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COMMUTATORS OF GENERALIZED CALDERÓN–ZYGMUND OPERATORS ON WEIGHTED HERZ–TYPE HARDY SPACES

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Abstract. In this paper, the authors establish the boundedness of commutators generated by the generalized Calderón-Zygmund operators and weighted BMO functions or weighted Lipschitz functions on weighted Herz-type Hardy spaces.

1. Introduction

The research of the generalized Calderón-Zygmund operator is motivated by the classical Calderón-Zygmund operator, whose theory is one of the greatest results of classical analysis in the last century.

Suppose that T is a generalized Calderón-Zygmund operator, whose accurate definition will be given later, and b is a locally integrable function on $\mathbb{R}^n$. The commutator $[b,T]$ generated by b and T is defined as following:

$$[b,T](f)(x) = b(x)Tf(x) - T(bf)(x).$$

In 1995, Pérez [17] studied the boundedness of the commutators generated by classical Calderón-Zygmund operators and BMO functions from Hardy type spaces to Lebesgue spaces. Lu, Wu and Yang in [11] established the boundedness of the commutators generated by classical Calderón-Zygmund operators and Lipschitz functions from Hardy spaces to Lebesgue spaces. Since Herz spaces and Herz-type Hardy spaces cover, respectively, the Lebesgue spaces and the Hardy spaces and their weighted versions with power weights, it is a natural idea to generalize the above results to the corresponding boundedness from Herz-type Hardy spaces to Herz spaces. The boundedness of the commutators generated by classical Calderón-Zygmund operators and BMO functions or Lipschitz functions from Herz-type Hardy spaces to Herz spaces was established in [14, 11], respectively.

The authors in [1] discussed the boundedness of generalized Calderón-Zygmund operators on weighted Lebesgue spaces and weighted Hardy spaces. In 2011, Lin [6] proved the boundedeness of $[b,T]$ on Morrey spaces when b is a BMO function or a

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Lipschitz function respectively. Lin and Sun [8] showed the boundedness of the generalized Calderón-Zygmund operator and its commutators with weighted BMO functions on weighted Morrey spaces. In 2015, Lin, Liu and Cong [7] obtained the boundedness of commutators generated by the generalized Calderón-Zygmund operators and weighted Lipschitz functions on weighted Morrey spaces. Liu and Li [10] established the boundedness of the commutators generated by generalized Calderón-Zygmund operators and BMO functions or Lipschitz functions from Herz-type Hardy spaces to Herz spaces, respectively.

Inspired by the above results, with further development, it is naturally interesting to consider the boudedness of commutators generated by the generalized Calderón-Zygmund operator and a weighted BMO function or a weighted Lipschitz function from weighted Herz-type Hardy spaces to weighted Herz spaces in this paper.

DEFINITION 1. Let S be the space of all Schwartz functions on $\mathbb{R}^n$ and S' its dual space, the class of all tempered distributions on $\mathbb{R}^n$. Suppose $T : S \to S'$ is a linear operator with $K(.,.)$ defined initially by

$$T(f)(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy, \quad f \in C_c^\infty(\mathbb{R}^n),\ x \notin \operatorname{supp} f.$$

The operator T is called a generalized Calderón-Zygmund operator provided the following three conditions are satisfied.

(1) T can be extended into a continuous operator on $L^2(\mathbb{R}^n)$.

(2) K is smooth away from the diagonal $\{(x,y) : x = y\}$ with

$$\int_{|x-y|>2|z-y|} (|K(x,y)-K(x,z)|+|K(y,x)-K(z,x)|)dx \leqslant C,$$

where $C > 0$ is a constant independent of y and z.

(3) There is a sequence of positive constant numbers $\{C_j\}$ such that for each $j \in \mathbb{N}$,

$$\left(\int_{2^j|z-y|\leqslant|x-y|<2^{j+1}|z-y|} |K(x,y)-K(x,z)|^{q_0}dx\right)^{\frac{1}{q_0}} \leqslant C_j(2^j|z-y|)^{-\frac{n}{q_0'}},$$

and

$$\left(\int_{2^j|y-z|\leqslant|y-x|<2^{j+1}|y-z|} |K(y,x)-K(z,x)|^{q_0}dx\right)^{\frac{1}{q_0}} \leqslant C_j(2^j|z-y|)^{-\frac{n}{q_0'}},$$

where (q_0, q_0') is a fixed pair of positive numbers with $1/q_0 + 1/q_0' = 1$ and $1 < q_0' < 2$.

Comparing the generalized Calderón-Zygmund operator with the classical Calderón-Zygmund operator, whose kernel $K(x,y)$ enjoys the conditions:

$$|K(x,y)| \leqslant C|x-y|^{-n}$$

and

$$|K(x,y)-K(x,z)|+|K(y,x)-K(z,x)| \leqslant C|x-y|^{-n}\left(\frac{|z-y|}{|x-y|}\right)^{\delta},$$

where $|x-y| > 2|z-y|$ for some $\delta > 0$, we can find out that the classical Calderón-Zygmund operator is actually a generalized Calderón-Zygmund operator defined as in Definition 1 with $C_j = 2^{-j\delta}, j \in \mathbb{N}$, and any $1 < q_0 < \infty$.

DEFINITION 2. ([15]) A non-negative measurable function ω is said to be in the Muckenhoupt class A_p with $1 < p < \infty$ if for every cube Q in $\mathbb{R}^n$, there exists a positive constant C independent of Q such that

$$\left(\frac{1}{|Q|}\int_Q \omega(x)dx\right)\left(\frac{1}{|Q|}\int_Q \omega(x)^{1-p'}\right)^{p-1} \leqslant C,$$

where Q denotes a cube in $\mathbb{R}^n$ with the side parallel to the coordinate axes and $1/p + 1/p' = 1$. When $p = 1$, a non-negative measurable function ω is said to belong to A_1, if there exists a constant $C > 0$ such that for any cube Q,

$$\frac{1}{|Q|}\int_Q \omega(y)dy \leqslant C\omega(x),\ a.e. x \in Q.$$

It is well known that if $\omega \in A_p$ with $1 < p < \infty$, then $\omega \in A_r$ for all $r > p$, and $\omega \in A_q$ for some $1 < q < p$. In particularly, $A_\infty = \cup_{1 \leqslant p < \infty} A_p$.

For $k \in \mathbb{Z}$, let $B_k = \{x \in \mathbb{R}^n : |x| \leqslant 2^k\}$, $E_k = B_k \backslash B_{k-1}$ and χ_k denote the characteristic function of the set E_k. For any given weight function ω on $\mathbb{R}^n$ and $0 < p < \infty$, we denote by $L^p_\omega(\mathbb{R}^n)$ the space of all functions f satisfying

$$\|f\|_{L^p(\omega)} = \left(\int_{\mathbb{R}^n} |f(x)|^p \omega(x)dx\right)^{1/p} < \infty.$$

DEFINITION 3. ([12]) Let $\alpha \in \mathbb{R}, 0 < p < \infty, 1 < q < \infty$ and ω_1, ω_2 be two weight functions on $\mathbb{R}^n$. The homogeneous weighted Herz space $\dot{K}^{\alpha,p}_q(\omega_1, \omega_2)$ is defined by

$$\dot{K}^{\alpha,p}_q(\omega_1,\omega_2) = \left\{f \in L^q_{loc}(\mathbb{R}^n \setminus \{0\};\omega_2) : \|f\|_{\dot{K}^{\alpha,p}_q(\omega_1,\omega_2)} < \infty\right\},$$

where

$$\|f\|_{\dot{K}^{\alpha,p}_q(\omega_1,\omega_2)} = \left(\sum_{k\in\mathbb{Z}} \omega_1(B_k)^{\frac{\alpha p}{n}} \|f\chi_k\|^p_{L^q(\omega_2)}\right)^{\frac{1}{p}}.$$

DEFINITION 4. ([13]) Let $0 < \alpha < \infty, 0 < p < \infty, 1 < q < \infty$ and ω_1, ω_2 be two weight functions on $\mathbb{R}^n$. Let $G(f)$ is the grand maximal function of f defined by

$$Gf(x) = \sup_{\varphi\in A_N} \sup_{|x-y|<t} |f * \varphi_t(y)|,$$

where $A_N = \{\varphi \in S(\mathbb{R}^n) : \sup_{|\alpha|,|\beta|\leqslant N} |x^\alpha D^\beta \varphi(x)| \leqslant 1\}$ and $N > n+1$.

The homogeneous weighted Herz-type Hardy space $H\dot{K}^{\alpha,p}_q(\omega_1,\omega_2)$ is defined by

$$H\dot{K}^{\alpha,p}_q(\omega_1,\omega_2) = \left\{f \in S' : G(f) \in \dot{K}^{\alpha,p}_q(\omega_1,\omega_2)\right\},$$

and we define $\|f\|_{H\dot{K}^{\alpha,p}_q(\omega_1,\omega_2)} = \|G(f)\|_{\dot{K}^{\alpha,p}_q(\omega_1,\omega_2)}$.

DEFINITION 5. ([13]) Let $1<q<\infty$, $n(1-1/q)\leqslant\alpha<\infty$ and $s\geqslant[\alpha+n(1/q-1)]$.

