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NEW TYPES OF INEQUALITIES FOR FUSION FRAMES

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Abstract. In this paper, we establish a more general inequality for fusion frames, which involves a scalar $\lambda \in [0,1]$. It is shown that the result we obtained covers the existing corresponding results recently given by Guo, Leng and Li. We also present several new inequalities for fusion frames, which are different in structure from previous ones.

1. Introduction

Frames (classical frames), which provide stable but generally non-unique expansions for a given vector, were formally defined by Duffin and Schaeffer [10] in their work to deal with some deep problems on nonharmonic Fourier series. Since the publication of the fundamental paper [9] on wavelet theory due to Daubechies, Grossmann and Meyer, frames have become the focus of active research. Today, owing to the flexibility, frames have been used in a variety of fields, both in theory and in applications, such as the characterization of function spaces, sampling theory, digital signal processing etc. For more details on frame theory and its applications, we refer the reader to [2, 4, 5, 8, 16].

Being an extension of frames, the notion of fusion frames was proposed by Casazza and Kutyniok in [6] and Fornasier in [11], which offers us a useful tool to work on some large systems. It should be remarked that, although most properties of fusion frames are similar to those for classical frames, new phenomena do arise due to the complex structure of fusion frames. This makes the study of fusion frames interesting. Now, fusion frames have shown their applications in dozens of areas, see [3, 7], for example.

To proceed with this section, we need to recall some notations and basic definitions.

Throughout this paper, the symbols $\mathscr{H}$ and $\mathbb{J}$ refer, respectively, to a separable Hilbert space and a finite or countable index set, $\{W_j\}_{j\in\mathbb{J}}$ and $\{V_j\}_{j\in\mathbb{J}}$ are two sequences of closed subspaces of $\mathscr{H}$, and π_{W_j} is used to denote the orthogonal projection onto W_j. We denote by $\{\omega_j\}_{j\in\mathbb{J}}$ and $\{\upsilon_j\}_{j\in\mathbb{J}}$ the sequences of weights, i.e., $\omega_j, \upsilon_j > 0$ for each $j \in \mathbb{J}$, the notation $\mathrm{Id}_{\mathscr{H}}$ is reserved for the identity operator on $\mathscr{H}$.

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We call the pair $\{(W_j,\omega_j)\}_{j\in\mathbb{J}}$ a fusion frame for $\mathscr{H}$, if there exist two constants $0<C\leqslant D<\infty$ such that

$$C\|f\|^2\leqslant\sum_{j\in\mathbb{J}}\omega_j^2\|\pi_{W_j}(f)\|^2\leqslant D\|f\|^2,\quad\forall f\in\mathscr{H}. \tag{1.1}$$

The numbers C and D are called fusion frame bounds. We call $\{(W_j,\omega_j)\}_{j\in\mathbb{J}}$ a μ-tight fusion frame if $C=D=\mu$, and a Parseval fusion frame if $C=D=1$. If only the right hand inequality in (1.1) is satisfied, then we call $\{(W_j,\omega_j)\}_{j\in\mathbb{J}}$ a Bessel fusion sequence with Bessel bound D.

Let $\{(W_j,\omega_j)\}_{j\in\mathbb{J}}$ be a fusion frame for $\mathscr{H}$, the fusion frame operator for $\{(W_j,\omega_j)\}_{j\in\mathbb{J}}$ is defined by

$$S:\mathscr{H}\to\mathscr{H},\quad Sf=\sum_{j\in\mathbb{J}}\omega_j^2\pi_{W_j}(f),\quad\forall f\in\mathscr{H}. \tag{1.2}$$

It is easy to check that S is positive, self-adjoint and invertible. It has been proved in [12] that $\{(S^{-1}W_j,\omega_j)\}_{j\in\mathbb{J}}$ is still a fusion frame for $\mathscr{H}$, which is called the dual fusion frame of $\{(W_j,\omega_j)\}_{j\in\mathbb{J}}$. Moreover, from equation (1.2) we see that

$$\begin{aligned}\sum_{j\in\mathbb{J}}\omega_j^2S^{-1}\pi_{W_j}(f)&=S^{-1}Sf=f\\&=SS^{-1}f=\sum_{j\in\mathbb{J}}\omega_j^2\pi_{W_j}(S^{-1}f)\end{aligned} \tag{1.3}$$

is valid for each $f\in\mathscr{H}$.

