{\gamma,1,\lambda}(z) + \left( \frac{8 + 4\gamma}{p} + 2 \right) f_{\gamma,2,\lambda}(z).
\]
Then
\[
g_\lambda(z) = \left( \frac{8+4\gamma}{p} + 4 \right) - \left( \frac{16+8\gamma}{p} + 6 \right) \frac{1 - |\lambda|^2}{1 - \lambda z} + \left( \frac{8+4\gamma}{p} + 2 \right) \frac{(1 - |\lambda|^2)^2}{(1 - \lambda z)^2} \right] f_{\gamma,0,\lambda}(z)
= \rho_\lambda(z)f_{\gamma,0,\lambda}(z).
\]

Since
\[
\sup_{z \in \mathbb{D}} |\rho_\lambda(z)| = \sup_{z \in \mathbb{D}} \left| \left( \frac{8+4\gamma}{p} + 4 \right) - \left( \frac{16+8\gamma}{p} + 6 \right) \frac{1 - |\lambda|^2}{1 - \lambda z} + \left( \frac{8+4\gamma}{p} + 2 \right) \frac{(1 - |\lambda|^2)^2}{(1 - \lambda z)^2} \right| \leq 36 \left( \frac{2 + \gamma}{p} \right) + 24,
\]
so \( \rho_\lambda \in H^p \). Therefore, we have that \( g_\lambda \in \mathcal{A}^p(\sigma) \) and \( \sup_{\lambda \in \mathbb{D}} \|g_\lambda\|_{\mathcal{A}^p(\sigma)} \lesssim 1 \). Moreover,
\[
g'_\lambda(z) = \left( \frac{8+4\gamma}{p} + 4 \right) f'_{\gamma,0,\lambda}(z) - \left( \frac{16+8\gamma}{p} + 6 \right) f'_{\gamma,1,\lambda}(z) + \left( \frac{8+4\gamma}{p} + 2 \right) f'_{\gamma,2,\lambda}(z)
\]
\[
g''_\lambda(z) = \left( \frac{8+4\gamma}{p} + 4 \right) f''_{\gamma,0,\lambda}(z) - \left( \frac{16+8\gamma}{p} + 6 \right) f''_{\gamma,1,\lambda}(z) + \left( \frac{8+4\gamma}{p} + 2 \right) f''_{\gamma,2,\lambda}(z).
\]
Therefore, by using (2), (3) and (4), we have that
\[
g_\lambda(\lambda) = 0, \ g''_\lambda(\lambda) = 0 \text{ and } g'_\lambda(\lambda) = \frac{-2\lambda}{\sigma^{1/p}(\lambda)(1 - |\phi(\lambda)|^2)^{1+2/p}}
\]
and so
\[
\|W_{\psi,\varphi}\|_{\mathcal{A}^p(\sigma) \rightarrow X_v} \gtrsim \|W_{\psi,\varphi}g_\lambda\|_{X_v} \\
\geq \nu(\xi)|\psi''(\xi)g_\lambda(\phi(\xi)) + 2\psi'(\xi)\phi'(\xi) + \psi(\xi)\phi''(\xi)|g'_\lambda(\phi(\xi)) + \psi(\xi)(\phi'(\xi))^2g''_\lambda(\phi(\xi))| \\
\geq 2 \frac{\nu(\xi)|2\psi'(\xi)\phi'(\xi) + \psi(\xi)\phi''(\xi)|}{\sigma^{1/p}(\phi(\xi))(1 - |\phi(\xi)|^2)^{1+1/p}}|\phi(\xi)|.
\]
Thus we have
\[
\sup_{\xi \in \mathbb{D}} \frac{\nu(\xi)|2\psi'(\xi)\phi'(\xi) + \psi(\xi)\phi''(\xi)|}{\sigma^{1/p}(\phi(\xi))(1 - |\phi(\xi)|^2)^{1+1/p}} \lesssim \|W_{\psi,\varphi}\|_{\mathcal{A}^p(\sigma) \rightarrow X_v}.
\]
Thus there exist some \( \delta_1 \in (0, 1) \), such that
\[
\sup_{|\phi(\xi)| > \delta_1} \frac{\nu(\xi)|2\psi'(\xi)\phi'(\xi) + \psi(\xi)\phi''(\xi)|}{\sigma^{1/p}(\phi(\xi))(1 - |\phi(\xi)|^2)^{1+1/p}} \lesssim \|W_{\psi,\varphi}\|_{\mathcal{A}^p(\sigma) \rightarrow X_v}.
\]
Since \( \sigma \) is monotonically increasing, so by (8), we have that
\[
\sup_{|\varphi(\zeta)| \leq \delta_1} \frac{\nu(\zeta)|2\psi'(\zeta)\varphi'(\zeta) + \psi(\zeta)\varphi''(\zeta)|}{\sigma^{1/p}(\varphi(\zeta))(1 - |\varphi(\zeta)|^2)^{1+1/p}} \leq \frac{1}{\sigma(\delta_1)(1 - \delta_1^2)^{1+2/p}} ||W_{\psi, \varphi}||_{\mathcal{A}^p(\sigma) \rightarrow \mathcal{L}_p}.
\] (15)