(i) A function $a(x)$ on $\mathbb{R}^n$ is said to be a central $(\alpha,q,\omega_1,\omega_2)$-atom if it satisfies

(a) $\operatorname{supp} a\subset B(0,r)=\{x\in\mathbb{R}^n:|x|<r\}, r>0$;

(b) $\|a\|_{L^q(\omega_2)}\leqslant(\omega_1(B(0,r)))^{-\frac{\alpha}{n}}$;

(c) $\int_{\mathbb{R}^n}a(x)x^\beta dx=0$ for every multi-index β with $|\beta|\leqslant s$.

Here and in what follows, for $t\in\mathbb{R}$, $[t]$ is the largest integer no more than t.

(ii) Let $b(x)$ be a locally integrable function on $\mathbb{R}^n$. A function $a(x)$ on $\mathbb{R}^n$ is said to be a central $(\alpha,q,b,\omega_1,\omega_2)$-atom if it satisfies

(d) *a is a* $(\alpha,q,\omega_1,\omega_2)$-atom;

(e) $\int_{\mathbb{R}^n}a(x)b(x)dx=0$.

DEFINITION 6. Let $\omega_1,\ \omega_2\in A_1$, $0<p<\infty$, $1<q<\infty$, $n(1-1/q)\leqslant\alpha<\infty$ and b be a locally integrable function on $\mathbb{R}^n$. Define

$$H\dot{K}_{q,b}^{\alpha,p}(\omega_1,\omega_2)=\left\{f:f=\sum_{j\in\mathbb{Z}}\lambda_j a_j \text{ in } S'(\mathbb{R}^n) \text{ and } \sum_{j\in\mathbb{Z}}|\lambda_j|^p<\infty\right\},$$

where each a_j is a central $(\alpha,q,b,\omega_1,\omega_2)$-atom supported on $B_j=B(0,2^j)$. Moreover,

$$\|f\|_{H\dot{K}_{q,b}^{\alpha,p}(\omega_1,\omega_2)}\sim\inf\left\{\left(\sum_{j\in\mathbb{Z}}|\lambda_j|^p\right)^{1/p}\right\}.$$

Here the infimum is taken over all decompositions of f as above.

DEFINITION 7. Let $1\leqslant p<\infty$ and ω be a weighted function. A locally integrable function b is said to be in the weighted BMO space $BMO_p(\omega)$ if

$$\|b\|_{BMO_p(\omega)}=\sup_Q\left(\frac{1}{\omega(Q)}\int_Q|b(x)-b_Q|^p\omega(x)^{1-p}dx\right)^{\frac{1}{p}}<\infty,$$

where $b_Q=\dfrac{1}{|Q|}\int_Q b(y)dy, \omega(Q)=\int_Q\omega(y)dy$ and the supremum is taken over all cubes $Q\subset\mathbb{R}^n$. Moreover, we denote simply by $BMO(\omega)$ when $p=1$.

DEFINITION 8. Let $1\leqslant p<\infty$, $0<\beta<1$ and ω be a weighted function. A locally integrable function b is said to be in the weighted Lipschitz space $Lip_\beta^p(\omega)$ if

$$\|b\|_{Lip_\beta^p(\omega)}=\sup_Q\frac{1}{\omega(Q)^{\frac{\beta}{n}}}\left(\frac{1}{\omega(Q)}\int_Q|b(x)-b_Q|^p\omega(x)^{1-p}dx\right)^{\frac{1}{p}}<\infty,$$

where $b_Q=\frac{1}{|Q|}\int_Q b(y)dy, \omega(Q)=\int_Q\omega(y)dy$ and the supremum is taken over all cubes $Q\subset\mathbb{R}^n$. Moreover, we denote simply by $Lip_\beta(\omega)$ when $p=1$.

DEFINITION 9. The Hardy-Littlewood maximal operator M is defined by

$$M(f)(x)=\sup_{Q\ni x}\frac{1}{|Q|}\int_Q|f(y)|dy.$$

We set $M_s(f)=M(|f|^s)^{\frac{1}{s}}$, where $0<s<\infty$.

The sharp maximal operator $M^\sharp$ is defined by

$$M^\sharp(f)(x)=\sup_{Q\ni x}\frac{1}{|Q|}\int_Q|f(y)-f_Q|dy\sim\sup_{Q\ni x}\inf_{a\in\mathbb{C}}\frac{1}{|Q|}\int_Q|f(y)-a|dy.$$

We define the t-sharp maximal operator $M_t^\sharp(f)=M^\sharp(|f|^t)^{\frac{1}{t}}$, where $0<t<1$.

Let ω be a weight. The weighted maximal operator M_ω is defined by

$$M_\omega(f)(x)=\sup_{Q\ni x}\frac{1}{\omega(Q)}\int_Q|f(y)|\omega(y)dy.$$

We also set $M_{s,\omega}(f)=M_\omega(|f|^s)^{\frac{1}{s}}$, where $0<s<\infty$.

DEFINITION 10. ([3]) A weighted function ω belongs to the reverse Hölder class RH_r if there exists two constants $r>1$ and $C>0$ such that the following reverse Hölder inequality

$$\left(\frac{1}{|Q|}\int_Q\omega(x)^r dx\right)^{\frac{1}{r}}\leqslant C\left(\frac{1}{|Q|}\int_Q\omega(x)dx\right)$$

holds for every cube Q in $\mathbb{R}^n$. Denote $r_\omega=\sup\{r>1:\omega\in RH_r\}$.

DEFINITION 11. For $0<\alpha<n, 1\leqslant l<\infty$, the fractional maximal operator $M_{\alpha,l}$ is defined by

$$M_{\alpha,l}(f)(x)=\sup_{Q\ni x}\left(\frac{1}{|Q|^{1-\frac{\alpha l}{n}}}\int_Q|f(y)|^l dy\right)^{\frac{1}{l}},$$

where the supremum is taken over all cubes Q containing x.

DEFINITION 12. For $0<\beta<n, 1\leqslant r<\infty$ and a weight ω, the weighted fractional maximal operator $M_{\beta,r,\omega}$ is defined by

$$M_{\beta,r,\omega}(f)(x)=\sup_{Q\ni x}\left(\frac{1}{\omega(Q)^{1-\frac{\beta r}{n}}}\int_Q|f(y)|^r\omega(y)dy\right)^{\frac{1}{r}},$$

where the supremum is taken over all cubes Q containing x.

It follows from [18] that $M_{\beta,r,\omega}$ is bounded from $L^p(\omega)$ to $L^q(\omega)$, where $0<\beta<n$, $1<r<p<n/\beta$, $1/q=1/p-\beta/n$ and $\omega\in A_1$.

2. Main results

Firstly, we establish the boundedness of commutators of generalized Calderón-Zygmund operators on weighted Lebesgue spaces.

THEOREM 1. *Let T be a generalized Calderón-Zygmund operator, q_0' be the same as in Definition 1 and the sequence $\{jC_j\} \in l^1$. Suppose $q_0' < p < \infty$, $\omega \in A_1$, and $r_\omega > \frac{(p-1)q_0'}{p-q_0'}$. If $b \in BMO(\omega)$, then $[b,T]$ is bounded from $L^p(\omega)$ to $L^p(\omega^{1-p})$.*

THEOREM 2. *Let T be a generalized Calderón-Zygmund operator, q_0' be the same as in Definition 1 and the sequence $\{jC_j\} \in l^1$. Suppose $0 < \beta < \min\{1, n/q_0'\}$, $q_0' < p < n/\beta$, $1/q = 1/p - \beta/n$, $\omega^{q/p} \in A_1$ and $r_\omega > \frac{(p-1)q_0'}{p-q_0'}$. If $b \in Lip_\beta(\omega)$, then $[b,T]$ is bounded from $L^p(\omega)$ to $L^q(\omega^{1-q})$.*

Secondly, we can also obtain the boundedness of the commutators on weighted Herz-type Hardy spaces.

THEOREM 3. *Let T be a generalized Calderón-Zygmund operator, q_0' be the same as in Definition 1. Suppose $q_0' < q < q_0$, $0 < p < \infty$, $n(1-1/q) \leqslant \alpha < \infty$, $\varepsilon > \alpha + n/q$, $\omega \in A_1$, $r_\omega > \max\left\{\frac{(q-1)q_0'}{q-q_0'}, \frac{q_0}{q_0-q}\right\}$ and the sequence $\{C_j 2^{j\varepsilon}\} \in l^1$. If $b \in BMO(\omega)$, then $[b,T]$ is continuous from $H\dot{K}_{q,b}^{\alpha,p}(\omega,\omega)$ to $\dot{K}_q^{\alpha,p}(\omega,\omega^{1-q})$.*

THEOREM 4. *Let T be a generalized Calderón-Zygmund operator, q_0' be the same as in Definition 1. Suppose $0 < \beta < \min\{n(1-2/q_0), 1\}$, $q_0' < q_1 < \dfrac{1}{1/q_0 + \beta/n}$, $1/q_2 = 1/q_1 - \beta/n$, $\omega^{q_2/q_1} \in A_1$, $r_\omega > \max\left\{\frac{(q_1-1)q_0'}{q_1-q_0'}, \frac{q_0}{q_0-q_2}\right\}$, $0 < p < \infty$, $n(1-1/q_1) \leqslant \alpha < \infty$, $\varepsilon > \alpha + n/q_1$ and the sequence $\{C_j 2^{j\varepsilon}\} \in l^1$. If $b \in Lip_\beta(\omega)$, then $[b,T]$ is continuous from $H\dot{K}_{q_1,b}^{\alpha,p}(\omega,\omega)$ to $\dot{K}_{q_2}^{\alpha,p}(\omega,\omega^{1-q_2})$.*

REMARK 1. The corresponding results of Theorem 3 and Theorem 4 on non-homogeneous weighted Herz-type Hardy spaces can also be obtained. we omit the details since their similarity.

