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ON THE RATE OF STRONG CONVERGENCE FOR A RECURSIVE PROBABILITY DENSITY ESTIMATOR OF END SAMPLES AND ITS APPLICATIONS

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Abstract. The purpose of this paper is to consider a kind of recursive density estimator of the probability density function for a sequence of extended negatively dependent random variables. Under some suitable conditions, we establish the strong convergence rate for the recursive density estimator. As application, we discuss the strong convergence rate for a kind of hazard rate function estimator.

1. Introduction

A finite family of random variables $\{X_1,\cdots,X_n\}$ is said to be extended negatively dependent (END) if there exists a constant $M>0$ such that both inequalities

$$P(X_1>x_1,X_2>x_2,\cdots,X_n>x_n)\leqslant M\prod_{i=1}^{n}P(X_i>x_i)$$

and

$$P(X_1\leqslant x_1,X_2\leqslant x_2,\cdots,X_n\leqslant x_n)\leqslant M\prod_{i=1}^{n}P(X_i\leqslant x_i)$$

hold for all real numbers $x_1,\cdots,x_n$. An infinite sequence $\{X_n,n\geqslant 1\}$ is said to be END if every finite subcollection is END.

This definition of END random variables was proposed by Liu ([1], 2009), and the notion of END received some attention recently. See, for example, Chen et al. ([2], 2010) gave more detailed discussion and some examples, Liu ([3], 2010) studied the sufficient and necessary conditions of moderate deviations for dependent random variables with heavy tails, Shen ([4], 2011) provided some probability inequalities, Wu and Guan ([5], 2012) studied the convergence properties under uniform integrability, Wang et al ([6], 2013) studied the complete convergence for weighted sums, Wang et al

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([7], 2015) obtained complete consistency for the estimator of nonparametric regression models.

Throughout the paper, let $X_1,\cdots,X_n$ be a strictly stationary sequence of END random variables with the unknown marginal probability density function $f(x)$, and distribution function $F(x)$. The hazard rate $r(x)=f(x)/(1-F(x))$. We consider the following recursive kernel estimator for $f(x)$

$$f_n(x)=\frac{1}{n}\sum_{j=1}^{n}\frac{1}{h_j}K\left(\frac{x-X_j}{h_j}\right), \tag{1.1}$$

and the hazard rate estimator for $r(x)$

$$r_n(x)=\frac{f_n(x)}{1-F_n(x)}, \tag{1.2}$$

where $0<h_n\to 0$ are bandwidths, $K(\cdot)$ is some kernel function, and $F_n(x)$ is the empirical distribution function for $F(x)$. The recursive kernel estimator (1.1) was introduced by Wolverton and Wagner ([8], 1969). Note that (1.1) can be computed recursively by

$$f_n(x)=\frac{n-1}{n}f_{n-1}(x)+(nh_n)^{-1}K\left(\frac{x-X_n}{h_n}\right). \tag{1.3}$$

This property of (1.3) is particularly useful in large sample size since $f_n(x)$ can be easily updated with each additional observation. Liang and Baek ([9], 2004) discussed the point asymptotic normality for $f_n(x)$ under negatively associated random variables. Li and Yang ([10], 2005) studied the strong convergence rate of recursive probability density estimator based NA samples. Li et al ([11], 2010) discuss the asymptotic bias, quadratic-mean convergence and establish the pointwise asymptotic normality of $f_n(x)$ for a stationary sequence of negatively associated sequences.

Since END random variables are much weaker than independent random variables, and NA random variables, studying the large sample character of the kernel density estimate for END sequence is of interest. So in this article, we will discuss the strong convergence rate for the recursive kernel estimator $f_n(x)$ based on a stationary and END sequence. As application, we will discuss the strong convergence rate for the hazard rate estimator $r_n(x)$.

In the sequel, let $C^2(f)$ stand for a point set in where the second-order derivative f'' exists and is bounded and continuous, $a^+=\max\{a,0\}$. All limits are taken as the sample size n tends to ∞, C_1, $C_2,\cdots$ and k_0 denote positive constants whose values may change from one place to another, unless specified otherwise.

2. Main results

For easy reference, the assumptions used in this paper are listed below.

ASSUMPTIONS.
(A1) $\int_{-\infty}^{+\infty}K(u)du=1$, $\int_{-\infty}^{+\infty}uK(u)du=0$, $\int_{-\infty}^{+\infty}u^2K(u)du<\infty$, $K(\cdot)\in L_1$;

(A2) The sequence of bandwidths $\{h_j; j \geqslant 1\}$ satisfies the requirements $0 < h_n \to 0$ and $nh_n \to \infty$.

Based on the assumptions above, we can get the following results.

