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# THE DAVIS–GUT LAW AND LAI LAW FOR FINITELY INHOMOGENEOUS WALKS

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*Abstract.* Let $\{X_n, \geqslant 1\}$ be a sequence of independent and identically distributed random variables with partial sums $S_n = \sum_{k=1}^{n} X_k$, $n \geqslant 1$. Davis-Gut law states that

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left\{|S_n| > (1+\varepsilon)\sqrt{2n\log\log n}\right\} \begin{cases} < \infty, & \text{if } \varepsilon > 0, \\ = \infty, & \text{if } \varepsilon < 0 \end{cases}$$

if and only if $EX_1 = 0$ and $EX_1^2 = 1$. Lai law states that

$$\sum_{n=1}^{\infty} n^{r-1} P\{|S_n| > (1+\varepsilon)\sqrt{2rn\log n}\} \begin{cases} < \infty, & \text{if } \varepsilon > 0, \\ = \infty, & \text{if } \varepsilon < 0 \end{cases}$$

if and only if $EX_1 = 0$, $EX_1^2 = 1$ and $E(X_1^2/\log|X_1|)^{r+1} < \infty$, where $r > 0$. The paper will extend those results to the case where $\{X_n, n \geqslant 1\}$ are no longer identically distributed, but rather their distributions come from a finite set of distributions.

## 1. Introduction and the main result

The classical Hartman-Wintner law of the iterated logarithm (see Hartman and Wintner [6]) states that for $\{X_n, n \geqslant 1\}$, a sequence of independent and identically distributed random variables, if

$$EX_1 = 0 \ \text{ and } \ EX_1^2 = 1, \tag{1.1}$$

then

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n\log\log n}} = 1 \ \text{ a.s. and } \ \liminf_{n\to\infty} \frac{S_n}{\sqrt{2n\log\log n}} = -1 \ \text{ a.s.,} \tag{1.2}$$

where $S_n = \sum_{k=1}^{n} X_k$, $n \geqslant 1$, $\log x = \log_e \max\{x, e\}$ for $x > 0$. The converse, the implication (1.2)$\Rightarrow$(1.1), was proved by Strassen [15].

The following theorem, related to the Hartman-Wintner law of the iterated logarithm, is well-known.

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THEOREM A. *Let $\{X_n, n \geqslant 1\}$ be a sequence of independent and identically distributed random variables with partial sums $S_n = \sum_{k=1}^n X_k$, $n \geqslant 1$. The following statements are equivalent*

$$(1.1) \text{ holds,}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left\{|S_n| > (1+\varepsilon)\sqrt{2n\log\log n}\right\} \begin{cases} < \infty, & \text{if } \varepsilon > 0, \\ = \infty, & \text{if } \varepsilon < 0. \end{cases} \tag{1.3}$$

One can call this result the Davis-Gut law. The implication (1.1)⇒(1.3) should be due to Theorem 4 of Davis [4] which was remedied by Corollary 2.3 of Li et al. [10]. For the implication (1.3)⇒(1.1), see Theorem 6.2 of Gut [5]. The sufficient part of Theorem A for the moving processes, i.e. (1.1)⇒(1.3), is obtained by Chen and Wang [2].

The following theorem, related to the law of single logarithm, is also well-known.

THEOREM B. *Let $r > 0$ and $\{X_n, n \geqslant 1\}$ be a sequence of independent and identically distributed random variables with partial sums $S_n = \sum_{k=1}^n X_k$, $n \geqslant 1$. Suppose that*

$$EX_1 = 0, \ EX_1^2 = 1 \text{ and } E(X_1^2/\log|X_1|)^{r+1} < \infty. \tag{1.4}$$

*Then*

$$\sum_{n=1}^{\infty} n^{r-1} P\{|S_n| > (1+\varepsilon)\sqrt{2rn\log n}\} < \infty, \ \text{ for all } \ \varepsilon > 0. \tag{1.5}$$