3. Preliminaries

Before giving the proof of our main results, we need some lemmas.

LEMMA 1. ([13]) *Let ω_1, $\omega_2 \in A_1$, $0 < p < \infty$, $1 < q < \infty$, $n(1-1/q) \leqslant \alpha < \infty$ and $s \geqslant [\alpha + n(1/q-1)]$. Then $f \in H\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$ if and only if $f(x) = \sum_{j\in\mathbb{Z}} \lambda_j a_j(x)$ in the sense of $S'(\mathbb{R}^n)$, where $\sum_{j\in\mathbb{Z}} |\lambda_j|^p < \infty$ with each a_j a central $(\alpha,q,\omega_1,\omega_2)$-atom supported on $B_j = B(0,2^j)$. Moreover,*

$$\|f\|_{H\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)} \sim \inf\left\{\left(\sum_{j\in\mathbb{Z}} |\lambda_j|^p\right)^{1/p}\right\},$$

where the infimum is taken over all the decompositions of f.

REMARK 2. By Definition 6 and Lemma 1, it is easy to see $H\dot{K}^{\alpha,p}_{q,b}(\omega_1,\omega_2) \subset H\dot{K}^{\alpha,p}_{q}(\omega_1,\omega_2)$.

LEMMA 2. ([16, 2]) *Let $\omega \in A_1$. Then for any $1 \leqslant p < \infty$, there exists an absolute constant $C > 0$ such that $\|b\|_{BMO_p(\omega)} \leqslant C\|b\|_{BMO(\omega)}$.*

LEMMA 3. ([16, 2]) *Let $\omega \in A_1$ and $0 < \beta < 1$. Then for any $1 \leqslant p < \infty$, there exists an absolute constant $C > 0$ such that $\|b\|_{Lip^p_\beta(\omega)} \leqslant C\|b\|_{Lip_\beta(\omega)}$.*

LEMMA 4. ([4]) *Let $\omega \in A_1$, then there are constants C_1, C_2 and $0 < \eta < 1$ depending only on A_1-constant of ω, such that for any measurable subset E of a ball B,*

$$C_1\frac{|E|}{|B|} \leqslant \frac{\omega(E)}{\omega(B)} \leqslant C_2\left(\frac{|E|}{|B|}\right)^{\eta}.$$

LEMMA 5. ([9]) *If $1 < p < \infty$ and $\omega \in A_\infty$, then for any $1 < s < p < \infty$,*

$$\|M_{s,\omega}(f)\|_{L^p(\omega)} \leqslant C\|f\|_{L^p(\omega)}.$$

LEMMA 6. ([8]) *Let T be a generalized Calderón-Zygmund operator, q_0' be the same as in Definition 1 and the sequence $\{C_j\} \in l^1$. If $0 < \delta < 1$ and $q_0' \leqslant s < \infty$, there exists a positive constant C such that*

$$M^\sharp_\delta(Tf)(x) \leqslant CM_s(f)(x), \quad x \in \mathbb{R}^n,$$

for every bounded and compactly supported function f.

LEMMA 7. *If $1 < r < p < \infty$ and $\omega \in A_{p/r}$, then,*

$$\|M_r(f)\|_{L^p(\omega)} \leqslant C\|f\|_{L^p(\omega)}.$$

Proof. Because of the $(L^{p/r}(\omega), L^{p/r}(\omega))$ bundedness of M, we have

$$\|M_r(f)\|_{L^p(\omega)} = \|M(|f|^r)\|^{1/r}_{L^{p/r}(\omega)} \leqslant C\||f|^r\|^{1/r}_{L^{p/r}(\omega)} = C\|f\|_{L^p(\omega)}. \quad \square$$

LEMMA 8. *Let T be a generalized Calderón-Zygmund operator, q_0' be the same as in Definition 1 and the sequence $\{C_j\} \in l^1$. Suppose $q_0' < p < \infty$ and $\omega \in A_{p/q_0'}$, then we have*

$$\|Tf\|_{L^p(\omega)} \leqslant C\|f\|_{L^p(\omega)}.$$

Proof. Since $q_0' < p < \infty$ and $\omega \in A_{p/q_0'}$, there exists an l such that $1 \leqslant l < p/q_0'$ and $\omega \in A_l$. So $q_0' < p/l \leqslant p$, and there exists an s such that $q_0' < s < p/l \leqslant p$. By Lemma 6 and Lemma 7, we can get

$$\|Tf\|_{L^p(\omega)} \leqslant \|M_\delta(Tf)\|_{L^p(\omega)} \leqslant C\|M_\delta^\sharp(Tf)\|_{L^p(\omega)} \leqslant C\|M_s(f)\|_{L^p(\omega)} \leqslant C\|f\|_{L^p(\omega)}. \qquad \square$$

LEMMA 9. ([5]) *If $0 < \alpha < n$, $1 < l < p < n/\alpha$, $1/q = 1/p - \alpha/n$ and $\omega \in A_1$, then there is a constant $C > 0$, independent of f, such that*

$$\left(\int_{\mathbb{R}^n} (M_{\alpha,l}f(x))^q \omega(x)dx\right)^{\frac{1}{q}} \leqslant C\left(\int_{\mathbb{R}^n} |f(x)|^p \omega(x)^{\frac{p}{q}} dx\right)^{\frac{1}{p}}.$$

LEMMA 10. ([8]) *Let T be a generalized Calderón-Zygmund operator, q_0' be the same as in Definition 1 and the sequence $\{jC_j\} \in l^1$. Let $0 < \delta < 1$, $\omega \in A_1 \cap RH_r$ with $r > q_0'$ and $b \in BMO(\omega)$, then for all $s > \dfrac{(r-1)q_0'}{r-q_0'}$ and a.e. $x \in \mathbb{R}^n$, we have*

$$M_\delta^\sharp([b,T]f)(x) \leqslant C\|b\|_{BMO(\omega)}(\omega(x)M_{s,\omega}(Tf)(x) + \omega(x)M_{s,\omega}(f)(x) + \omega(x)M_s(f)(x)).$$

LEMMA 11. ([7]) *Let T be a generalized Calderón-Zygmund operator, q_0' be the same as in Definition 1 and the sequence $\{jC_j\} \in l^1$. Let $0 < \delta < 1$, $0 < \beta < 1$, $\omega \in A_1$, $r_\omega > q_0'$ and $b \in Lip_\beta(\omega)$, then for all $s > \dfrac{(r_\omega-1)q_0'}{r_\omega-q_0'}$ and a.e. $x \in \mathbb{R}^n$, we have*

$$\begin{aligned}M_\delta^\sharp([b,T]f)(x) \leqslant C\|b\|_{Lip_\beta(\omega)}\big(&\omega(x)M_{\beta,s,\omega}(Tf)(x) + \omega(x)M_{\beta,s,\omega}(f)(x)\\ &+ \omega(x)^{1+\frac{\beta}{n}}M_{\beta,s}(f)(x)\big).\end{aligned}$$

4. Proof of main results

Now we are able to prove our main results.

Proof of Theorem 1. Since $r_\omega > \frac{(p-1)q_0'}{p-q_0'}$, there exists a r such that $r > \frac{(p-1)q_0'}{p-q_0'}$ and $\omega \in RH_r$. It follows from $r > \frac{(p-1)q_0'}{p-q_0'}$ that $p > \frac{(r-1)q_0'}{r-q_0'}$. Then we can choose an s such that $p > s > \frac{(r-1)q_0'}{r-q_0'}$. Since $\omega \in A_1$, then $\omega^{1-p} \in A_p$. Applying Lemma 10, Lemma 7, Lemma 5 and Lemma 8, we have

$$\begin{aligned}&\|[b,T](f)\|_{L^p(\omega^{1-p})}\\ &\leqslant \|M_\delta([b,T]f)\|_{L^p(\omega^{1-p})}\\ &\leqslant C\|M_\delta^\sharp([b,T]f)\|_{L^p(\omega^{1-p})}\\ &\leqslant C\|b\|_{BMO(\omega)}\|\omega M_{s,\omega}(Tf) + \omega M_{s,\omega}(f) + \omega M_s(f)\|_{L^p(\omega^{1-p})}\end{aligned}$$

$$\leqslant C\|b\|_{BMO(\omega)}(\|M_{s,\omega}(Tf))\|_{L^p(\omega)}+\|M_{s,\omega}(f))\|_{L^p(\omega)}+\|M_s(f)\|_{L^p(\omega)})$$
$$\leqslant C\|b\|_{BMO(\omega)}(\|Tf\|_{L^p(\omega)}+\|f\|_{L^p(\omega)})$$
$$\leqslant C\|b\|_{BMO(\omega)}\|f\|_{L^p(\omega)}.$$