Combining (14) and (15) we have that
\[
\sup_{\zeta \in \mathbb{D}} \frac{\nu(\zeta)|2\psi'(\zeta)\varphi'(\zeta) + \psi(\zeta)\varphi''(\zeta)|}{\sigma^{1/p}(\varphi(\zeta))(1 - |\varphi(\zeta)|^2)^{1+1/p}} \lesssim ||W_{\psi, \varphi}||_{\mathcal{A}^p(\sigma) \rightarrow \mathcal{L}_p}.
\] (16)

Again, let \( \lambda = \varphi(\zeta) \) and consider the family of functions
\[
h_\lambda(z) = 2 \left( \frac{2 + \gamma}{p} + 1 \right) \left( \frac{4 + 2\gamma}{p} + 1 \right) f_{\gamma,0,\lambda}(z)
- 8 \left( \frac{2 + \gamma}{p} \right) \left( \frac{2 + \gamma}{p} + 1 \right) f_{\gamma,1,\lambda}(z)
+ 2 \left( \frac{2 + \gamma}{p} \right) \left( \frac{4 + 2\gamma}{p} + 1 \right) f_{\gamma,2,\lambda}(z).
\]

Then
\[
h_\lambda(z) = \left[ 2 \left( \frac{2 + \gamma}{p} + 1 \right) \left( \frac{4 + 2\gamma}{p} + 1 \right) - 8 \left( \frac{2 + \gamma}{p} \right) \left( \frac{2 + \gamma}{p} + 1 \right) \frac{1 - \lambda}{1 - \lambda z} \right]
+ 2 \left( \frac{2 + \gamma}{p} \right) \left( \frac{4 + 2\gamma}{p} + 1 \right) \frac{1}{(1 - \lambda z)^2} f_{\gamma,0,\lambda}(z)
= g_1(z) f_{\gamma,0,\lambda}(z).
\]

Proceeding as above we can show that \( g_1 \in H^\infty, h_\lambda \in \mathcal{A}^p(\sigma) \) and \( \sup_{\lambda \in \mathbb{D}} ||h_\lambda||_{\mathcal{A}^p(\sigma)} \lesssim 1 \) and
\[
h'_\lambda(\lambda) = 0, h''_\lambda(\lambda) = 0 \quad \text{and} \quad h_\lambda(\lambda) = \frac{2}{\sigma^{1/p}(\lambda)(1 - |\lambda|^2)^{2/p}}.
\]

Therefore,
\[
||W_{\psi, \varphi}||_{\mathcal{A}^p(\sigma) \rightarrow \mathcal{L}_p} \gtrsim ||W_{\psi, \varphi}h_\lambda||_{\mathcal{L}_p} \geq \frac{\nu(\zeta)|\psi''(\zeta)|}{\sigma^{1/p}(\varphi(\zeta))(1 - |\varphi(\zeta)|^2)^{2/p}}.
\]

Taking the supermum over \( \zeta \in \mathbb{D} \), we have that
\[
\sup_{\zeta \in \mathbb{D}} \frac{\nu(\zeta)|\psi''(\zeta)|}{\sigma^{1/p}(\varphi(\zeta))(1 - |\varphi(\zeta)|^2)^{2/p}} \lesssim ||W_{\psi, \varphi}||_{\mathcal{A}^p(\sigma) \rightarrow \mathcal{L}_p}.
\] (17)