*Conversely, if (1.5) holds for some $\varepsilon > -1$, then $EX_1 = 0$ and $E(X_1^2/\log|X_1|)^{r+1} < \infty$.*

One can label this result as the Lai law which first established by Lai [9]. Chen and Wang [2] extended it to the moving processes partly, and furthermore showed that

$$\sum_{n=1}^{\infty} n^{r-1} P\{|S_n| > (1+\varepsilon)\sqrt{2rn\log n}\} = \infty, \ \text{ for all } \ \varepsilon < 0.$$

Combining the results of Lai [9] and Chen and Wang [2], we have

$$\sum_{n=1}^{\infty} n^{r-1} P\{|S_n| > (1+\varepsilon)\sqrt{2rn\log n}\} \begin{cases} < \infty, & \text{if } \varepsilon > 0, \\ = \infty, & \text{if } \varepsilon < 0 \end{cases} \tag{1.6}$$

if and only if (1.4) holds.

When $r = 0$, an analog of (1.6) is discussed by Chen and Qi [1].

Recently, Spătaru [14] obtained the classical Hartman-Wintner law of the iterated logarithm for finitely inhomogeneous walks. The term *finitely inhomogeneous walk* designs a sequence of sums $S_n = \sum_{k=1}^n X_k, n \geqslant 1$, where the steps $X_n$, $n \geqslant 1$, are independent random variables having a finite number of possible distributions. This setting arises naturally in the study of some type Galton-Watson process, and was proposed by Kesten and Lawler [8]. Due to the work of Spătaru [14], the purpose of this paper is to generalize Theorem A and Theorem B to the finitely inhomogeneous walks.

In the following, we always assume that $\{X_n, n \geqslant 1\}$ is a sequence of independent random variables having a finite number of possible distributions as $Y_1, \cdots, Y_p$, $p \geqslant 1$, set $S_n = \sum_{k=1}^n X_k$, $n \geqslant 1$. Some lemmas and the proofs of the main results will be presented in the next section.

THEOREM 1.1. *Assume that* $EY_i = 0$ *and* $EY_i^2 = 1$ *for* $1 \leqslant i \leqslant p$. *Then (1.3) holds.*

THEOREM 1.2. *Let* $r > 0$. *Assume that* $EY_i = 0$, $EY_i^2 = 1$ *and* $E(Y_i^2/\log|Y_i|)^{r+1} < \infty$ *for* $1 \leqslant i \leqslant p$. *Then (1.6) holds.*

THEOREM 1.3. *Let* $\{X_{nk}, 1 \leqslant k \leqslant, n \geqslant 1\}$ *be an array of independent random variables having a finite number of possible distributions as* $Y_1, \cdots, Y_p$, $p \geqslant 1$, *set* $S_{nn} = \sum_{k=1}^{n} X_{nk}$, $n \geqslant 1$. *Assume that* $EY_i = 0$, $EY_i^2 = 1$ *and* $E(Y_i^2/\log|Y_i|)^2 < \infty$ *for* $1 \leqslant i \leqslant p$. *Then*

$$\limsup_{n\to\infty} \frac{|S_{nn}|}{\sqrt{2n\log n}} = 1 \text{ a.s.} \tag{1.7}$$

Throughout this paper, $C$ always stands for a positive constant which may differ from one place to another and $I(A)$ denotes the indicator function of the event $A$.

## 2. Lemmas and proofs of main results

The main idea in the proof of the main result is from the invariance principle' way to estimate the rate of convergence (see Sakhanenko [11, 12, 13]), which is a powerful tool in the field of limit theory (for example, see Csörgo, Szyszkowicz and Wu [3], Jiang and Zhang [7], Chen and Wang [2], etc.) and is listed as the following lemma.