This completes the proof of Theorem 1. □

Proof of Theorem 2. It follows from $r_\omega > \frac{(p-1)q_0'}{p-q_0'}$ that $p > \frac{(r_\omega-1)q_0'}{r_\omega-q_0'}$. Thus there exists a r such that $p > r > \frac{(r_\omega-1)q_0'}{r_\omega-q_0'}$. Since $\omega^{q/p} \in A_1$, then $\omega \in A_1$ and $\omega^{1-q} \in A_q$. By Lemma 11, we have

$$\begin{aligned}
&\|[b,T](f)\|_{L^q(\omega^{1-q})}\\
&\leqslant C\|M_\delta^\sharp([b,T]f)\|_{L^q(\omega^{1-q})}\\
&\leqslant C\|b\|_{Lip_\beta(\omega)}\|\omega M_{\beta,r,\omega}(Tf)+\omega M_{\beta,r,\omega}(f)+\omega^{1+\frac{\beta}{n}}M_{\beta,r}(f)\|_{L^q(\omega^{1-q})}\\
&\leqslant C\|b\|_{Lip_\beta(\omega)}(\|M_{\beta,r,\omega}(Tf)\|_{L^q(\omega)}+\|M_{\beta,r,\omega}(f)\|_{L^q(\omega)}+\|M_{\beta,r}(f)\|_{L^q(\omega^{\frac{q}{p}})}).
\end{aligned}$$

Since $M_{\beta,r,\omega}$ is bounded from $L^p(\omega)$ to $L^q(\omega)$, then by Lemma 9 and Lemma 8, we can get

$$\begin{aligned}
&\|[b,T](f)\|_{L^q(\omega^{1-q})}\\
&\leqslant C\|b\|_{Lip_\beta(\omega)}(\|Tf\|_{L^p(\omega)}+\|f\|_{L^p(\omega)})\\
&\leqslant C\|b\|_{Lip_\beta(\omega)}\|f\|_{L^p(\omega)},
\end{aligned}$$

which completes the proof of Theorem 2. □

Proof of Theorem 3. For $f \in H\dot{K}_{q,b}^{\alpha,p}(\omega,\omega)$, we can write $f=\sum_{k\in\mathbb{Z}}\lambda_k a_k$ with each a_k a $(\alpha,q,b,\omega,\omega)$-atom supported on $B_k = B(0,2^k)$ and $\sum_{k\in\mathbb{Z}}|\lambda_k|^p<\infty$. Then, $\|a_k\|_{L^q(\omega)} \leqslant \omega(B_k)^{-\frac{\alpha}{n}}$ and $\int_{\mathbb{R}^n} a_k(x)dx = \int_{\mathbb{R}^n} a_k(x)b(x)dx = 0$. Write

$$\begin{aligned}
&\|[b,T](f)\|^p_{\dot{K}_q^{\alpha,p}(\omega,\omega^{1-q})}\\
&\leqslant C\sum_{j\in\mathbb{Z}}\omega(B_j)^{\frac{\alpha p}{n}}\Big\|\sum_{k=-\infty}^{j-2}\lambda_k[b,T](a_k)\chi_j\Big\|^p_{L^q(\omega^{1-q})}\\
&\quad+C\sum_{j\in\mathbb{Z}}\omega(B_j)^{\frac{\alpha p}{n}}\Big\|\sum_{k=j-1}^{+\infty}\lambda_k[b,T](a_k)\chi_j\Big\|^p_{L^q(\omega^{1-q})}\\
&:=V_1+V_2.
\end{aligned}$$

Let us estimate V_2 first. Since $\varepsilon > \alpha+n/q$, $\alpha \geqslant n(1-1/q)$ and the sequence $\{C_j 2^{j\varepsilon}\}\in l^1$, then $2^{j\varepsilon} \geqslant 2^{jn} \geqslant j$ and the sequence $\{jC_j\} \in l^1$. It follows from Theorem 1 that $[b,T]$ is a continuous operator from $L^q(\omega)$ to $L^q(\omega^{1-q})$.

By Lemma 4, we have $\frac{\omega(B_j)}{\omega(B_k)} \leqslant C2^{(j-k)n\eta}$ for $k \geqslant j-1$. It follows from Theorem 1 and Hölder's inequality that

$$\begin{aligned}
V_2 &\leqslant C \sum_{j\in\mathbb{Z}} \omega(B_j)^{\frac{\alpha p}{n}} \left(\sum_{k=j-1}^{+\infty} |\lambda_k| \|[b,T](a_k)\|_{L^q(\omega^{1-q})} \right)^p \\
&\leqslant C\|b\|_{BMO(\omega)}^p \sum_{j\in\mathbb{Z}} \omega(B_j)^{\frac{\alpha p}{n}} \left(\sum_{k=j-1}^{+\infty} |\lambda_k| \omega(B_k)^{-\frac{\alpha}{n}} \right)^p \\
&\leqslant C\|b\|_{BMO(\omega)}^p \sum_{j\in\mathbb{Z}} \left(\sum_{k=j-1}^{+\infty} |\lambda_k| 2^{(j-k)\alpha\eta} \right)^p \\
&\leqslant \begin{cases} C\|b\|_{BMO(\omega)}^p \sum_{j\in\mathbb{Z}} \left(\sum_{k=j-1}^{+\infty} |\lambda_k|^p 2^{(j-k)\alpha\eta p} \right), & \text{if } 0 < p \leqslant 1 \\ C\|b\|_{BMO(\omega)}^p \sum_{j\in\mathbb{Z}} \left(\sum_{k=j-1}^{+\infty} |\lambda_k|^p 2^{(j-k)\alpha\eta} \right) \left(\sum_{k=j-1}^{+\infty} 2^{(j-k)\alpha\eta} \right)^{\frac{p}{p'}}, & \text{if } p > 1 \end{cases} \\
&\leqslant \begin{cases} C\|b\|_{BMO(\omega)}^p \sum_{k\in\mathbb{Z}} |\lambda_k|^p \sum_{j=-\infty}^{k+1} 2^{(j-k)\alpha\eta p}, & \text{if } 0 < p \leqslant 1 \\ C\|b\|_{BMO(\omega)}^p \sum_{k\in\mathbb{Z}} |\lambda_k|^p \sum_{j=-\infty}^{k+1} 2^{(j-k)\alpha\eta}, & \text{if } p > 1 \end{cases} \\
&\leqslant C\|b\|_{BMO(\omega)}^p \left(\sum_{k\in\mathbb{Z}} |\lambda_k|^p \right).
\end{aligned}$$

Now we estimate V_1. Observe that

$$[b,T](a_k)(x) = (b(x) - b_{B_j})Ta_k(x) + T((b_{B_j} - b)a_k)(x),$$

then,

$$\begin{aligned}
\|[b,T](a_k)\chi_j\|_{L^q(\omega^{1-q})} &\leqslant \|(b-b_{B_j})T(a_k)\chi_j\|_{L^q(\omega^{1-q})} + \|T((b_{B_j}-b)a_k)\chi_j\|_{L^q(\omega^{1-q})} \\
&:= G_1 + G_2.
\end{aligned}$$

By the cancellation condition of a_k and the Minkowski inequality, we then obtain

$$\begin{aligned}
G_1 &= \left(\int_{E_j} |b(x) - b_{B_j}|^q \left| \int_{B_k} (K(x,y) - K(x,0)) a_k(y) dy \right|^q \omega(x)^{1-q} dx \right)^{\frac{1}{q}} \\
&\leqslant \int_{B_k} |a_k(y)| \left(\int_{E_j} |b(x) - b_{B_j}|^q |K(x,y) - K(x,0)|^q \omega(x)^{1-q} dx \right)^{\frac{1}{q}} dy.
\end{aligned}$$

Since $r_\omega > \frac{q_0}{q_0-q}$, there exists an s such that $\omega \in RH_s$ and $s > \frac{q_0}{q_0-q}$. Let $r = (s-1)\frac{q_0}{q}$, then we can get $s = 1 + \frac{rq}{q_0} > \frac{q_0}{q_0-q}$. Furthermore, we have $r > \frac{q_0}{q_0-q} = (\frac{q_0}{q})'$. Since $\frac{1}{\frac{q_0}{q}} + \frac{1}{r} < 1$, there exists an $l > 1$ such that $\frac{1}{\frac{q_0}{q}} + \frac{1}{r} + \frac{1}{l} = 1$. It follows from $r(1-\frac{1}{l}) = \frac{rq}{q_0} + 1 = s$ that $\omega \in RH_s = RH_{r(1-\frac{1}{l})}$.