Let $\{(W_j,\omega_j)\}_{j\in\mathbb{J}}$ be a fusion frame for $\mathscr{H}$ with fusion frame operator S, and $\{(V_j,\upsilon_j)\}_{j\in\mathbb{J}}$ be a Bessel fusion sequence for $\mathscr{H}$. Then $\{(V_j,\upsilon_j)\}_{j\in\mathbb{J}}$ is said to be an alternate dual fusion frame of $\{(W_j,\omega_j)\}_{j\in\mathbb{J}}$, if for any $f\in\mathscr{H}$ we have

$$f=\sum_{j\in\mathbb{J}}\upsilon_j\omega_j\pi_{V_j}S^{-1}\pi_{W_j}(f). \tag{1.4}$$

Let $\{(W_j,\omega_j)\}_{j\in\mathbb{J}}$ be a Bessel fusion sequence for $\mathscr{H}$. For any $\mathbb{K}\subset\mathbb{J}$, we let $\mathbb{K}^c=\mathbb{J}\backslash\mathbb{K}$, and define the operators $S_{\mathbb{K}},S_{\mathbb{K}^c}:\mathscr{H}\to\mathscr{H}$ by

$$S_{\mathbb{K}}f=\sum_{j\in\mathbb{K}}\omega_j^2\pi_{W_j}(f),\quad S_{\mathbb{K}^c}f=\sum_{j\in\mathbb{K}^c}\omega_j^2\pi_{W_j}(f),\quad\forall f\in\mathscr{H}. \tag{1.5}$$

It is trivial that $S_{\mathbb{K}}$ and $S_{\mathbb{K}^c}$ are positive, bounded linear and self-adjoint operators.

Balan et al. [1] discovered a remarkable identity for Parseval frames when working on efficient algorithms for signal reconstruction. Moreover, in [1] the following inequality was obtained:

THEOREM 1.1. *Let $\{f_j\}_{j\in\mathbb{J}}$ be a Parseval frame for $\mathscr{H}$, then for every $\mathbb{K}\subset\mathbb{J}$ and every $f\in\mathscr{H}$, we have*

$$\sum_{j\in\mathbb{K}}|\langle f,f_j\rangle|^2+\Big\|\sum_{j\in\mathbb{K}^c}\langle f,f_j\rangle f_j\Big\|^2=\sum_{j\in\mathbb{K}^c}|\langle f,f_j\rangle|^2+\Big\|\sum_{j\in\mathbb{K}}\langle f,f_j\rangle f_j\Big\|^2\geqslant\frac{3}{4}\|f\|^2. \tag{1.6}$$

Later on, Găvruţa in [13] generalized inequality (1.6) to general frames:

THEOREM 1.2. *Let* $\{f_j\}_{j\in\mathbb{J}}$ *be a frame for* $\mathscr{H}$ *with canonical dual frame* $\{\tilde{f}_j\}_{j\in\mathbb{J}}$. *Then for all* $\mathbb{K}\subset\mathbb{J}$ *and all* $f\in\mathscr{H}$, *we have*

$$\sum_{j\in\mathbb{K}}|\langle f,f_j\rangle|^2+\sum_{j\in\mathbb{J}}|\langle S_{\mathbb{K}^c}f,\tilde{f}_j\rangle|^2=\sum_{j\in\mathbb{K}^c}|\langle f,f_j\rangle|^2+\sum_{j\in\mathbb{J}}|\langle S_{\mathbb{K}}f,\tilde{f}_j\rangle|^2\geqslant\frac{3}{4}\sum_{j\in\mathbb{J}}|\langle f,f_j\rangle|^2. \tag{1.7}$$

Recently, Guo, Leng and Li [14] extended inequalities (1.6) and (1.7) to the case of fusion frames:

THEOREM 1.3. *Let* $\{(W_j,\omega_j)\}_{j\in\mathbb{J}}$ *be an* A*-tight fusion frame for* $\mathscr{H}$, *then for any* $\mathbb{K}\subset\mathbb{J}$ *and any* $f\in\mathscr{H}$, *we have*

$$\begin{aligned}&A\sum_{j\in\mathbb{K}}\omega_j^2\|\pi_{W_j}(f)\|^2+\|S_{\mathbb{K}^c}f\|^2\\&\qquad=A\sum_{j\in\mathbb{K}^c}\omega_j^2\|\pi_{W_j}(f)\|^2+\|S_{\mathbb{K}}f\|^2\geqslant\frac{3}{4}A^2\langle f,f\rangle.\end{aligned} \tag{1.8}$$