LEMMA 2.1. *For any* $q > 2$, *there exists* $B = B(q) > 0$ *satisfying that for any sequence of independent random variables* $\{\xi_k, 1 \leqslant k \leqslant n\}$ *with mean zero and* $E|\xi_k|^q < \infty, 1 \leqslant k \leqslant n$, *there is a sequence of independent normal random variables* $\{\eta_k, 1 \leqslant k \leqslant n\}$ *with* $E\eta_k = 0$, $E\eta_k^2 = E\xi_k^2$ *and for all* $y > 0$,

$$P\left\{\max_{1\leqslant m\leqslant n} \left|\sum_{k=1}^{m} \xi_k - \sum_{k=1}^{m} \eta_k\right| > y\right\} \leqslant By^{-q} \sum_{k=1}^{n} E|\xi_k|^q. \tag{2.1}$$

The following two lemmas are well-known.

LEMMA 2.2. *Let* $Y$ *be a random variable with* $EY^2 < \infty$. *Then*

$$\sum_{n=1}^{\infty} P\{|Y| > \sqrt{n}\} \leqslant CEY^2 < \infty$$

*and*

$$\sum_{n=1}^{\infty} n^{-q/2} E|Y|^q I(|Y| \leqslant \sqrt{n}) \leqslant CE|Y|^2 < \infty,$$

*for any* $q > 2$.

LEMMA 2.3. *Let* $Y$ *be a random variable with* $E(Y^2/\log|Y|)^{r+1} < \infty$ *for some* $r > 0$. *Then*

$$\sum_{n=1}^{\infty} n^r P\{|Y| > \sqrt{n\log n}\} \leqslant CE(Y^2/\log|Y|)^{r+1} < \infty$$

*and*

$$\sum_{n=1}^{\infty} n^r (n\log n)^{-q/2} E|Y|^q I(|Y| \leqslant \sqrt{n\log n}) \leqslant CE(Y^2/\log|Y|)^{r+1} < \infty$$

*for any* $q > 2(r+1)$.

*Proof of Theorem 1.1.* Set $a_n = \sqrt{2n\log\log n}$, $X_{nk} = X_k I(|X_k| \leqslant \sqrt{n})$ and $T_n = \sum_{k=1}^{n} X_{nk}$. We first prove that

$$\sum_{n=1}^{\infty} \frac{1}{n} P\{|S_n| > (1+\varepsilon)a_n\} < \infty, \ \ \forall\, \varepsilon > 0. \tag{2.2}$$

Note that

$$\{|S_n| > (1+\varepsilon)a_n\} \subset \{\max_{1\leqslant k\leqslant n} |X_k| > \sqrt{n}\} \cup \{|T_n| > (1+\varepsilon)a_n\}$$

and by Lemma 2.2,

$$\sum_{n=1}^{\infty} \frac{1}{n} P\{\max_{1\leqslant k\leqslant n} |X_k| > \sqrt{n}\} \leqslant \sum_{i=1}^{p}\sum_{n=1}^{\infty} P\{|Y_i| > \sqrt{n}\} \leqslant C\sum_{i=1}^{p} EY_i^2 < \infty \tag{2.3}$$

and

$$\frac{1}{a_n}|ET_n| \leqslant \frac{1}{a_n}\sum_{k=1}^{n} E|X_k| I(|X_k| > \sqrt{n}) \leqslant \frac{1}{\sqrt{2\log\log n}} \to 0 \tag{2.4}$$

as $n \to \infty$. Hence to prove (2.2), it is enough to prove that

$$\sum_{n=1}^{\infty} \frac{1}{n} P\{|T_n - ET_n| > (1+\varepsilon)a_n\} < \infty, \ \ \forall\, \varepsilon > 0. \tag{2.5}$$