Thus, by Hölder's inequality for the three numbers $\frac{q_0}{q}, l$ and $r,$ we have

$$G_1 \leqslant \int_{B_k} |a_k(y)| \left(\int_{E_j} |b(x)-b_{B_j}|^q \omega(x)^{\frac{1}{l}-q} |K(x,y)-K(x,0)|^q \omega(x)^{1-\frac{1}{l}} dx \right)^{\frac{1}{q}} dy$$
$$\leqslant \int_{B_k} |a_k(y)| \left(\int_{E_j} |K(x,y)-K(x,0)|^{q_0} dx \right)^{\frac{1}{q_0}} \left(\int_{E_j} |b(x)-b_{B_j}|^{lq} \omega(x)^{1-lq} dx \right)^{\frac{1}{lq}}$$
$$\times \left(\int_{E_j} \omega(x)^{r(1-\frac{1}{l})} dx \right)^{\frac{1}{rq}} dy.$$

When $x \in E_j, y \in B_k$ and $k \leqslant j-2,$ we have $|x| \geqslant 2^{j-1} \geqslant 2^{k+1} \geqslant 2|y|.$ By the condition (3) of Definition 1 and the sequence $\{C_j 2^{j\varepsilon}\} \in l^1$, we have

$$\left(\int_{E_j} |K(x,y)-K(x,0)|^{q_0} |x|^{n\frac{q_0}{q_0'}+q_0\varepsilon} dx \right)^{\frac{1}{q_0}}$$
$$\leqslant \sum_{i=1}^{+\infty} \left(\int_{2^i|y| \leqslant |x| < 2^{i+1}|y|} |K(x,y)-K(x,0)|^{q_0} |x|^{n\frac{q_0}{q_0'}+q_0\varepsilon} dx \right)^{\frac{1}{q_0}}$$
$$\leqslant \sum_{i=1}^{+\infty} (2^{i+1}|y|)^{\frac{n}{q_0'}+\varepsilon} \left(\int_{2^i|y| \leqslant |x| < 2^{i+1}|y|} |K(x,y)-K(x,0)|^{q_0} dx \right)^{\frac{1}{q_0}}$$
$$\leqslant C \sum_{i=1}^{+\infty} (2^{i+1}|y|)^{\frac{n}{q_0'}+\varepsilon} C_i (2^i|y|)^{-\frac{n}{q_0'}}$$
$$\leqslant C|y|^{\varepsilon}.$$

Then,

$$\left(\int_{E_j} |K(x,y)-K(x,0)|^{q_0} dx \right)^{\frac{1}{q_0}}$$
$$\leqslant C 2^{-j(\frac{n}{q_0'}+\varepsilon)} \left(\int_{E_j} |K(x,y)-K(x,0)|^{q_0} |x|^{n\frac{q_0}{q_0'}+q_0\varepsilon} dx \right)^{\frac{1}{q_0}}$$
$$\leqslant C 2^{-j(\frac{n}{q_0'}+\varepsilon)} |y|^{\varepsilon}. \tag{1}$$

Using the definition of the weighted BMO function and Lemma 2, we can easily get

$$\left(\int_{B_j} |b(x)-b_{B_j}|^{lq} \omega(x)^{1-lq} dx \right)^{\frac{1}{lq}} \leqslant C\|b\|_{BMO(\omega)} \omega(B_j)^{\frac{1}{lq}}. \tag{2}$$

Since $\omega \in RH_{r(1-\frac{1}{l})},$ we have

$$\left(\frac{1}{|B_j|} \int_{B_j} \omega(x)^{r(1-\frac{1}{l})} dx \right)^{\frac{1}{r(1-\frac{1}{l})}} \leqslant C \frac{\omega(B_j)}{|B_j|}.$$

Then,

$$\left(\int_{B_j}\omega(x)^{r(1-\frac{1}{l})}dx\right)^{\frac{1}{rq}}\leqslant C\omega(B_j)^{\frac{1-\frac{1}{l}}{q}}|B_j|^{\frac{1}{rq}-\frac{1-\frac{1}{l}}{q}}. \tag{3}$$

Note that

$$\begin{aligned}\int_{B_k}|a_k(y)|dy &\leqslant C\left(\int_{B_k}|a_k(y)|^q\omega(y)dy\right)^{\frac{1}{q}}\left(\int_{B_k}\omega(y)^{-\frac{q'}{q}}dy\right)^{\frac{1}{q'}}\\ &\leqslant C|B_k|\omega(B_k)^{-\frac{1}{q}}\omega(B_k)^{-\frac{\alpha}{n}}. \end{aligned}\tag{4}$$

By (1)–(4) and $\frac{\omega(B_j)}{\omega(B_k)}\leqslant C2^{(j-k)n}$ for $k\leqslant j-2$, we can get

$$\begin{aligned}G_1 &\leqslant C\|b\|_{BMO(\omega)}2^{-j(\frac{n}{q_0'}+\varepsilon)}|B_j|^{\frac{1}{rq}-\frac{1}{q}+\frac{1}{lq}}\omega(B_j)^{\frac{1}{q}}\int_{B_k}|a_k(y)||y|^{\varepsilon}dy\\ &\leqslant C\|b\|_{BMO(\omega)}2^{-j(\frac{n}{q_0'}+\varepsilon)+k\varepsilon}|B_k|\omega(B_k)^{-\frac{1}{q}}\omega(B_k)^{-\frac{\alpha}{n}}\omega(B_j)^{\frac{1}{q}}|B_j|^{\frac{1}{rq}-\frac{1}{q}+\frac{1}{lq}}\\ &\leqslant C\|b\|_{BMO(\omega)}2^{-j(\frac{n}{q_0'}+\varepsilon)+k\varepsilon+kn+jn(\frac{1}{rq}-\frac{1}{q}+\frac{1}{lq})}\omega(B_k)^{-\frac{\alpha}{n}}\left(\frac{\omega(B_j)}{\omega(B_k)}\right)^{\frac{1}{q}}\\ &\leqslant C\|b\|_{BMO(\omega)}2^{-j(\frac{n}{q_0'}+\varepsilon)+k\varepsilon+kn+jn(\frac{1}{rq}-\frac{1}{q}+\frac{1}{lq})+(j-k)\frac{n}{q}}\omega(B_k)^{-\frac{\alpha}{n}}\\ &=C\|b\|_{BMO(\omega)}2^{(k-j)(\varepsilon+\frac{n}{q'})}\omega(B_k)^{-\frac{\alpha}{n}}. \end{aligned}\tag{5}$$

Next we estimate G_2.

By the cancellation condition of a_k and the Minkowski inequality, we can obtain that

$$\begin{aligned}G_2 &=\left(\int_{E_j}\left|\int_{\mathbb{R}^n}(K(x,y)-K(x,0))(b_{B_j}-b(y))a_k(y)dy\right|^q\omega(x)^{1-q}dx\right)^{\frac{1}{q}}\\ &\leqslant\left(\int_{E_j}\left|\int_{\mathbb{R}^n}(K(x,y)-K(x,0))(b_{B_j}-b_{B_k})a_k(y)dy\right|^q\omega(x)^{1-q}dx\right)^{\frac{1}{q}}\\ &\quad+\left(\int_{E_j}\left|\int_{\mathbb{R}^n}(K(x,y)-K(x,0))(b_{B_k}-b(y))a_k(y)dy\right|^q\omega(x)^{1-q}dx\right)^{\frac{1}{q}}\\ &:=G_{21}+G_{22}.\end{aligned}$$

Notice that $(\frac{q_0}{q})'=\frac{q_0}{q_0-q}$ and $(1-q)\frac{q_0}{q_0-q}=1-\frac{q(q_0-1)}{q_0-q}$. Let $u'=\frac{q(q_0-1)}{q_0-q}$. The fact $\frac{1}{u}+\frac{1}{u'}=1$ implies $u=\frac{q(q_0-1)}{q_0(q-1)}$. It follows from $\omega\in A_1$ that $\omega\in A_u$. Thus,

$$\left(\frac{1}{|B_j|}\int_{B_j}\omega(x)dx\right)\left(\frac{1}{|B_j|}\int_{B_j}\omega(x)^{1-u'}dx\right)^{u-1}\leqslant C.$$

Note that $u-1=\frac{q_0-q}{q_0(q-1)}$ and

$$\left(\frac{1}{|B_j|}\int_{B_j}\omega(x)^{1-u'}dx\right)^{u-1}\leqslant C\frac{|B_j|}{\omega(B_j)}.$$

We have

$$\left(\frac{1}{|B_j|}\int_{B_j}\omega(x)^{(1-q)(\frac{q_0}{q})'}dx\right)^{\frac{q_0-q}{q_0(q-1)}}\leqslant C\frac{|B_j|}{\omega(B_j)}.$$

Then,

$$\left(\int_{B_j}\omega(x)^{(1-q)(\frac{q_0}{q})'}dx\right)^{\frac{1}{q(\frac{q_0}{q})'}}\leqslant C|B_j|^{\frac{1}{q_0'}}\omega(B_j)^{-(1-\frac{1}{q})}. \tag{6}$$

Since

$$\begin{aligned}|b_{B_{i+1}}-b_{B_i}|&\leqslant\frac{1}{|B_i|}\left(\int_{B_i}|b(y)-b_{B_{i+1}}|^q\omega(y)^{1-q}dy\right)^{\frac{1}{q}}\left(\int_{B_i}\omega(y)^{-\frac{(1-q)q'}{q}}dy\right)^{\frac{1}{q'}}\\&\leqslant C\|b\|_{BMO(\omega)}\frac{\omega(B_{i+1})}{|B_i|},\end{aligned}$$

we have

$$\begin{aligned}|b_{B_j}-b_{B_k}|&\leqslant\sum_{i=k}^{j-1}|b_{B_{i+1}}-b_{B_i}|\\&\leqslant C\|b\|_{BMO(\omega)}(j-k)\omega(B_j)2^{(j-k)n}\frac{1}{|B_j|}.\end{aligned} \tag{7}$$