THEOREM 1.4. *Let* $\{(W_j,\omega_j)\}_{j\in\mathbb{J}}$ *be a fusion frame for* $\mathscr{H}$ *with fusion frame operator* S *and* $\{(S^{-1}W_j,\omega_j)\}_{j\in\mathbb{J}}$ *be the dual fusion frame of* $\{(W_j,\omega_j)\}_{j\in\mathbb{J}}$. *Then for any* $\mathbb{K}\subset\mathbb{J}$ *and any* $f\in\mathscr{H}$, *we have*

$$\begin{aligned}&\sum_{j\in\mathbb{K}}\omega_j^2\|\pi_{W_j}(f)\|^2+\sum_{j\in\mathbb{J}}\omega_j^2\|\pi_{W_j}(S^{-1}S_{\mathbb{K}^c}f)\|^2\\&\quad=\sum_{j\in\mathbb{K}^c}\omega_j^2\|\pi_{W_j}(f)\|^2+\sum_{j\in\mathbb{J}}\omega_j^2\|\pi_{W_j}(S^{-1}S_{\mathbb{K}}f)\|^2\geqslant\frac{3}{4}\langle Sf,f\rangle\end{aligned} \tag{1.9}$$

In this work, we generalize the inequalities (1.8) and (1.9) to a more general form where a scalar $\lambda\in[0,1]$ is involved. We also present several new inequalities for fusion frames, which differ in structure from those in Theorems 1.3 and 1.4.

2. Main results

To derive our main results, we need the following lemmas due to Găvruţa [13] and Poria [15], respectively.

LEMMA 2.1. (see [13]) *If* P *and* Q *are two bounded linear operators on* $\mathscr{H}$ *satisfying* $P+Q=\mathrm{Id}_{\mathscr{H}}$, *then*

$$P^*P+\frac{1}{2}(Q^*+Q)=Q^*Q+\frac{1}{2}(P^*+P)\geqslant\frac{3}{4}\mathrm{Id}_{\mathscr{H}}. \tag{2.1}$$

LEMMA 2.2. (see [15]) *Let P and Q be two bounded linear and self-adjoint operators on $\mathscr{H}$ such that $P+Q=\mathrm{Id}_{\mathscr{H}}$, then for any $\lambda\in[0,1]$ and any $f\in\mathscr{H}$ we have*

$$\|Pf\|^2+2\lambda\langle Qf,f\rangle=\|Qf\|^2+2(1-\lambda)\langle Pf,f\rangle+(2\lambda-1)\|f\|^2\geqslant(2\lambda-\lambda^2)\|f\|^2. \tag{2.2}$$

THEOREM 2.3. *Let $\{(W_j,\omega_j)\}_{j\in\mathbb{J}}$ be a fusion frame for $\mathscr{H}$ with the fusion frame operator S and $\{(S^{-1}W_j,\omega_j)\}_{j\in\mathbb{J}}$ be the dual fusion frame of $\{(W_j,\omega_j)\}_{j\in\mathbb{J}}$. Then for any $\lambda\in[0,1]$, for all $\mathbb{K}\subset\mathbb{J}$ and all $f\in\mathscr{H}$, we have*

$$\begin{aligned}\langle Sf,f\rangle&\geqslant\sum_{j\in\mathbb{K}}\omega_j^2\|\pi_{W_j}(f)\|^2+\sum_{j\in\mathbb{J}}\omega_j^2\|\pi_{W_j}(S^{-1}S_{\mathbb{K}^c}f)\|^2\\&=\sum_{j\in\mathbb{K}^c}\omega_j^2\|\pi_{W_j}(f)\|^2+\sum_{j\in\mathbb{J}}\omega_j^2\|\pi_{W_j}(S^{-1}S_{\mathbb{K}}f)\|^2\\&\geqslant(2\lambda-\lambda^2)\langle S_{\mathbb{K}}f,f\rangle+(1-\lambda^2)\langle S_{\mathbb{K}^c}f,f\rangle.\end{aligned}\tag{2.3}$$

Proof. Since $S_{\mathbb{K}}+S_{\mathbb{K}^c}=S$, it follows that $S^{-\frac12}S_{\mathbb{K}}S^{-\frac12}+S^{-\frac12}S_{\mathbb{K}^c}S^{-\frac12}=\mathrm{Id}_{\mathscr{H}}$. Taking $S^{-\frac12}S_{\mathbb{K}}S^{-\frac12}$, $S^{-\frac12}S_{\mathbb{K}^c}S^{-\frac12}$ and $S^{\frac12}f$ instead of P, Q and f respectively in Lemma 2.2 yields

$$\begin{aligned}\langle S^{-1}S_{\mathbb{K}}f,S_{\mathbb{K}}f\rangle&=\langle S^{-1}S_{\mathbb{K}^c}f,S_{\mathbb{K}^c}f\rangle+2(1-\lambda)\langle S_{\mathbb{K}}f,f\rangle+(2\lambda-1)\langle Sf,f\rangle-2\lambda\langle S_{\mathbb{K}^c}f,f\rangle\\&\geqslant(2\lambda-\lambda^2)\langle Sf,f\rangle-2\lambda\langle S_{\mathbb{K}^c}f,f\rangle\\&=2\lambda(\langle Sf,f\rangle-\langle S_{\mathbb{K}^c}f,f\rangle)-\lambda^2\langle Sf,f\rangle\\&=2\lambda\langle S_{\mathbb{K}}f,f\rangle-\lambda^2\langle Sf,f\rangle.\end{aligned}\tag{2.4}$$