From (13), (16) and (17) we have that
\[
M_1 + M_2 + M_3 \lesssim ||W_{\psi, \varphi}||_{\mathcal{A}^p(\sigma) \rightarrow \mathcal{L}_p}.
\] (18)
Conversely, suppose that the conditions (1)–(3) hold. Then
\[
\nu(z)|(W_{\psi,\varphi}f)''(z)| = \nu(z)|\psi''(z)f(\varphi(z)) + (2\psi'(z)\varphi'(z)
+ \psi(z)\varphi''(z))f'(\varphi(z)) + \psi(z)(\varphi'(z))^2f''(\varphi(z))|
\leq \left( \frac{\nu(z)|\psi''(z)|}{\sigma^{1/p}(\varphi(z))(1 - |\varphi(z)|^2)^{2/p}} + \frac{\nu(z)|2\psi'(z)\varphi'(z) + \psi(z)\varphi''(z)|}{\sigma^{1/p}(\varphi(z))(1 - |\varphi(z)|^2)^{1+2/p}} + \frac{\nu(z)|\psi(z)|\varphi'(z)^2}{\sigma^{1/p}(\varphi(z))(1 - |\varphi(z)|^2)^{2+2/p}} \right) \|f\|_{A^p(\sigma)}.
\]
Taking the supremum over \(z \in \mathbb{D}\), we get
\[
\sup_{z \in \mathbb{D}} \nu(z)|(W_{\psi,\varphi}f)''(z)| \lesssim (M_1 + M_2 + M_3)\|f\|_{A^p(\sigma)}.
\]
Further, we have
\[
\|W_{\psi,\varphi}\|_{A^p(\sigma) \to L^\nu} \lesssim M_1 + M_2 + M_3, \tag{19}
\]
\[
\|W_{\psi,\varphi}f(0)\| = |\psi(0)||f(\varphi(0))| \lesssim \frac{|\psi(0)||f|_{A^p(\sigma)}}{\sigma^{1/p}(\varphi(0))(1 - |\varphi(0)|^2)^{2/p}} \tag{20}
\]
and
\[
|(W_{\psi,\varphi}f)'(0)| = |\psi'(0)f(\varphi(0)) + \psi(0)\varphi'(0)f'(\varphi(0))|
\leq \left( \frac{|\psi'(0)|}{\sigma^{1/p}(\varphi(0))(1 - |\varphi(0)|^2)^{2/p}} + \frac{|\psi(0)\varphi'(0)|}{\sigma^{1/p}(\varphi(0))(1 - |\varphi(0)|^2)^{1+2/p}} \right) \|f\|_{A^p(\sigma)}. \tag{21}
\]
Combining (19), (20) and (21), we have that
\[
\|W_{\psi,\varphi}\|_{A^p(\sigma) \to L^\nu} \lesssim \frac{|\psi(0)| + |\psi'(0)|}{\sigma^{1/p}(\varphi(0))(1 - |\varphi(0)|^2)^{2/p}}
+ \frac{|\psi(0)\varphi'(0)|}{\sigma^{1/p}(\varphi(0))(1 - |\varphi(0)|^2)^{1+2/p}} + M_1 + M_2 + M_3. \tag{22}
\]
From (18) and (22), (6) holds. \(\square\)

**Theorem 2.2.** Let \(p > 0\), \(\sigma\) an admissible weight, \(\nu\) a normal weight, \(\psi \in H(\mathbb{D})\) and \(\varphi\) be a holomorphic self map of \(\mathbb{D}\) such that \(||\varphi||_{\infty} < 1\). Let \(W_{\psi,\varphi} : A^p(\sigma) \to L^\nu\) is bounded. Then following conditions are equivalent:

1. \(W_{\psi,\varphi} : A^p(\sigma) \to L^\nu\) is bounded.

2. \(W_{\psi,\varphi} : A^p(\sigma) \to L^\nu\) is compact.

3. \(N_1 = \sup_{z \in \mathbb{D}} \nu(z)|\psi''(z)| < \infty,\)
\(N_2 = \sup_{z \in \mathbb{D}} \nu(z)|2\psi'(z)\varphi'(z) + \psi(z)\varphi''(z)| < \infty,\)
\(N_3 = \sup_{z \in \mathbb{D}} \nu(z)|\psi(z)||\varphi'(z)|^2 < \infty.\)
Proof. (2) ⇒ (1) is trivially true. To complete the theorem we only need to prove that (1) ⇒ (3) and (3) ⇒ (2).

(1) ⇒ (3) By taking \( f(z) = 1, f(z) = z \) and \( f(z) = z^2 / 2 \), respectively in \( \| W_{\psi, \phi} f \|_{\mathcal{A}^\sigma} \leqslant \| W_{\psi, \phi} \| \| f \|_{\mathcal{A}^\sigma} \). We see that all the conditions in (3) hold.