The fact $y\in B_k$ implies $|y|\leqslant 2^k$. By the estimate of (4), we have

$$\int_{B_k}|a_k(y)||y|^{\varepsilon}dy\leqslant C2^{k\varepsilon}|B_k|\omega(B_k)^{-\frac{1}{q}}\omega(B_k)^{-\frac{\alpha}{n}}. \tag{8}$$

By (1), (6)–(8) and $\frac{\omega(B_j)}{\omega(B_k)}\leqslant C2^{(j-k)n}$ for $k\leqslant j-2$, we get

$$\begin{aligned}G_{21}&\leqslant|b_{B_j}-b_{B_k}|\int_{B_k}|a_k(y)|\left(\int_{E_j}|K(x,y)-K(x,0)|^q\omega(x)^{1-q}dx\right)^{\frac{1}{q}}dy\\&\leqslant|b_{B_j}-b_{B_k}|\int_{B_k}|a_k(y)|\left(\int_{E_j}|K(x,y)-K(x,0)|^{q_0}dx\right)^{\frac{1}{q_0}}\\&\qquad\times\left(\int_{E_j}\omega(x)^{(1-q)(\frac{q_0}{q})'}dx\right)^{\frac{1}{q(\frac{q_0}{q})'}}dy\\&\leqslant C\|b\|_{BMO(\omega)}(j-k)\omega(B_j)2^{(j-k)n}\frac{1}{|B_j|}2^{-j(\frac{n}{q_0'}+\varepsilon)}\omega(B_j)^{-\frac{1}{q'}}|B_j|^{\frac{1}{q_0'}}\\&\qquad\times\int_{B_k}|a_k(y)||y|^{\varepsilon}dy\end{aligned}$$

$$
\begin{aligned}
&\leqslant C\|b\|_{BMO(\omega)}(j-k)2^{(k-j)\varepsilon}\left(\frac{\omega(B_j)}{\omega(B_k)}\right)^{\frac{1}{q}}\omega(B_k)^{-\frac{\alpha}{n}}\\
&\leqslant C\|b\|_{BMO(\omega)}(j-k)2^{(j-k)(\frac{n}{q}-\varepsilon)}\omega(B_k)^{-\frac{\alpha}{n}}. \qquad (9)
\end{aligned}
$$

Using $\frac{\omega(B_k)}{\omega(B_j)}\leqslant C2^{(k-j)n\eta}$ for $k\leqslant j-2$, (1), (6) and the Minkowski inequality, we obtain

$$
\begin{aligned}
G_{22}&\leqslant \int_{B_k}|b(y)-b_{B_k}||a_k(y)|\left(\int_{E_j}|K(x,y)-K(x,0)|^q\omega(x)^{1-q}dx\right)^{\frac{1}{q}}dy\\
&\leqslant \int_{B_k}|b(y)-b_{B_k}||a_k(y)|\left(\int_{E_j}|K(x,y)-K(x,0)|^{q_0}dx\right)^{\frac{1}{q_0}}\\
&\qquad\times\left(\int_{E_j}\omega(x)^{(1-q)(\frac{q_0}{q})'}dx\right)^{\frac{1}{q(\frac{q_0}{q})'}}dy\\
&\leqslant C\omega(B_j)^{-\frac{1}{q'}}|B_j|^{\frac{1}{q_0'}}2^{-j(\frac{n}{q_0'}+\varepsilon)}\int_{B_k}|b(y)-b_{B_k}||a_k(y)||y|^{\varepsilon}dy\\
&\leqslant C\omega(B_j)^{-\frac{1}{q'}}|B_j|^{\frac{1}{q_0'}}2^{(k-j)\varepsilon}2^{-j\frac{n}{q_0'}}\left(\int_{B_k}|b(y)-b_{B_k}|^{q'}\omega(y)^{-\frac{q'}{q}}dy\right)^{\frac{1}{q'}}\\
&\qquad\times\left(\int_{B_k}|a_k(y)|^q\omega(y)dy\right)^{\frac{1}{q}}\\
&\leqslant C\omega(B_j)^{-\frac{1}{q'}}|B_j|^{\frac{1}{q_0'}}2^{(k-j)\varepsilon}2^{-j\frac{n}{q_0'}}\left(\int_{B_k}|b(y)-b_{B_k}|^{q'}\omega(y)^{-\frac{q'}{q}}dy\right)^{\frac{1}{q'}}\omega(B_k)^{-\frac{\alpha}{n}}\\
&\leqslant C\|b\|_{BMO(\omega)}2^{(k-j)\varepsilon}\omega(B_k)^{-\frac{\alpha}{n}}\left(\frac{\omega(B_k)}{\omega(B_j)}\right)^{\frac{1}{q'}}\\
&\leqslant C\|b\|_{BMO(\omega)}2^{(k-j)(\varepsilon+\frac{n\eta}{q'})}\omega(B_k)^{-\frac{\alpha}{n}}. \qquad (10)
\end{aligned}
$$

Finally, by (5), (9) and (10), we have for $k\leqslant j-2$

$$
\begin{aligned}
&\|[b,T](a_k)\chi_j\|_{L^q(\omega^{1-q})}\\
&\leqslant C\|b\|_{BMO(\omega)}2^{(k-j)\varepsilon}\omega(B_k)^{-\frac{\alpha}{n}}\left(2^{(k-j)\frac{n}{q'}}+(j-k)2^{(j-k)\frac{n}{q}}+2^{(k-j)\frac{n\eta}{q'}}\right)\\
&\leqslant C\|b\|_{BMO(\omega)}\omega(B_k)^{-\frac{\alpha}{n}}(j-k)2^{(j-k)(\frac{n}{q}-\varepsilon)}.
\end{aligned}
$$

So we obtain the following estimate,

$$
V_1\leqslant C\|b\|^p_{BMO(\omega)}\sum_{j\in\mathbb{Z}}\omega(B_j)^{\frac{\alpha p}{n}}\left(\sum_{k=-\infty}^{j-2}|\lambda_k|(j-k)2^{(j-k)(\frac{n}{q}-\varepsilon)}\omega(B_k)^{-\frac{\alpha}{n}}\right)^p.
$$

It follows from $\alpha+\frac{n}{q}-\varepsilon<0$ that

$$
\begin{aligned}
V_1 &\leqslant C\|b\|_{BMO(\omega)}^p \sum_{j\in\mathbb{Z}}\Big(\sum_{k=-\infty}^{j-2}|\lambda_k|(j-k)2^{(j-k)(\alpha-\varepsilon+\frac{n}{q})}\Big)^p\\
&\leqslant \begin{cases} C\|b\|_{BMO(\omega)}^p \sum_{j\in\mathbb{Z}}\sum_{k=-\infty}^{j-2}|\lambda_k|^p(j-k)^p 2^{(j-k)(\alpha-\varepsilon+\frac{n}{q})p}, & \text{if } 0<p\leqslant 1\\ C\|b\|_{BMO(\omega)}^p \sum_{j\in\mathbb{Z}}\Big(\sum_{k=-\infty}^{j-2}|\lambda_k|^p(j-k)^p 2^{(j-k)(\alpha-\varepsilon+\frac{n}{q})}\Big) & \\ \qquad \times\Big(\sum_{k=-\infty}^{j-2}2^{(j-k)(\alpha-\varepsilon+\frac{n}{q})}\Big)^{\frac{p}{p'}}, & \text{if } p>1 \end{cases}\\
&\leqslant \begin{cases} C\|b\|_{BMO(\omega)}^p \sum_{k\in\mathbb{Z}}|\lambda_k|^p \sum_{j=k+2}^{+\infty}(j-k)^p 2^{(j-k)(\alpha-\varepsilon+\frac{n}{q})p}, & \text{if } 0<p\leqslant 1\\ C\|b\|_{BMO(\omega)}^p \sum_{k\in\mathbb{Z}}|\lambda_k|^p \sum_{j=k+2}^{+\infty}(j-k)^p 2^{(j-k)(\alpha-\varepsilon+\frac{n}{q})}, & \text{if } p>1 \end{cases}\\
&\leqslant C\|b\|_{BMO(\omega)}^p\Big(\sum_{k\in\mathbb{Z}}|\lambda_k|^p\Big).
\end{aligned}
$$

Combining V_1 and V_2, we have

$$\|[b,T](f)\|_{\dot{K}_q^{\alpha,p}(\omega,\omega^{1-q})}^p \leqslant C\|b\|_{BMO(\omega)}^p\Big(\sum_{k\in\mathbb{Z}}|\lambda_k|^p\Big).$$

By taking infimum over all decompositions of f, we get

$$\|[b,T](f)\|_{\dot{K}_q^{\alpha,p}(\omega,\omega^{1-q})} \leqslant C\|b\|_{BMO(\omega)}\|f\|_{H\dot{K}_{q,b}^{\alpha,p}(\omega,\omega)}.$$