Noting that

$$\begin{aligned}&\langle S^{-1}S_{\mathbb{K}}f,S_{\mathbb{K}}f\rangle+\langle S_{\mathbb{K}^c}f,f\rangle\\&\quad=\langle S^{-\frac12}S_{\mathbb{K}}S^{-\frac12}S^{\frac12}f,S^{-\frac12}S_{\mathbb{K}}S^{-\frac12}S^{\frac12}f\rangle+\langle S_{\mathbb{K}^c}f,f\rangle\\&\quad=\langle(\mathrm{Id}_{\mathscr{H}}-S^{-\frac12}S_{\mathbb{K}^c}S^{-\frac12})S^{\frac12}f,(\mathrm{Id}_{\mathscr{H}}-S^{-\frac12}S_{\mathbb{K}^c}S^{-\frac12})S^{\frac12}f\rangle+\langle S_{\mathbb{K}^c}f,f\rangle\\&\quad=\langle S^{\frac12}f,S^{\frac12}f\rangle-\langle S^{\frac12}f,S^{-\frac12}S_{\mathbb{K}^c}S^{-\frac12}S^{\frac12}f\rangle-\langle S^{-\frac12}S_{\mathbb{K}^c}S^{-\frac12}S^{\frac12}f,S^{\frac12}f\rangle\\&\qquad+\langle S^{-\frac12}S_{\mathbb{K}^c}S^{-\frac12}S^{\frac12}f,S^{-\frac12}S_{\mathbb{K}^c}S^{-\frac12}S^{\frac12}f\rangle+\langle S_{\mathbb{K}^c}f,f\rangle\\&\quad=\langle Sf,f\rangle-\langle f,S_{\mathbb{K}^c}f\rangle-\langle S_{\mathbb{K}^c}f,f\rangle+\langle S^{-1}S_{\mathbb{K}^c}f,S_{\mathbb{K}^c}f\rangle+\langle S_{\mathbb{K}^c}f,f\rangle\\&\quad=\langle S^{-1}S_{\mathbb{K}^c}f,S_{\mathbb{K}^c}f\rangle+\langle S_{\mathbb{K}}f,f\rangle,\end{aligned}\tag{2.5}$$

we have

$$\begin{aligned}&\sum_{j\in\mathbb{K}}\omega_j^2\|\pi_{W_j}(f)\|^2+\sum_{j\in\mathbb{J}}\omega_j^2\|\pi_{W_j}(S^{-1}S_{\mathbb{K}^c}f)\|^2\\&\qquad=\langle S_{\mathbb{K}}f,f\rangle+\langle SS^{-1}S_{\mathbb{K}^c}f,S^{-1}S_{\mathbb{K}^c}f\rangle\end{aligned}$$

$$
\begin{aligned}
&= \langle S_{\mathbb{K}} f, f\rangle + \langle S^{-1} S_{\mathbb{K}^c} f, S_{\mathbb{K}^c} f\rangle \\
&= \langle S_{\mathbb{K}^c} f, f\rangle + \langle S^{-1} S_{\mathbb{K}} f, S_{\mathbb{K}} f\rangle \\
&= \langle S_{\mathbb{K}^c} f, f\rangle + \langle S S^{-1} S_{\mathbb{K}} f, S^{-1} S_{\mathbb{K}} f\rangle \\
&= \sum_{j\in\mathbb{K}^c} \omega_j^2 \|\pi_{W_j}(f)\|^2 + \sum_{j\in\mathbb{J}} \omega_j^2 \|\pi_{W_j}(S^{-1} S_{\mathbb{K}} f)\|^2. \qquad (2.6)
\end{aligned}
$$

Now, from (2.4) it follows that

$$
\begin{aligned}
&\sum_{j\in\mathbb{K}} \omega_j^2 \|\pi_{W_j}(f)\|^2 + \sum_{j\in\mathbb{J}} \omega_j^2 \|\pi_{W_j}(S^{-1} S_{\mathbb{K}^c} f)\|^2 \\
&\quad= \sum_{j\in\mathbb{K}^c} \omega_j^2 \|\pi_{W_j}(f)\|^2 + \sum_{j\in\mathbb{J}} \omega_j^2 \|\pi_{W_j}(S^{-1} S_{\mathbb{K}} f)\|^2 \\
&\quad\geqslant 2\lambda \langle S_{\mathbb{K}} f, f\rangle - \lambda^2 \langle S f, f\rangle + \langle S_{\mathbb{K}^c} f, f\rangle \\
&\quad= (2\lambda - \lambda^2)\langle S_{\mathbb{K}} f, f\rangle + (1-\lambda^2)\langle S_{\mathbb{K}^c} f, f\rangle. \qquad (2.7)
\end{aligned}
$$