(3) ⇒ (2). Suppose that conditions in (3) hold. Let \( \{ f_j \} \) be a bounded sequence in \( \mathcal{A}^p(\sigma) \) such that \( \sup_j \| f_j \|_{\mathcal{A}^\sigma} \leqslant M \) and \( f_j \to 0 \) uniformly on compact subsets of \( \mathbb{D} \). From this and by the Cauchy inequality we have that \( f_j' \to 0 \) and \( f_j'' \to 0 \) uniformly on compact subsets of \( \mathbb{D} \). Thus for fixed \( z \in \mathbb{D} \), we have that

\[
|v(z)(W_{\psi, \phi} f_j)''(z)| \leqslant v(z)|\psi''(z)||f_j'(\phi(z))| + v(z)|2\psi'(z)\phi'(z) + \psi(z)\phi''(z)||f_j'(\phi(z))|
\]

\[
+ v(z)|\psi'(z)||\phi'(z)|^2|f_j''(\phi(z))|
\]

\[
\leqslant v(z)|\psi''(z)| \max_{|\xi| \leqslant \| \phi \|_\infty} |f_j(\xi)| + v(z)|2\psi'(z)\phi'(z) + \psi(z)|\phi'(z)|^2 \max_{|\xi| \leqslant \| \phi \|_\infty} |f_j''(\xi)|.
\]

Thus

\[
\sup_{z \in \mathbb{D}} v(z)(W_{\psi, \phi} f_j)''(z) \leqslant N_1 \max_{|\xi| \leqslant \| \phi \|_\infty} |f_j(\xi)| + N_2 \max_{|\xi| \leqslant \| \phi \|_\infty} |f_j'(\xi)|
\]

\[
+ N_3 \max_{|\xi| \leqslant \| \phi \|_\infty} |f_j''(\xi)| \to 0 \text{ as } j \to \infty
\]

and so \( \lim_{j \to \infty} \sup_{z \in \mathbb{D}} v(z)(W_{\psi, \phi} f_j)''(z) = 0 \). Moreover, \( |\psi(0)||f_j(\phi(0))| \to 0 \) and \( |\psi'(0)f_j(\phi(0)) + \psi(0)\phi'(0)f_j(\phi(0))| \to 0 \) as \( j \to \infty \). Thus

\[
\lim_{j \to \infty} \| W_{\psi, \phi} f_j \|_{\mathcal{Z}_v} = 0.
\]

Hence by Lemma 1.4, \( W_{\psi, \phi} : \mathcal{A}^p(\sigma) \to \mathcal{Z}_v \) is compact. \( \square \)

3. Essential norm of \( W_{\psi, \phi} : \mathcal{A}^p(\sigma) \to \mathcal{Z}_v \).

In this section, we give some upper and lower bounds for the essential norm of the operators.

Recall that if \( X \) and \( Y \) are two Banach spaces, then the essential norm \( \| T \|_{e, X \to Y} \) of a bounded linear operator \( T : X \to Y \) is defined as

\[
\| T \|_{e, X \to Y} = \inf\{ \| T - K \| : K \text{ is compact from } X \text{ to } Y \},
\]

where \( \| T \| \) denote the usual operator norm. Clearly \( T \) is compact if and only if \( \| T \|_{e, X \to Y} = 0 \).

**Theorem 3.1.** Let \( p > 1, \sigma \) an admissible weight, \( v \) a normal weight, \( \psi \in H(\mathbb{D}) \) and \( \phi \) be a holomorphic self map of \( \mathbb{D} \) such that \( \| \phi \|_\infty = 1 \). Let \( W_{\psi, \phi} : \mathcal{A}^p(\sigma) \to \mathcal{Z}_v \). Then...
\( \mathcal{A}^p(\sigma) \to \mathcal{L}_v \) is bounded. Then
\[
\| W_{\psi, \varphi} \|_{c, \mathcal{A}^p(\sigma) \to \mathcal{L}_v} \geq \limsup_{r \to 1} \frac{v(z) |\psi''(z)|}{r^{1/p}(\varphi(z))(1 - |\varphi(z)|^2)^{2/p}} + \limsup_{r \to 1} \frac{v(z)|2\psi'(z)\psi(z) + \psi(z)\varphi''(z)|}{r^{1/p}(\varphi(z))(1 - |\varphi(z)|^2)^{1+2/p}} + \limsup_{r \to 1} \frac{v(z)|\psi(z)||\varphi'(z)|^2}{r^{1/p}(\varphi(z))(1 - |\varphi(z)|^2)^{2+2/p}}.
\]