We can get from Lemma 2.1 that for any $n \geqslant 1$, there exists normal random variables $Z_{nk}$ with $EZ_{nk} = 0$ and $EZ_{nk}^2 = E(X_{nk} - EX_{nk})^2$, $1 \leqslant k \leqslant n$, such that for any $q > 2$ and all $y > 0$

$$P\left\{|(T_n - ET_n) - \sum_{k=1}^{n} Z_{nk}| > y\right\} \leqslant Ay^{-q} \sum_{k=1}^{n} E|X_{nk} - EX_{nk}|^q. \tag{2.6}$$

Note that

$$\{|T_n - ET_n| > (1+\varepsilon)a_n\} \subset \left\{|(T_n - ET_n) - \sum_{k=1}^{k} Z_{nk}| > \varepsilon_1 a_n\right\} \cup \left\{|\sum_{k=1}^{n} Z_{nk}| > (1+\varepsilon_2)a_n\right\}$$

where $\varepsilon_1 > 0, \varepsilon_1 > 0$ with $\varepsilon = \varepsilon_1 + \varepsilon_2$. Hence

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{n} P\{|T_n - ET_n| > (1+\varepsilon)a_n\} \\
&\leqslant \sum_{n=1}^{\infty} \frac{1}{n} P\left\{|(T_n - ET_n) - \sum_{k=1}^{n} Z_{nk}| > \varepsilon_1 a_n\right\} + \sum_{n=1}^{\infty} \frac{1}{n} P\left\{|\sum_{k=1}^{n} Z_{nk}| > (1+\varepsilon_2)a_n\right\} \\
&= I_1 + I_2.
\end{aligned}$$

By (2.6) and Lemma 2.2 we can derive that for $q > 2$

$$\begin{aligned}
I_1 &\leqslant C \sum_{n=1}^{\infty} \frac{1}{n a_n^q} \sum_{k=1}^{n} E|X_{nk} - EX_{nk}|^q \\
&\leqslant C \sum_{n=1}^{\infty} \frac{1}{n a_n^q} \sum_{k=1}^{n} E|X_k|^q I(|X_k| \leqslant \sqrt{n}) \\
&\leqslant C \sum_{i=1}^{p} \sum_{n=1}^{\infty} \frac{1}{a_n^q} E|Y_i|^q I(|Y_i| \leqslant \sqrt{n}) \\
&\leqslant C \sum_{i=1}^{p} EY_i^2 < \infty.
\end{aligned}$$

Let $N$ be a standard normal random variable. Note that $E(X_{nk} - EX_{nk})^2 \leqslant 1$ for all $1 \leqslant k \leqslant n$ and $n \geqslant 1$, and $P\{|N| > x\} \sim \sqrt{2/\pi}\, x^{-1} e^{-x^2/2}$. Hence for large enough $n$,

$$\begin{aligned}
P\left\{|\sum_{k=1}^{n} Z_{nk}| > (1+\varepsilon_2)a_n\right\} &= P\left\{|N| > \frac{(1+\varepsilon_2)a_n}{\sqrt{\sum_{k=1}^{n} E(X_{nk} - EX_{nk})^2}}\right\} \\
&\leqslant C \exp\left\{-\frac{(1+\varepsilon_2)^2 n \log\log n}{\sum_{k=1}^{n} E(X_{nk} - EX_{nk})^2}\right\} \\
&\leqslant C \exp\left\{-(1+\varepsilon_2)^2 \log\log n\right\} \\
&= C(\log n)^{-t_1},
\end{aligned}$$

where $t_1 = (1+\varepsilon_2)^2 > 1$. Then $I_2 < \infty$.