It remains to prove the left hand inequality in (2.3). It is easily seen that $P = S^{-\frac{1}{2}} S_{\mathbb{K}} S^{-\frac{1}{2}}$ and $Q = S^{-\frac{1}{2}} S_{\mathbb{K}^c} S^{-\frac{1}{2}}$ are positive operators and that $PQ = QP$. Thus,

$$
0 \leqslant PQ = P(\mathrm{Id}_{\mathscr{H}} - P) = P - P^2 = S^{-\frac{1}{2}}(S_{\mathbb{K}} - S_{\mathbb{K}} S^{-1} S_{\mathbb{K}}) S^{-\frac{1}{2}},
$$

implying that $S_{\mathbb{K}} - S_{\mathbb{K}} S^{-1} S_{\mathbb{K}} \geqslant 0$. Hence

$$
\begin{aligned}
&\sum_{j\in\mathbb{K}} \omega_j^2 \|\pi_{W_j}(f)\|^2 + \sum_{j\in\mathbb{J}} \omega_j^2 \|\pi_{W_j}(S^{-1} S_{\mathbb{K}^c} f)\|^2 \\
&\quad= \sum_{j\in\mathbb{K}^c} \omega_j^2 \|\pi_{W_j}(f)\|^2 + \sum_{j\in\mathbb{J}} \omega_j^2 \|\pi_{W_j}(S^{-1} S_{\mathbb{K}} f)\|^2 \\
&\quad= \langle S_{\mathbb{K}^c} f, f\rangle + \langle S^{-1} S_{\mathbb{K}} f, S_{\mathbb{K}} f\rangle \\
&\quad\leqslant \langle S_{\mathbb{K}^c} f, f\rangle + \langle S_{\mathbb{K}} f, f\rangle = \langle S f, f\rangle. \qquad (2.8)
\end{aligned}
$$

This completes the proof. □

If $\{(W_j, \omega_j)\}_{j\in\mathbb{J}}$ is an A-tight fusion frame for $\mathscr{H}$ with the fusion frame operator S, then $S = A\mathrm{Id}_{\mathscr{H}}$. Combining Theorem 2.3 with $\sum_{j\in\mathbb{J}} \omega_j^2 \|\pi_{W_j}(S^{-1} S_{\mathbb{K}^c} f)\|^2 = \frac{1}{A}\|S_{\mathbb{K}^c} f\|^2$ and $\sum_{j\in\mathbb{J}} \omega_j^2 \|\pi_{W_j}(S^{-1} S_{\mathbb{K}} f)\|^2 = \frac{1}{A}\|S_{\mathbb{K}} f\|^2$, we immediately obtain the following result.

COROLLARY 2.4. *Let $\{(W_j, \omega_j)\}_{j\in\mathbb{J}}$ be an A-tight fusion frame for $\mathscr{H}$, then for any $\lambda \in [0,1]$, for all $\mathbb{K} \subset \mathbb{J}$ and all $f \in \mathscr{H}$, we have*

$$
\begin{aligned}
A^2\|f\|^2 &\geqslant A \sum_{j\in\mathbb{K}} \omega_j^2 \|\pi_{W_j}(f)\|^2 + \|S_{\mathbb{K}^c} f\|^2 \\
&= A \sum_{j\in\mathbb{K}^c} \omega_j^2 \|\pi_{W_j}(f)\|^2 + \|S_{\mathbb{K}} f\|^2 \\
&\geqslant A(2\lambda - \lambda^2)\langle S_{\mathbb{K}} f, f\rangle + A(1-\lambda^2)\langle S_{\mathbb{K}^c} f, f\rangle. \qquad (2.9)
\end{aligned}
$$

REMARK 2.5. If we take $\lambda = \frac{1}{2}$ in Theorem 2.3 and Corollary 2.4, then we can obtain the inequalities in Theorems 1.4 and 1.3, respectively.

The next two theorems give several new inequalities for fusion frames, which possess different structure comparing with the inequalities shown in Theorems 1.3 and 1.4.