Proof. Lower Bound. Let \( \{z_j\} \) be a sequence in \( \mathbb{D} \) such that \( |\varphi(z_j)| \to 1 \) as \( j \to \infty \). Consider the functions \( f_j, g_j \) and \( h_j \) defined, respectively as
\[
\begin{align*}
    f_j(z) &= f_{\gamma,0,\varphi(z_j)}(z) - 2f_{\gamma,1,\varphi(z_j)}(z) + f_{\gamma,2,\varphi(z_j)}(z), \\
    g_j(z) &= \left(\frac{8 + 4\gamma}{p} + 4\right)f_{\gamma,0,\varphi(z_j)}(z) - \left(\frac{16 + 8\gamma}{p} + 6\right)f_{\gamma,1,\varphi(z_j)}(z) + \left(\frac{8 + 4\gamma}{p} + 2\right)f_{\gamma,2,\varphi(z_j)}(z), \\
    h_j(z) &= 2\left(\frac{2 + \gamma}{p} + 1\right)\left(\frac{4 + 2\gamma}{p} + 1\right)f_{\gamma,0,\varphi(z_j)}(z) - 8\left(\frac{2 + \gamma}{p} + 1\right)f_{\gamma,1,\varphi(z_j)}(z) + 2\left(\frac{2 + \gamma}{p} + 1\right)f_{\gamma,2,\varphi(z_j)}(z),
\end{align*}
\]
where \( f_{\gamma,0,\varphi(z_j)} \), \( f_{\gamma,1,\varphi(z_j)} \), and \( f_{\gamma,2,\varphi(z_j)} \) are defined as in (1). As in Theorem 2.1, we have that \( \{f_j\}, \{g_j\} \) and \( \{h_j\} \) are bounded in \( \mathcal{A}^p(\sigma) \) and \( \sup_j \|K_j\| \leq M \), where \( K_j = f_j \) or \( g_j \) or \( h_j \). Moreover \( K_j \to 0 \) as \( j \to \infty \) uniformly on compact subsets of \( \mathbb{D} \). Let \( K : \mathcal{A}^p(\sigma) \to \mathcal{L}_v \) is compact. Then by Lemma 1.3, \( \|Kf_j\|_{\mathcal{L}_v} \to 0 \) as \( j \to \infty \). As in Theorem 2.1,
\[
f_j(\varphi(z_j)) = 0, f_j'(\varphi(z_j)) = 0 \text{ and } f_j''(\varphi(z_j)) = \frac{2(\varphi(z_j))^2}{\sigma^{1/p}(\varphi(z_j))(1 - |\varphi(z_j)|^2)^{2+2/p}}.
\]

Therefore,
\[
\begin{align*}
    \| W_{\psi, \varphi} \|_{c, \mathcal{A}^p(\sigma) \to \mathcal{L}_v} &\geq \limsup_{j \to \infty} \| W_{\psi, \varphi} f_j - Kf_j \|_{\mathcal{L}_v} \\
    &\geq \limsup_{j \to \infty} \| W_{\psi, \varphi} f_j \|_{\mathcal{L}_v} - \limsup_{j \to \infty} \|Kf_j\|_{\mathcal{L}_v} \\
    &\geq \limsup_{j \to \infty} \frac{v(z)|\psi(z)||\varphi'(z)|^2}{\sigma^{1/p}(\varphi(z_j))(1 - |\varphi(z_j)|^2)^{2+2/p}}.
\end{align*}
\]

(23)

Proceeding as in Theorem 2.1, we have that
\[
g_j(\varphi(z_j)) = 0, g_j''(\varphi(z_j)) = 0 \text{ and } g_j'(\varphi(z_j)) = -\frac{2\varphi(z_j)}{\sigma^{1/p}(\varphi(z_j))(1 - |\varphi(z_j)|^2)^{1+2/p}}.
\]
and so
\[\|W_{\psi,\varphi}\|_{e,\mathcal{A}^p(\sigma) \to \mathcal{X}_v} \geq C \limsup_{j \to \infty} \|W_{\psi,\varphi} g_j - Kg_j\|_{\mathcal{X}_v} \geq C \limsup_{j \to \infty} \|W_{\psi,\varphi} g_j\|_{\mathcal{X}_v} - \limsup_{j \to \infty} \|Kg_j\|_{\mathcal{X}_v} \geq C \limsup_{j \to \infty} \frac{v(z)|\psi'(z_j)\psi'(z_j) + \psi(z_j)\psi''(z_j)|}{\sigma^{1/p}(\varphi(z_j))(1 - |\varphi(z_j)|^2)^{1+2/p}}.\]

(24)

Once again as in Theorem 2.1, we have that
\[h_j'(\varphi(z_j)) = 0, \quad h_j''(\varphi(z_j)) = 0 \quad \text{and} \quad h_j(\varphi(z_j)) = \frac{2}{\sigma^{1/p}(\varphi(z_j))(1 - |\varphi(z_j)|^2)^{2/p}}.\]