This completes the proof of Theorem 3. □

Proof of Theorem 4. For $f\in H\dot{K}_{q_1,b}^{\alpha,p}(\omega,\omega)$, we can write $f=\sum_{k\in\mathbb{Z}}\lambda_k a_k$ with each a_k a $(\alpha,q_1,b,\omega,\omega)$-atom supported on $B_k=B(0,2^k)$ and $\sum_{k\in\mathbb{Z}}|\lambda_k|^p<\infty$. Then, $\|a_k\|_{L^{q_1}(\omega)}\leqslant\omega(B_k)^{-\frac{\alpha}{n}}$ and $\int_{\mathbb{R}^n}a_k(x)dx=\int_{\mathbb{R}^n}a_k(x)b(x)dx=0$. Write

$$
\begin{aligned}
&\|[b,T](f)\|_{\dot{K}_{q_2}^{\alpha,p}(\omega,\omega^{1-q_2})}^p\\
&\leqslant C\sum_{j\in\mathbb{Z}}\omega(B_j)^{\frac{\alpha p}{n}}\Big\|\sum_{k=-\infty}^{j-2}\lambda_k[b,T](a_k)\chi_j\Big\|_{L^{q_2}(\omega^{1-q_2})}^p\\
&\qquad + C\sum_{j\in\mathbb{Z}}\omega(B_j)^{\frac{\alpha p}{n}}\Big\|\sum_{k=j-1}^{+\infty}\lambda_k[b,T](a_k)\chi_j\Big\|_{L^{q_2}(\omega^{1-q_2})}^p\\
&:= H_1+H_2.
\end{aligned}
$$

Let us estimate H_2 firstly. Since $\varepsilon > \alpha + n/q_1,\ \alpha \geqslant n(1-1/q_1)$ and the sequence $\{C_j 2^{j\varepsilon}\} \in l^1$, then $2^{j\varepsilon} \geqslant 2^{jn} \geqslant j$ and the sequence $\{jC_j\} \in l^1$. It follows from Theorem 2 that $[b,T]$ is a continuous operator from $L^{q_1}(\omega)$ to $L^{q_2}(\omega^{1-q_2})$.

By Lemma 4, we have $\frac{\omega(B_j)}{\omega(B_k)} \leqslant C2^{(j-k)n\eta}$ for $k \geqslant j-1$. As the estimate of V_2 in the proof of Theorem 3, we have

$$
\begin{aligned}
H_2 &\leqslant C \sum_{j\in\mathbb{Z}} \omega(B_j)^{\frac{\alpha p}{n}} \left(\sum_{k=j-1}^{+\infty} |\lambda_k| \|[b,T](a_k)\|_{L^{q_2}(\omega^{1-q_2})} \right)^p \\
&\leqslant C\|b\|^p_{Lip_\beta(\omega)} \sum_{j\in\mathbb{Z}} \omega(B_j)^{\frac{\alpha p}{n}} \left(\sum_{k=j-1}^{+\infty} |\lambda_k| \|a_k\|_{L^{q_1}(\omega)} \right)^p \\
&\leqslant C\|b\|^p_{Lip_\beta(\omega)} \left(\sum_{k\in\mathbb{Z}} |\lambda_k|^p \right).
\end{aligned}
$$

Now we estimate H_1. Observe that

$$
[b,T](a_k)(x) = (b(x) - b_{B_j})Ta_k(x) + T((b_{B_j} - b)a_k)(x),
$$

then,

$$
\begin{aligned}
&\|[b,T](a_k)\chi_j\|_{L^{q_2}(\omega^{1-q_2})}(x) \\
&\leqslant \|(b-b_{B_j})T(a_k)\chi_j\|_{L^{q_2}(\omega^{1-q_2})} + \|T((b_{B_j}-b)a_k)\chi_j\|_{L^{q_2}(\omega^{1-q_2})} \\
&:= F_1 + F_2.
\end{aligned}
$$

By the cancellation condition of a_k and the Minkowski inequality, similar to estimate G_1, we obtain that

$$
F_1 \leqslant \int_{B_k} |a_k(y)| \left(\int_{E_j} |b(x) - b_{B_j}|^{q_2} |K(x,y) - K(x,0)|^{q_2} \omega(x)^{1-q_2} dx \right)^{\frac{1}{q_2}} dy.
$$

The fact $q_1 < \frac{1}{\frac{1}{q_0} + \frac{\beta}{n}}$ and $\frac{1}{q_2} = \frac{1}{q_1} - \frac{\beta}{n}$ implies $q_2 < q_0$. Since $r_\omega > \frac{q_0}{q_0 - q_2}$, there exists an s such that $\omega \in RH_s$ and $s > \frac{q_0}{q_0-q_2}$. Let $r = (s-1)\frac{q_0}{q_2}$, then we can get $s = 1 + \frac{rq_2}{q_0} > \frac{q_0}{q_0-q_2}$. Furthermore, we have $r > \frac{q_0}{q_0-q_2} = (\frac{q_0}{q_2})'$. Since $\frac{1}{\frac{q_0}{q_2}} + \frac{1}{r} < 1$, there exists an $l > 1$ such that $\frac{1}{\frac{q_0}{q_2}} + \frac{1}{r} + \frac{1}{l} = 1$. It follows from $r(1-\frac{1}{l}) = \frac{rq_2}{q_0} + 1 = s$ that $\omega \in RH_s = RH_{r(1-\frac{1}{l})}$.

By Hölder's inequality for the three numbers $\frac{q_0}{q_2}, l$ and r, it is easy to estimate F_1 similarly to G_1. Thus,

$$
\begin{aligned}
F_1 \leqslant \int_{B_k} |a_k(y)| &\left(\int_{E_j} |K(x,y) - K(x,0)|^{q_0} dx \right)^{\frac{1}{q_0}} \left(\int_{E_j} |b(x) - b_{B_j}|^{lq_2} \omega(x)^{1-lq_2} dx \right)^{\frac{1}{lq_2}} \\
&\times \left(\int_{E_j} \omega(x)^{r(1-\frac{1}{l})} dx \right)^{\frac{1}{rq_2}} dy.
\end{aligned}
$$

Using the definition of the weighted Lipschitz function, we can easily get

$$\left(\int_{B_j}|b(x)-b_{B_j}|^{lq_2}\omega(x)^{1-lq_2}dx\right)^{\frac{1}{lq_2}} \leqslant C\|b\|_{Lip_\beta(\omega)}\omega(B_j)^{\frac{1}{lq_2}+\frac{\beta}{n}}. \tag{11}$$

As in the estimate of (3), we can get

$$\left(\int_{B_j}\omega(x)^{r(1-\frac{1}{l})}dx\right)^{\frac{1}{rq_2}} \leqslant C\omega(B_j)^{\frac{1-\frac{1}{l}}{q_2}}|B_j|^{\frac{1}{rq_2}-\frac{1-\frac{1}{l}}{q_2}}. \tag{12}$$

Note that

$$\begin{aligned}\int_{B_k}|a_k(y)|dy &\leqslant C\left(\int_{B_k}|a_k(y)|^{q_1}\omega(y)dy\right)^{\frac{1}{q_1}}\left(\int_{B_k}\omega(y)^{-\frac{q_1'}{q_1}}dy\right)^{\frac{1}{q_1'}}\\ &\leqslant C|B_k|\omega(B_k)^{-\frac{1}{q_1}}\omega(B_k)^{-\frac{\alpha}{n}}.\end{aligned} \tag{13}$$

By (1), (11)–(13) and $\frac{\omega(B_j)}{\omega(B_k)} \leqslant C2^{(j-k)n}$ for $k \leqslant j-2$, we have

$$\begin{aligned}F_1 &\leqslant C\|b\|_{Lip_\beta(\omega)}2^{-j(\frac{n}{q_0'}+\varepsilon)}\omega(B_j)^{\frac{1}{q_1}}|B_j|^{\frac{1}{rq_2}-\frac{1}{q_2}+\frac{1}{lq_2}}\int_{B_k}|a_k(y)||y|^{\varepsilon}dy\\ &\leqslant C\|b\|_{Lip_\beta(\omega)}2^{-j(\frac{n}{q_0'}+\varepsilon)+k\varepsilon+kn+jn(\frac{1}{rq_2}-\frac{1}{q_2}+\frac{1}{lq_2})}\omega(B_k)^{-\frac{\alpha}{n}}\left(\frac{\omega(B_j)}{\omega(B_k)}\right)^{\frac{1}{q_1}}\\ &\leqslant C\|b\|_{Lip_\beta(\omega)}2^{(k-j)(\varepsilon+\frac{n}{q_1'})}\omega(B_k)^{-\frac{\alpha}{n}}.\end{aligned} \tag{14}$$

Next we will estimate F_2.