THEOREM 2.6. *Let $\{(W_j, \omega_j)\}_{j\in\mathbb{J}}$ be an A-tight fusion frame for $\mathscr{H}$ with the fusion frame operator S, then for any $\mathbb{K}\subset\mathbb{J}$ and any $f\in\mathscr{H}$, we have*

$$0 \leqslant A\sum_{j\in\mathbb{K}}\omega_j^2\|\pi_{W_j}(f)\|^2 - \left\|\sum_{j\in\mathbb{K}}\omega_j^2\pi_{W_j}(f)\right\|^2 \leqslant \frac{A^2}{4}\|f\|^2. \tag{2.10}$$

$$\frac{A^2}{2}\|f\|^2 \leqslant \left\|\sum_{j\in\mathbb{K}}\omega_j^2\pi_{W_j}(f)\right\|^2 + \left\|\sum_{j\in\mathbb{K}^c}\omega_j^2\pi_{W_j}(f)\right\|^2 \leqslant A^2\|f\|^2. \tag{2.11}$$

Proof. Applying $S_{\mathbb{K}} + S_{\mathbb{K}^c} = S = A\mathrm{Id}_{\mathscr{H}}$, we have that $A^{-1}S_{\mathbb{K}} + A^{-1}S_{\mathbb{K}^c} = \mathrm{Id}_{\mathscr{H}}$. Noticing that $S_{\mathbb{K}}$ and $S_{\mathbb{K}^c}$ are positive and that $S_{\mathbb{K}}S_{\mathbb{K}^c} = S_{\mathbb{K}^c}S_{\mathbb{K}}$, we have

$$0 \leqslant (A^{-1}S_{\mathbb{K}})(A^{-1}S_{\mathbb{K}^c}) = A^{-1}S_{\mathbb{K}}(\mathrm{Id}_{\mathscr{H}} - A^{-1}S_{\mathbb{K}}) = A^{-1}S_{\mathbb{K}} - A^{-2}S_{\mathbb{K}}^2. \tag{2.12}$$

We also have

$$A^{-1}S_{\mathbb{K}} - A^{-2}S_{\mathbb{K}}^2 = -\left(A^{-1}S_{\mathbb{K}} - \frac{1}{2}\mathrm{Id}_{\mathscr{H}}\right)^2 + \frac{1}{4}\mathrm{Id}_{\mathscr{H}} \leqslant \frac{1}{4}\mathrm{Id}_{\mathscr{H}}. \tag{2.13}$$

Since

$$A^{-1}\langle S_{\mathbb{K}}f, f\rangle - A^{-2}\langle S_{\mathbb{K}}^2 f, f\rangle = A^{-1}\sum_{j\in\mathbb{K}}\omega_j^2\|\pi_{W_j}(f)\|^2 - A^{-2}\left\|\sum_{j\in\mathbb{K}}\omega_j^2\pi_{W_j}(f)\right\|^2$$

for each $f\in\mathscr{H}$, combination of (2.12) and (2.13) it follows that

$$0 \leqslant A\sum_{j\in\mathbb{K}}\omega_j^2\|\pi_{W_j}(f)\|^2 - \left\|\sum_{j\in\mathbb{K}}\omega_j^2\pi_{W_j}(f)\right\|^2 \leqslant \frac{A^2}{4}\|f\|^2. \tag{2.14}$$

We next prove that (2.11) holds. On one hand we have

$$\begin{aligned} A^{-2}S_{\mathbb{K}}^2 + A^{-2}S_{\mathbb{K}^c}^2 &= A^{-2}S_{\mathbb{K}}^2 + (\mathrm{Id}_{\mathscr{H}} - A^{-1}S_{\mathbb{K}})^2 \\ &= 2A^{-2}S_{\mathbb{K}}^2 - 2A^{-1}S_{\mathbb{K}} + \mathrm{Id}_{\mathscr{H}} \\ &= 2A^{-2}\left(S_{\mathbb{K}} - \frac{A}{2}\mathrm{Id}_{\mathscr{H}}\right)^2 + \frac{1}{2}\mathrm{Id}_{\mathscr{H}} \geqslant \frac{1}{2}\mathrm{Id}_{\mathscr{H}}. \end{aligned} \tag{2.15}$$

On the other hand we get

$$\begin{aligned} A^{-2}S_{\mathbb{K}}^2 + A^{-2}S_{\mathbb{K}^c}^2 &= A^{-2}S_{\mathbb{K}}^2 + (\mathrm{Id}_{\mathscr{H}} - A^{-1}S_{\mathbb{K}})^2 \\ &= \mathrm{Id}_{\mathscr{H}} - 2(A^{-1}S_{\mathbb{K}} - A^{-2}S_{\mathbb{K}}^2) \leqslant \mathrm{Id}_{\mathscr{H}}, \end{aligned} \tag{2.16}$$

since, as mentioned before, $A^{-1}S_{\mathbb{K}} - A^{-2}S_{\mathbb{K}}^2 \geqslant 0$. Therefore,

$$\frac{A^2}{2}\mathrm{Id}_{\mathscr{H}} \leqslant S_{\mathbb{K}}^2 + S_{\mathbb{K}^c}^2 \leqslant A^2\mathrm{Id}_{\mathscr{H}}. \tag{2.17}$$