Therefore,
\[\|W_{\psi,\varphi}\|_{e,\mathcal{A}^p(\sigma) \to \mathcal{X}_v} \geq C \limsup_{j \to \infty} \|W_{\psi,\varphi} h_j - Kh_j\|_{\mathcal{X}_v} \geq C \limsup_{j \to \infty} \|W_{\psi,\varphi} h_j\|_{\mathcal{X}_v} - \limsup_{j \to \infty} \|Kh_j\|_{\mathcal{X}_v} \geq C \limsup_{j \to \infty} \frac{v(z)|\psi(z_j)\varphi'(z_j)|^2}{\sigma^{1/p}(\varphi(z_j))(1 - |\varphi(z_j)|^2)^{2/p}}.\]

(25)

Combining (23), (24) and (25), we get
\[\|W_{\psi,\varphi}\|_{e,\mathcal{A}^p(\sigma) \to \mathcal{X}_v} \geq \limsup_{|\varphi(z)| \to 1} \frac{v(z)|\psi''(z)|}{\sigma^{1/p}(\varphi(z))(1 - |\varphi(z)|^2)^{2/p}} + \limsup_{|\varphi(z)| \to 1} \frac{v(z)|2\psi'(z)\varphi'(z) + \psi(z)\psi''(z)|}{\sigma^{1/p}(\varphi(z))(1 - |\varphi(z)|^2)^{1+2/p}} + \limsup_{|\varphi(z)| \to 1} \frac{v(z)|\psi(z)|\varphi'(z)|^2}{\sigma^{1/p}(\varphi(z))(1 - |\varphi(z)|^2)^{2+2/p}}.\]

(26)

Upper Bound. Let \(\varphi_k(z) = \frac{k}{k+1}z\). Then \(\|\varphi_k\|_\infty < 1\). Let
\[L_k f(z) = C_{\varphi_k} f(z) = f\left(\frac{k}{k+1}z\right).\]

Then by Corollary 4 in [13], we have that \(L_k : \mathcal{A}^p(\sigma) \to \mathcal{A}^p(\sigma)\) is compact. Since \(W_{\psi,\varphi} : \mathcal{A}^p(\sigma) \to \mathcal{X}_v\) is bounded, so \(W_{\psi,\varphi} L_k : \mathcal{A}^p(\sigma) \to \mathcal{X}_v\) is compact. Thus
\[\|W_{\psi,\varphi}\| \leq \|W_{\psi,\varphi} - W_{\psi,\varphi} L_k\| \leq \sup_{\|f\|_{\mathcal{A}^p(\sigma)} \leq 1} \|W_{\psi,\varphi} f - W_{\psi,\varphi} f\|_{\mathcal{X}_v} \leq \sup_{\|f\|_{\mathcal{A}^p(\sigma)} \leq 1} \left[|W_{\psi,\varphi} f(0)| + |W_{\psi,\varphi} f(0)|' + \sup_{z \in \mathbb{D}} v(z)|W_{\psi,\varphi} f''(z)|\right].\]

(27)
where $I$ is the identity operator on $\mathbb{A}^p(\sigma)$. For any $r \in (0, 1)$, we can write
\[
\sup_{z \in D} |W_{\psi, \varphi}(I - L_k)f''(z)| = \sup_{|\varphi(z)| \leq r} |W_{\psi, \varphi}(I - L_k)f''(z)| \\
+ \sup_{|\varphi(z)| > r} |W_{\psi, \varphi}(I - L_k)f''(z)|. \tag{28}
\]

Now
\[
|W_{\psi, \varphi}(I - L_k)f''(z)| = \left| \psi''(z) \left\{ f(\varphi(z)) - f\left( \frac{k}{k+1} \varphi(z) \right) \right\} + (2 \psi'(z) \varphi'(z) + \psi(z) \varphi''(z)) \left\{ f'(\varphi(z)) - \frac{k}{k+1} f'\left( \frac{k}{k+1} \varphi(z) \right) \right\} + \psi(z) \varphi'(z)^2 \left\{ f''(\varphi(z)) - \frac{k^2}{(k+1)^2} f''\left( \frac{k}{k+1} \varphi(z) \right) \right\} \right|. \tag{29}
\]