Now we prove that

$$\sum_{n=1}^{\infty} \frac{1}{n} P\{|S_n| > (1+\varepsilon)a_n\} = \infty, \quad \forall\, \varepsilon < 0. \tag{2.7}$$

By (2.3) and (2.4), (2.7) is equivalent to

$$\sum_{n=1}^{\infty} \frac{1}{n} P\{|T_n - ET_n| > (1+\varepsilon)a_n\} = \infty, \quad \forall\, \varepsilon < 0. \tag{2.8}$$

Note that for any $\varepsilon < 0$, if we take $\varepsilon_3 > 0$ and $\varepsilon_4 < 0$ with $\varepsilon_4 = \varepsilon_3 + \varepsilon$, then

$$\begin{aligned}
\left\{|\sum_{k=1}^{n} Z_{nk}| > (1+\varepsilon_4)a_n\right\} \subset &\left\{|(T_n - ET_n) - \sum_{k=1}^{n} Z_{nk}| > \varepsilon_3 a_n\right\} \\
&\cup \{|T_n - ET_n| > (1+\varepsilon)a_n\},
\end{aligned}$$

Then by $I_1 < \infty$, it is enough to show that

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left\{ |\sum_{k=1}^{n} Z_{nk}| > (1+\varepsilon_4)a_n \right\} = \infty. \tag{2.9}$$

Since $n^{-1}\sum_{k=1}^{n} E(X_{nk} - EX_{nk})^2 \to 1$ as $n \to \infty$, then there exists $\delta < 1$ close to 1 enough with $t_2 = (1+\varepsilon_4)^2/\delta < 1$ such that $\sum_{k=1}^{n} E(X_{nk} - EX_{nk})^2 \geqslant \delta n$ for $n$ large enough. Using $P\{|N| > x\} \sim \sqrt{2/\pi}\, x^{-1}e^{-x^2/2}$ again,

$$\begin{aligned} P\left\{ |\sum_{k=1}^{n} Z_{nk}| > (1+\varepsilon_4)a_n \right\} &= P\left\{ |N| > \frac{(1+\varepsilon_4)a_n}{\sqrt{\sum_{k=1}^{n} E(X_{nk} - EX_{nk})^2}} \right\} \\ &\geqslant P\left\{ |N| > \frac{(1+\varepsilon_4)a_n}{\sqrt{\delta n}} \right\} \sim \frac{C}{(\log n)^{t_2}(\log\log n)^{1/2}}. \end{aligned}$$

Therefore (2.9) holds by the fact that the series $\sum_{n=1}^{\infty} n^{-1}(\log n)^{-t_2}(\log\log n)^{-1/2} = \infty$. The proof is completed. $\square$

*Proof of Theorem 1.2.* For all $1 \leqslant k \leqslant n$ and $n \geqslant 1$, set

$$X_{nk} = X_k I(|X_{nk}| \leqslant \sqrt{n\log n}), \ \ T_n = \sum_{k=1}^{n} X_{nk}.$$

We first prove that

$$\sum_{n=1}^{\infty} n^{r-1} P\{|S_n| > (1+\varepsilon)\sqrt{2rn\log n}\} < \infty, \ \ \forall\, \varepsilon > 0. \tag{2.10}$$

Note that

$$P\{|S_n| > (1+\varepsilon)\sqrt{2rn\log n}\} \subset \{\max_{1\leqslant k\leqslant n} |X_k| > \sqrt{n\log n}\} \cup \{|T_n| > (1+\varepsilon)\sqrt{2rn\log n}\}$$

and by Lemma 2.3,

$$\begin{aligned} \sum_{n=1}^{\infty} n^{r-1} P\{\max_{1\leqslant k\leqslant n} |X_k| > \sqrt{n\log n}\} &\leqslant \sum_{i=1}^{p}\sum_{n=1}^{\infty} n^r P\{|Y_i| > \sqrt{n\log n}\}\} \\ &\leqslant C\sum_{i=1}^{p} E\left(Y_i^2/\log|Y_i|\right)^{r+1} < \infty \end{aligned} \tag{2.11}$$

and

$$\frac{1}{\sqrt{n\log n}}|ET_n| \leqslant \frac{1}{\sqrt{n\log n}} \sum_{k=1}^{n} E|X_k| I(|X_k| \leqslant \sqrt{n\log n}) \leqslant \frac{1}{\log n} \to 0 \tag{2.12}$$

as $n \to \infty$. Hence to prove (2.10), it is enough to prove that

$$\sum_{n=1}^{\infty} n^{r-1} P\{|T_n - ET_n| > (1+\varepsilon)\sqrt{2rn\log n}\} < \infty, \ \ \forall\, \varepsilon > 0. \tag{2.13}$$