Finally, for each $f \in \mathscr{H}$, we have

$$\begin{aligned}\langle S_{\mathbb{K}}^2 f, f\rangle + \langle S_{\mathbb{K}^c}^2 f, f\rangle &= \|S_{\mathbb{K}} f\|^2 + \|S_{\mathbb{K}^c} f\|^2 \\ &= \Big\|\sum_{j\in\mathbb{K}} \omega_j^2 \pi_{W_j}(f)\Big\|^2 + \Big\|\sum_{j\in\mathbb{K}^c} \omega_j^2 \pi_{W_j}(f)\Big\|^2. \end{aligned}\tag{2.18}$$

This along with (2.17) yields (2.11) and the proof is completed. □

The above theorem leads to a direct consequence as follows.

COROLLARY 2.7. *Let $\{(W_j, \omega_j)\}_{j\in\mathbb{J}}$ be a Parseval fusion frame for $\mathscr{H}$, then for any $\mathbb{K} \subset \mathbb{J}$ and any $f \in \mathscr{H}$, we have*

$$0 \leqslant \sum_{j\in\mathbb{K}} \omega_j^2 \|\pi_{W_j}(f)\|^2 - \Big\|\sum_{j\in\mathbb{K}} \omega_j^2 \pi_{W_j}(f)\Big\|^2 \leqslant \frac{1}{4}\|f\|^2. \tag{2.19}$$

$$\frac{1}{2}\|f\|^2 \leqslant \Big\|\sum_{j\in\mathbb{K}} \omega_j^2 \pi_{W_j}(f)\Big\|^2 + \Big\|\sum_{j\in\mathbb{K}^c} \omega_j^2 \pi_{W_j}(f)\Big\|^2 \leqslant \|f\|^2. \tag{2.20}$$

Let $\{(W_j, \omega_j)\}_{j\in\mathbb{J}}$ be a fusion frame for $\mathscr{H}$ with fusion frame operator S, and $\{(V_j, \upsilon_j)\}_{j\in\mathbb{J}}$ be the alternate dual fusion frame of $\{(W_j, \omega_j)\}_{j\in\mathbb{J}}$. We define the bounded linear operators $L_{\mathbb{K}}, L_{\mathbb{K}^c} : \mathscr{H} \to \mathscr{H}$ as follows:

$$L_{\mathbb{K}} f = \sum_{j\in\mathbb{K}} \upsilon_j \omega_j \pi_{V_j} S^{-1} \pi_{W_j}(f),\ L_{\mathbb{K}^c} f = \sum_{j\in\mathbb{K}^c} \upsilon_j \omega_j \pi_{V_j} S^{-1} \pi_{W_j}(f), \quad \forall f \in \mathscr{H}. \tag{2.21}$$

THEOREM 2.8. *Let $\{(W_j, \omega_j)\}_{j\in\mathbb{J}}$ be a fusion frame for $\mathscr{H}$ with fusion frame operator S, and $\{(V_j, \upsilon_j)\}_{j\in\mathbb{J}}$ be the alternate dual fusion frame of $\{(W_j, \omega_j)\}_{j\in\mathbb{J}}$. Then for any $\mathbb{K} \subset \mathbb{J}$ and any $f \in \mathscr{H}$, we have*

$$\begin{aligned}\frac{3 + \|L_{\mathbb{K}} - L_{\mathbb{K}^c}\|^2}{4}\|f\|^2 &\geqslant \Big\|\sum_{j\in\mathbb{K}} \upsilon_j \omega_j \pi_{V_j} S^{-1} \pi_{W_j}(f)\Big\|^2 + \mathrm{Re}\sum_{j\in\mathbb{K}^c} \upsilon_j \omega_j \langle S^{-1}\pi_{W_j}(f), \pi_{V_j}(f)\rangle \\ &= \Big\|\sum_{j\in\mathbb{K}^c} \upsilon_j \omega_j \pi_{V_j} S^{-1} \pi_{W_j}(f)\Big\|^2 + \mathrm{Re}\sum_{j\in\mathbb{K}} \upsilon_j \omega_j \langle S^{-1}\pi_{W_j}(f), \pi_{V_j}(f)\rangle \\ &\geqslant \frac{3}{4}\|f\|^2. \end{aligned}\tag{2.22}$$