Let $|\varphi(z)| \leq r$ and $w = \varphi(z)$. Denote the straight line segment from $kw/(k+1)$ to $w$ by $[kw/(k+1), w]$. Then $[kw/(k+1), w] \subset D(0, r)$, where $D(0, r) = \{ z : |z| \leq r \}$. Thus for $i \in \{ 0, 1, 2 \}$, by Lemma 1.1 and the fact that $\sigma$ is non-increasing, we have that
\[
\left| f^{(i)}(w) - f^{(i)}\left( \frac{k}{k+1} w \right) \right| = \left| \int_{[kw/(k+1), w]} f^{(i+1)}(\zeta) d\zeta \right| \\
\leq \frac{|w|}{k+1} \sup_{\zeta \in D(0, r)} |f^{(i+1)}(\zeta)| \\
\leq \frac{|w|}{k+1} \sup_{\zeta \in D(0, r)} \frac{\| f \|_{\mathbb{A}^p(\sigma)} \sigma^{1/p(\zeta)} (1 - |\zeta|^2)^{i+1+1/2}}{\sigma^{1/p(\zeta)} (1 - r^2)^{i+1+1/2/p}} \tag{30}
\]

Using Lemma 1.1, (29) and (30), we have that
\[
\sup_{\| f \|_{\mathbb{A}^p(\sigma)} \leq 1} \sup_{|\varphi(z)| \leq r} |W_{\psi, \varphi}(I - L_k)f''(z)| \rightarrow 0 \text{ as } k \rightarrow \infty. \tag{31}
\]

Using (30) with $i = 0$, we have that
\[
|W_{\psi, \varphi}(I - L_k)f(0)| = \left| \psi(0)f(\varphi(0)) - \psi(0)f\left( \frac{k}{k+1} \varphi(0) \right) \right| \\
\leq \frac{|\psi(0)\varphi(0)|}{k+1} \frac{\| f \|_{\mathbb{A}^p(\sigma)} \sigma^{1/p(r)} (1 - r^2)^{i+1+1/2}}{\sigma^{1/p(r)} (1 - r^2)^{i+1+1/2/p}}. \tag{32}
\]

On the other hand, by Lemma 1.1 and (30) with $i = 0$ and $i = 1$, we have that
\[
\left| (W_{\psi, \varphi}(I - L_k)f)'(0) \right| \\
\leq |\psi'(0)| \left| f(\varphi(0)) - f\left( \frac{k}{k+1} \varphi(0) \right) \right| + |\psi(0)\varphi'(0)| \left| f'(\varphi(0)) - \frac{k}{k+1} f'\left( \frac{k}{k+1} \varphi(0) \right) \right|.\]
\[ \leq |\psi'(0)| f(\phi(0)) - f\left(\frac{k}{k+1} \phi(0)\right) + \frac{|\psi(0)\phi'(0)|}{k+1} |f'(\phi(0))| \\
+ |\psi(0)\phi'(0)| \frac{k}{k+1} \left| f'(\phi(0)) - f'\left(\frac{k}{k+1} \phi(0)\right) \right| \]

\[ \lambda \left( \frac{|\psi'(0)|}{k+1} \right) \frac{1}{\sigma^{1/p}(r) \left(1 - r^2\right)^{1+2/p}} + \frac{|\psi(0)\phi'(0)|}{k+1} \frac{1}{\sigma^{1/p}(r) \left(1 - r^2\right)^{1+2/p}} \|f\|_{\mathcal{A}^p(\sigma)}. \]  

(33)

Combining (32) and (33), we have that

\[ \sup_{\|f\|_{\mathcal{A}^p(\sigma)} \leq 1} \left[ |W_{\psi, \phi}(I - L_k)f(0)| + |(W_{\psi, \phi}(I - L_k)f)'(0)| \right] \to 0 \]  

(34)
as \( k \to \infty \). The second term in the right hand side of (28) is dominated by

\[ \sup_{|\phi(z)| > r} \nu(z) |\psi''(z)| \left\{ |f(\phi(z))| + \left| f\left(\frac{k}{k+1} \phi(z)\right)\right| \right\} \]

\[ + \sup_{|\phi(z)| > r} \nu(z) \left| 2\psi'(z)\phi'(z) + \psi(z)\phi''(z) \right| \left\{ |f'(\phi(z))| + \frac{k}{k+1} \left| f'\left(\frac{k}{k+1} \phi(z)\right)\right| \right\} \]

\[ + \sup_{|\phi(z)| > r} \nu(z) |\psi(z)(\phi'(z))^2| \left\{ |f''(\phi(z))| + \frac{k^2}{(k+1)^2} \left| f''\left(\frac{k}{k+1} \phi(z)\right)\right| \right\}, \]