We can get from Lemma 2.1 that for any $n \geqslant 1$, there exists normal random variables $Z_{nk}$ with $EZ_{nk} = 0$ and $EZ_{nk}^2 = E(X_{nk} - EX_{nk})^2$, $1 \leqslant k \leqslant n$, such that for any $q > 2$ and all $y > 0$

$$P\left\{|(T_n - ET_n) - \sum_{k=1}^{n} Z_{nk}| > y\right\} \leqslant Ay^{-q} \sum_{k=1}^{n} E|X_{nk} - EX_{nk}|^q. \tag{2.14}$$

Note that

$$\left\{|T_n - ET_n| > (1+\varepsilon)\sqrt{2rn\log n}\right\} \subset \left\{|(T_n - ET_n) - \sum_{k=1}^{k} Z_{nk}| > \varepsilon_1 \sqrt{2rn\log n}\right\} \cup \left\{|\sum_{k=1}^{n} Z_{nk}| > (1+\varepsilon_2)\sqrt{2rn\log n}\right\}$$

where $\varepsilon_1 > 0, \varepsilon_1 > 0$ with $\varepsilon = \varepsilon_1 + \varepsilon_2$. Hence

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{r-1} P\left\{|T_n - ET_n| > (1+\varepsilon)\sqrt{2rn\log n}\right\} \\ &\leqslant \sum_{n=1}^{\infty} n^{r-1} P\left\{|(T_n - ET_n) - \sum_{k=1}^{n} Z_{nk}| > \varepsilon_1 \sqrt{2rn\log n}\right\} \\ &\quad + \sum_{n=1}^{\infty} n^{r-1} P\left\{|\sum_{k=1}^{n} Z_{nk}| > (1+\varepsilon_2)\sqrt{2rn\log n}\right\} = J_1 + J_2. \end{aligned}$$

By (2.14) and Lemma 2.3 we can derive that for any $q > 2(r+1)$,

$$\begin{aligned} J_1 &\leqslant C \sum_{n=1}^{\infty} n^{r-1-q/2} (\log n)^{-q/2} \sum_{k=1}^{n} E|X_{nk} - EX_{nk}|^q \\ &\leqslant C \sum_{n=1}^{\infty} n^{r-1-q/2} (\log n)^{-q/2} \sum_{k=1}^{n} E|X_k|^q I(|X_k| \leqslant \sqrt{n\log n}) \\ &\leqslant C \sum_{i=1}^{p} \sum_{n=1}^{\infty} n^{r-q/2} (\log n)^{-q/2} E|Y_i|^q I(|Y_i| \leqslant \sqrt{n\log n}) \\ &\leqslant C \sum_{i=1}^{p} E(Y_i^2/\log|Y_i|)^{r+1} < \infty. \end{aligned}$$

Let $N$ be a standard normal random variable. Note that $E(X_{nk} - EX_{nk})^2 \leqslant 1$ for all $1 \leqslant k \leqslant n$ and $n \geqslant 1$, and $P\{|N| > x\} \sim \sqrt{2/\pi}\, x^{-1} e^{-x^2/2}$. Hence for large enough $n$,

$$\begin{aligned} P\left\{|\sum_{k=1}^{n} Z_{nk}| > (1+\varepsilon_2)\sqrt{2rn\log n}\right\} &= P\left\{|N| > \frac{(1+\varepsilon_2)\sqrt{2rn\log n}}{\sqrt{\sum_{k=1}^{n} E(X_{nk} - EX_{nk})^2}}\right\} \\ &\leqslant C\exp\left\{-\frac{r(1+\varepsilon_2)^2 n\log n}{\sum_{k=1}^{n} E(X_{nk} - EX_{nk})^2}\right\} \\ &\leqslant C\exp\left\{-r(1+\varepsilon_2)^2 \log n\right\} = Cn^{-t_1}, \end{aligned}$$

where $t_1 = r(1+\varepsilon_2)^2 > r$. Then $J_2 < \infty$.