Proof. Clearly, $L_{\mathbb{K}} + L_{\mathbb{K}^c} = \mathrm{Id}_{\mathscr{H}}$. Applying this fact to Lemma 2.1, we obtain

$$\begin{aligned}&\Big\|\sum_{j\in\mathbb{K}} \upsilon_j \omega_j \pi_{V_j} S^{-1} \pi_{W_j}(f)\Big\|^2 + \mathrm{Re}\sum_{j\in\mathbb{K}^c} \upsilon_j \omega_j \langle S^{-1}\pi_{W_j}(f), \pi_{V_j}(f)\rangle \\ &\quad = \langle L_{\mathbb{K}} f, L_{\mathbb{K}} f\rangle + \mathrm{Re}\langle L_{\mathbb{K}^c} f, f\rangle \end{aligned}$$

$$
\begin{aligned}
&= \langle L_{\mathbb{K}}^{*} L_{\mathbb{K}} f, f\rangle + \frac{1}{2}\langle (L_{\mathbb{K}^c}^{*} + L_{\mathbb{K}^c}) f, f\rangle \\
&= \langle L_{\mathbb{K}^c}^{*} L_{\mathbb{K}^c} f, f\rangle + \frac{1}{2}\langle (L_{\mathbb{K}}^{*} + L_{\mathbb{K}}) f, f\rangle \\
&= \langle L_{\mathbb{K}^c} f, L_{\mathbb{K}^c} f\rangle + \mathrm{Re}\langle L_{\mathbb{K}} f, f\rangle \\
&= \Big\| \sum_{j\in\mathbb{K}^c} \upsilon_j \omega_j \pi_{V_j} S^{-1} \pi_{W_j}(f) \Big\|^2 + \mathrm{Re} \sum_{j\in\mathbb{K}} \upsilon_j \omega_j \langle S^{-1}\pi_{W_j}(f), \pi_{V_j}(f)\rangle \\
&\geqslant \frac{3}{4}\|f\|^2,
\end{aligned} \tag{2.23}
$$

for each $f \in \mathscr{H}$. One the other hand we have

$$
\begin{aligned}
&\Big\| \sum_{j\in\mathbb{K}^c} \upsilon_j \omega_j \pi_{V_j} S^{-1} \pi_{W_j}(f) \Big\|^2 + \mathrm{Re} \sum_{j\in\mathbb{K}} \upsilon_j \omega_j \langle S^{-1}\pi_{W_j}(f), \pi_{V_j}(f)\rangle \\
&\quad = \mathrm{Re}\langle L_{\mathbb{K}} f, f\rangle + \langle L_{\mathbb{K}^c} f, L_{\mathbb{K}^c} f\rangle = \mathrm{Re}(\langle f, f\rangle - \langle L_{\mathbb{K}^c} f, f\rangle) + \langle L_{\mathbb{K}^c} f, L_{\mathbb{K}^c} f\rangle \\
&\quad = \langle f, f\rangle - \mathrm{Re}\langle L_{\mathbb{K}^c} f, f\rangle + \langle L_{\mathbb{K}^c} f, L_{\mathbb{K}^c} f\rangle = \langle f, f\rangle - \mathrm{Re}\langle L_{\mathbb{K}^c} f, L_{\mathbb{K}} f\rangle \\
&\quad = \langle f, f\rangle - \frac{1}{2}\langle L_{\mathbb{K}} f, L_{\mathbb{K}^c} f\rangle - \frac{1}{2}\langle L_{\mathbb{K}^c} f, L_{\mathbb{K}} f\rangle \\
&\quad = \frac{3}{4}\|f\|^2 + \frac{1}{4}\langle (L_{\mathbb{K}} + L_{\mathbb{K}^c}) f, (L_{\mathbb{K}} + L_{\mathbb{K}^c}) f\rangle - \frac{1}{2}\langle L_{\mathbb{K}} f, L_{\mathbb{K}^c} f\rangle - \frac{1}{2}\langle L_{\mathbb{K}^c} f, L_{\mathbb{K}} f\rangle \\
&\quad = \frac{3}{4}\|f\|^2 + \frac{1}{4}\langle L_{\mathbb{K}} f, L_{\mathbb{K}} f\rangle + \frac{1}{4}\langle L_{\mathbb{K}^c} f, L_{\mathbb{K}^c} f\rangle - \frac{1}{4}\langle L_{\mathbb{K}} f, L_{\mathbb{K}^c} f\rangle - \frac{1}{4}\langle L_{\mathbb{K}^c} f, L_{\mathbb{K}} f\rangle \\
&\quad = \frac{3}{4}\|f\|^2 + \frac{1}{4}\langle (L_{\mathbb{K}} - L_{\mathbb{K}^c}) f, (L_{\mathbb{K}} - L_{\mathbb{K}^c}) f\rangle \\
&\quad \leqslant \frac{3}{4}\|f\|^2 + \frac{1}{4}\|L_{\mathbb{K}} - L_{\mathbb{K}^c}\|^2 \|f\|^2 = \frac{3 + \|L_{\mathbb{K}} - L_{\mathbb{K}^c}\|^2}{4}\|f\|^2.
\end{aligned} \tag{2.24}
$$

Now, combining this with (2.23), we obtain the relation stated in the theorem. □

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