which is further dominated by a constant multiple of

\[ \sup_{|\phi(z)| > r} \nu(z) |\psi''(z)| \left\{ \frac{\|f\|_{\mathcal{A}^p(\sigma)}}{\sigma^{1/p}(\phi(z)) \left(1 - |\phi(z)|^2\right)^{2/p}} + \frac{\|f\|_{\mathcal{A}^p(\sigma)}}{\sigma^{1/p}(\phi(z)) \left(1 - \frac{k^2}{(k+1)^2} |\phi(z)|^2\right)^{2/p}} \right\} \]

\[ + \sup_{|\phi(z)| > r} \nu(z) \left| 2\psi'(z)\phi'(z) + \psi(z)\phi''(z) \right| \left\{ \frac{\|f\|_{\mathcal{A}^p(\sigma)}}{\sigma^{1/p}(\phi(z)) \left(1 - |\phi(z)|^2\right)^{1+2/p}} \right\} \]

\[ + \frac{k}{k+1} \frac{\|f\|_{\mathcal{A}^p(\sigma)}}{\sigma^{1/p}(\phi(z)) \left(1 - \frac{k^2}{(k+1)^2} |\phi(z)|^2\right)^{1+2/p}} \}

\[ + \sup_{|\phi(z)| > r} \nu(z) |\psi(z)(\phi'(z))^2| \left\{ \frac{\|f\|_{\mathcal{A}^p(\sigma)}}{\sigma^{1/p}(\phi(z)) \left(1 - |\phi(z)|^2\right)^{2+2/p}} \right\} \]

\[ + \frac{k^2}{(k+1)^2} \frac{\|f\|_{\mathcal{A}^p(\sigma)}}{\sigma^{1/p}(\phi(z)) \left(1 - \frac{k^2}{(k+1)^2} |\phi(z)|^2\right)^{2+2/p}} \}, \]  

(35)

Letting \( k \to \infty \) in (35), we get

\[ \limsup_{k \to \infty} \sup_{\|f\|_{\mathcal{A}^p(\sigma)} \leq 1} \sup_{|\phi(z)| > r} \nu(z) |W_{\psi, \phi}(I - L_k)f''(z)| \]

\[ \lesssim \sup_{|\phi(z)| > r} \frac{\nu(z) |\psi''(z)|}{\sigma^{1/p}(\phi(z)) \left(1 - |\phi(z)|^2\right)^{2/p}} \]
Finally, letting $r \to 1$, then we get

$$\|W_{\psi, \varphi}\|_{e, \mathcal{A}^P(\sigma) \to \mathcal{Z}_V} \lesssim \limsup_{|\varphi(z)| \to 1} \frac{v(z)|\psi''(z)|}{\sigma^{1/p}(\varphi(z))(1 - |\varphi(z)|^2)^{2/p}} + \limsup_{|\varphi(z)| \to 1} \frac{v(z)|2\psi'(z)\varphi'(z) + \psi(z)\varphi''(z)|}{\sigma^{1/p}(\varphi(z))(1 - |\varphi(z)|^2)^{1+2/p}} + \limsup_{|\varphi(z)| \to 1} \frac{v(z)|\varphi'(z)|^2}{\sigma^{1/p}(\varphi(z))(1 - |\varphi(z)|^2)^{2+2/p}}. \quad (37)$$

Combining (26) and (37), we get the desired result. \[\square\]

**Corollary 3.2.** Let $p > 1$, $\sigma$ an admissible weight, $v$ a normal weight, $\psi \in H(D)$ and $\varphi$ be a holomorphic self map of $D$, such that $\|\varphi\|_\infty = 1$. Let $W_{\psi, \varphi}: \mathcal{A}^P(\sigma) \to \mathcal{Z}_V$ is bounded. Then $W_{\psi, \varphi}: \mathcal{A}^P(\sigma) \to \mathcal{Z}_V$ is compact if and only if the following conditions are satisfied

1. \[\limsup_{|\varphi(z)| \to 1} \frac{v(z)|\psi''(z)|}{\sigma^{1/p}(\varphi(z))(1 - |\varphi(z)|^2)^{2/p}} = 0.\]

2. \[\limsup_{|\varphi(z)| \to 1} \frac{v(z)|2\psi'(z)\varphi'(z) + \psi(z)\varphi''(z)|}{\sigma^{1/p}(\varphi(z))(1 - |\varphi(z)|^2)^{1+2/p}} = 0.\]

3. \[\limsup_{|\varphi(z)| \to 1} \frac{v(z)|\varphi'(z)|^2}{\sigma^{1/p}(\varphi(z))(1 - |\varphi(z)|^2)^{2+2/p}} = 0.\]

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