Now we prove that

$$\sum_{n=1}^{\infty} n^{r-1} P\left\{|S_n| > (1+\varepsilon)\sqrt{2rn\log n}\right\} = \infty, \ \forall\, \varepsilon < 0. \tag{2.15}$$

By (2.11) and (2.12), (2.15) is equivalent to

$$\sum_{n=1}^{\infty} n^{r-1} P\left\{|T_n - ET_n| > (1+\varepsilon)\sqrt{2rn\log n}\right\} = \infty, \ \forall\, \varepsilon < 0. \tag{2.16}$$

Note that for any $\varepsilon < 0$, if we take $\varepsilon_3 > 0$ and $\varepsilon_4 < 0$ with $\varepsilon_4 = \varepsilon_3 + \varepsilon$, then

$$\begin{aligned}\left\{|\sum_{k=1}^{n} Z_{nk}| > (1+\varepsilon_4)\sqrt{2rn\log n}\right\} \subset &\left\{|(T_n - ET_n) - \sum_{k=1}^{n} Z_{nk}| > \varepsilon_3\sqrt{2rn\log n}\right\} \\ &\cup \left\{|T_n - ET_n| > (1+\varepsilon)\sqrt{2rn\log n}\right\},\end{aligned}$$

Then by $J_1 < \infty$, it is enough to show that

$$\sum_{n=1}^{\infty} n^{r-1} P\left\{|\sum_{k=1}^{n} Z_{nk}| > (1+\varepsilon_4)\sqrt{2rn\log n}\right\} = \infty. \tag{2.17}$$

Since $n^{-1}\sum_{k=1}^{n} E(X_{nk} - EX_{nk})^2 \to 1$ as $n \to \infty$, then there exists $\delta < 1$ close to 1 enough with $t_2 = r(1+\varepsilon_4)^2/\delta < r$ such that $\sum_{k=1}^{n} E(X_{nk} - EX_{nk})^2 \geqslant \delta n$ for $n$ large enough. Using $P\{|N| > x\} \sim \sqrt{2/\pi}\, x^{-1} e^{-x^2/2}$ again,

$$\begin{aligned}P\left\{|\sum_{k=1}^{n} Z_{nk}| > (1+\varepsilon_4)\sqrt{2rn\log n}\right\} &= P\left\{|N| > \frac{(1+\varepsilon_4)\sqrt{2rn\log n}}{\sqrt{\sum_{k=1}^{n} E(X_{nk} - EX_{nk})^2}}\right\} \\ &\geqslant P\left\{|N| > \frac{(1+\varepsilon_4)\sqrt{2rn\log n}}{\sqrt{\delta n}}\right\} \\ &\sim \frac{C}{n^{t_2}(\log n)^{1/2}}.\end{aligned}$$

Therefore (2.17) holds by the fact that the series $\sum_{n=1}^{\infty} n^{r-1-t_2}(\log n)^{-1/2} = \infty$. The proof is completed. □

*Proof of Theorem 1.3.* By the same argument as Theorem 1.2, we have

$$\sum_{n=1}^{\infty} P\{|S_{nn}| > (1+\varepsilon)\sqrt{2n\log n}\} \begin{cases} < \infty, & \text{if } \varepsilon > 0, \\ = \infty, & \text{if } \varepsilon < 0. \end{cases}$$

Then by the Borel-Cantelli lemma, (1.7) holds at once. □

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