By the cancellation condition of a_k and the Minkowski inequality, we can obtain that

$$\begin{aligned}F_2 &= \left(\int_{E_j}\left|\int_{\mathbb{R}^n}(K(x,y)-K(x,0))(b_{B_j}-b(y))a_k(y)dy\right|^{q_2}\omega(x)^{1-q_2}dx\right)^{\frac{1}{q_2}}\\ &\leqslant \left(\int_{E_j}\left|\int_{\mathbb{R}^n}(K(x,y)-K(x,0))(b_{B_j}-b_{B_k})a_k(y)dy\right|^{q_2}\omega(x)^{1-q_2}dx\right)^{\frac{1}{q_2}}\\ &\quad+\left(\int_{E_j}\left|\int_{\mathbb{R}^n}(K(x,y)-K(x,0))(b_{B_k}-b(y))a_k(y)dy\right|^{q_2}\omega(x)^{1-q_2}dx\right)^{\frac{1}{q_2}}\\ &:= F_{21}+F_{22}.\end{aligned}$$

Notice that $(\frac{q_0}{q_2})' = \frac{q_0}{q_0-q_2}$ and $(1-q_2)\frac{q_0}{q_0-q_2} = 1-\frac{q_2(q_0-1)}{q_0-q_2}$. Let $v' = \frac{q_2(q_0-1)}{q_0-q_2}$. The fact $\frac{1}{v}+\frac{1}{v'}=1$ implies $v = \frac{q_2(q_0-1)}{q_0(q_2-1)}$. It follows from $\omega \in A_1$ that $\omega \in A_v$.

As in the estimate of (6), we have

$$\left(\int_{B_j}\omega(x)^{(1-q_2)(\frac{q_0}{q_2})'}dx\right)^{\frac{1}{q_2(\frac{q_0}{q_2})'}} \leqslant C|B_j|^{\frac{1}{q_0'}}\omega(B_j)^{-(1-\frac{1}{q_2})}. \tag{15}$$

Since

$$|b_{B_{i+1}}-b_{B_i}|\leqslant\frac{1}{|B_i|}\left(\int_{B_i}|b(y)-b_{B_{i+1}}|^{q_2}\omega(y)^{1-q_2}dy\right)^{\frac{1}{q_2}}\left(\int_{B_i}\omega(y)^{-\frac{(1-q_2)q_2'}{q_2}}dy\right)^{\frac{1}{q_2'}}$$
$$\leqslant C\|b\|_{Lip_\beta(\omega)}\frac{\omega(B_{i+1})^{1+\frac{\beta}{n}}}{|B_i|},$$

we have

$$|b_{B_j}-b_{B_k}|\leqslant\sum_{i=k}^{j-1}|b_{B_{i+1}}-b_{B_i}|$$
$$\leqslant C\|b\|_{Lip_\beta(\omega)}(j-k)\omega(B_j)^{1+\frac{\beta}{n}}2^{(j-k)n}\frac{1}{|B_j|}. \tag{16}$$

The fact $y\in B_k$ implies $|y|\leqslant 2^k$. Using (13), we have

$$\int_{B_k}|a_k(y)||y|^{\varepsilon}dy\leqslant C2^{k\varepsilon}|B_k|\omega(B_k)^{-\frac{1}{q_1}}\omega(B_k)^{-\frac{\alpha}{n}}. \tag{17}$$

By (1), (15)–(17) and $\frac{\omega(B_j)}{\omega(B_k)}\leqslant C2^{(j-k)n}$ for $k\leqslant j-2$, similar to (9), we get

$$F_{21}\leqslant C\|b\|_{Lip_\beta(\omega)}(j-k)\omega(B_j)^{1+\frac{\beta}{n}}2^{(j-k)n}\frac{1}{|B_j|}2^{-j(\frac{n}{q_0'}+\varepsilon)}\omega(B_j)^{-\frac{1}{q_2'}}|B_j|^{\frac{1}{q_0'}}$$
$$\times\int_{B_k}|a_k(y)||y|^{\varepsilon}dy$$
$$\leqslant C\|b\|_{Lip_\beta(\omega)}(j-k)2^{(j-k)(\frac{n}{q_1}-\varepsilon)}\omega(B_k)^{-\frac{\alpha}{n}}. \tag{18}$$

It follows from (1), (15) and Hölder's inequality that

$$\left(\int_{E_j}|K(x,y)-K(x,0)|^{q_2}\omega(x)^{1-q_2}dx\right)^{\frac{1}{q_2}}$$
$$\leqslant\left(\int_{E_j}|K(x,y)-K(x,0)|^{q_0}dx\right)^{\frac{1}{q_0}}\left(\int_{E_j}\omega(x)^{(1-q_2)(\frac{q_0}{q_2})'}dx\right)^{\frac{1}{q_2(\frac{q_0}{q_2})'}}$$
$$\leqslant C\omega(B_j)^{-\frac{1}{q_2'}}|B_j|^{\frac{1}{q_0'}}2^{-j(\frac{n}{q_0'}+\varepsilon)}|y|^{\varepsilon}. \tag{19}$$

By $\frac{\omega(B_k)}{\omega(B_j)}\leqslant C2^{(k-j)n\eta}$ for $k\leqslant j-2$, (19) and the Minkowski inequality, we obtain

$$F_{22}\leqslant\int_{B_k}|b(y)-b_{B_k}||a_k(y)|\left(\int_{E_j}|K(x,y)-K(x,0)|^{q_2}\omega(x)^{1-q_2}dx\right)^{\frac{1}{q_2}}dy$$
$$\leqslant C\omega(B_j)^{-\frac{1}{q_2'}}|B_j|^{\frac{1}{q_0'}}2^{-j(\frac{n}{q_0'}+\varepsilon)}\int_{B_k}|b(y)-b_{B_k}||a_k(y)||y|^{\varepsilon}dy$$

$$\begin{aligned}
&\leqslant C\omega(B_j)^{-\frac{1}{q_2'}}|B_j|^{\frac{1}{q_0'}}2^{(k-j)\varepsilon}2^{-j\frac{n}{q_0'}}\left(\int_{B_k}|b(y)-b_{B_k}|^{q_1'}\omega(y)^{-\frac{q_1'}{q_1}}dy\right)^{\frac{1}{q_1'}}\\
&\qquad\times\left(\int_{B_k}|a_k(y)|^{q_1}\omega(y)dy\right)^{\frac{1}{q_1}}\\
&\leqslant C\|b\|_{Lip_\beta(\omega)}2^{(k-j)\varepsilon}\omega(B_k)^{-\frac{\alpha}{n}}\left(\frac{\omega(B_k)}{\omega(B_j)}\right)^{\frac{1}{q_2'}}\\
&\leqslant C\|b\|_{Lip_\beta(\omega)}2^{(k-j)(\varepsilon+\frac{n\eta}{q_2'})}\omega(B_k)^{-\frac{\alpha}{n}}. \qquad (20)
\end{aligned}$$

Finally, by (14), (18) and (20), we have

$$\begin{aligned}
&\|[b,T](a_k)\chi_j\|_{L^{q_2}(\omega^{1-q_2})}\\
&\leqslant C\|b\|_{Lip_\beta(\omega)}2^{(k-j)\varepsilon}\omega(B_k)^{-\frac{\alpha}{n}}\left(2^{(k-j)\frac{n}{q_1'}}+(j-k)2^{(j-k)\frac{n}{q_1}}+2^{(k-j)\frac{n\eta}{q_2'}}\right)\\
&\leqslant C\|b\|_{Lip_\beta(\omega)}(j-k)2^{(j-k)(\frac{n}{q_1}-\varepsilon)}\omega(B_k)^{-\frac{\alpha}{n}}.
\end{aligned}$$

So we obtain the following estimate

$$\begin{aligned}
H_1 &\leqslant C\sum_{j\in\mathbb{Z}}\omega(B_j)^{\frac{\alpha p}{n}}\left(\sum_{k=-\infty}^{j-2}|\lambda_k|\|[b,T](a_k)\chi_j\|_{L^{q_2}(\omega^{1-q_2})}\right)^p\\
&\leqslant C\|b\|^p_{Lip_\beta(\omega)}\sum_{j\in\mathbb{Z}}\omega(B_j)^{\frac{\alpha p}{n}}\left(\sum_{k=-\infty}^{j-2}|\lambda_k|(j-k)2^{(j-k)(\frac{n}{q_1}-\varepsilon)}\omega(B_k)^{-\frac{\alpha}{n}}\right)^p.
\end{aligned}$$

Similarly to estimate V_1 in the proof of Theorem 3, it follows from $\alpha+\frac{n}{q_1}-\varepsilon<0$ that

$$H_1 \leqslant C\|b\|^p_{Lip_\beta(\omega)}\left(\sum_{k\in\mathbb{Z}}|\lambda_k|^p\right).$$

Combining H_1 and H_2, we have

$$\|[b,T](f)\|^p_{\dot{K}^{\alpha,p}_{q_2}(\omega,\omega^{1-q_2})} \leqslant C\|b\|^p_{Lip_\beta(\omega)}\left(\sum_{k\in\mathbb{Z}}|\lambda_k|^p\right).$$

By taking infimum over all decompositions of f, we get

$$\|[b,T](f)\|_{\dot{K}^{\alpha,p}_{q_2}(\omega,\omega^{1-q_2})} \leqslant C\|b\|_{Lip_\beta(\omega)}\|f\|_{H\dot{K}^{\alpha,p}_{q_1,b}(\omega,\omega)}.$$

This completes the proof of Theorem 4. □

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