THEOREM 2.1. *Let $\{X_n; n \geqslant 1\}$ be END sequence, and let assumptions (A1) and (A2) hold true. Suppose that the kernel $K(\cdot)$ is a bounded variation function, and the bandwidth $h_n = O(n^{-\nu})$ where $\nu \geqslant 1/6$. Then, for $l > 0$ and $x \in C^2(f)$,*

$$[nh_n^2/(\log n(\log\log n)^l)]^{1/2}(f_n(x) - f(x)) \to 0, \ \ a.s.$$

THEOREM 2.2. *Let $\{X_n; n \geqslant 1\}$ be END sequence, and assumptions (A1) and (A2) be satisfied. If the kernel $K(\cdot)$ is a bounded monotonic density function, and the bandwidth $h_n = O(n^{-\nu})$ where $\nu \geqslant 1/5$. Then, for $l > 0$ and $x \in C^2(f)$,*

$$[nh_n/(\log n(\log\log n)^l)]^{1/2}(f_n(x) - f(x)) \to 0, \ \ a.s.$$

THEOREM 2.3. *Under the conditions of Theorem 2.1, if there exists a point x_0 such that $F(x_0) < 1$, then for $x \leqslant x_0$ and $x \in C^2(f)$,*

$$[nh_n^2/(\log n(\log\log n))]^{1/2}(r_n(x) - r(x)) \to 0, \ \ a.s.$$

THEOREM 2.4. *Under the conditions of Theorem 2.2, if there exists a point x_0 such that $F(x_0) < 1$, then for $x \leqslant x_0$ and $x \in C^2(f)$,*

$$[nh_n/(\log n(\log\log n))]^{1/2}(r_n(x) - r(x)) \to 0, \ \ a.s.$$

3. Some auxiliary results

In this section, we will present some important lemmas which will be used to prove the above main results.

LEMMA 3.1. (see [3]) *Let $\{X_1, \cdots, X_n\}$ be END random variables.*

(i) If $f_1, \cdots, f_n$ are all nondecreasing (or nonincreasing) functions, then random variables $f_1(X_1), \cdots, f_n(X_n)$ are END.

(ii) For each $n \geqslant 1$, there exists a constant $M > 0$ such that

$$E\left(\prod_{j=1}^{n} X_j^+\right) \leqslant M \prod_{j=1}^{n} EX_j^+.$$

LEMMA 3.2. (see [7]) *Let $\{X_n, n \geqslant 1\}$ be a sequence of END random variables with $EX_n = 0$ and $|X_n| \leqslant d_n$ a.s. for each $n \geqslant 1$, where $\{d_n, n \geqslant 1\}$ is a sequence of positive constants. Assume that $t > 0$ such that $t \cdot \max_{1 \leqslant i \leqslant n} d_i \leqslant 1$. Then for any $\varepsilon > 0$, there exists a constant $M > 0$ such that*

$$P(|\sum_{i=1}^{n} X_i| > \varepsilon) \leqslant 2M \exp\{-t\varepsilon + t^2 \sum_{i=1}^{n} EX_i^2\}.$$

LEMMA 3.3. (see [10]) *Suppose the assumption (A1) hold for $K(\cdot)$. Then for $x \in C(f)$,*

$$\lim_{h\to 0}\int_R K(u)f(x-hu)du = f(x).$$

LEMMA 3.4. (see [10]) *Suppose the assumption (A1) hold. Then for $x \in C^2(f)$,*

$$\left(\frac{1}{n}\sum_{i=1}^{n} h_i^2\right)^{-1} |Ef_n(x)-f(x)| \leqslant C < \infty.$$

LEMMA 3.5. *Let $\{X_1,\cdots,X_n\}$ be END random variables having unknown distribution function $F(x)$ and bounded probability density function $f(x)$, and let $F_n(x)$ is the empirical distribution function. If there exists a sequence of positive constants $\{\tau_n\}$ such that $\tau_n \to 0$, and $n\tau_n^2/\log n \to \infty$, then*

$$\sup_x |F_n(x)-F(x)| = o(\tau_n), \ a.s.$$

Particularly, taking $\tau_n = n^{-1/2}(\log n)^{1/2}\log\log n$, then

$$\sup_x |F_n(x)-F(x)| = o(n^{-1/2}(\log n)^{1/2}\log\log n), \ a.s.$$

Proof. By the Lemma 2 in Yang ([12], 2003), let $\{x_{n,k}\}$ satisfy $F(x_{n,k}) = k/n$ for $n \geqslant 3$ and $k = 1,\cdots,n-1$, then we have

$$\sup_{-\infty<x<+\infty} |F_n(x)-F(x)| \leqslant \max_{1\leqslant k\leqslant n-1} |F_n(x_{n,k})-F(x_{n,k})| + \frac{2}{n}. \tag{3.1}$$

It is easily seen that $n\tau_n \to \infty$. Then for $\varepsilon > 0$ and all n large enough, we have $2/n < \varepsilon\tau_n/2$. Thus, by (3.1), we get

$$\begin{aligned} P\left(\sup_x |F_n(x)-F(x)| > \varepsilon\tau_n\right) &\leqslant P\left(\max_{1\leqslant j\leqslant n-1} |F_n(x_{n,j})-F(x_{n,j})| > \varepsilon\tau_n/2\right) \\ &\leqslant \sum_{j=1}^{n-1} P\left(|F_n(x_{n,j})-F(x_{n,j})| > \varepsilon\tau_n/2\right). \end{aligned} \tag{3.2}$$

Set $\xi_i = I(X_i < x_{n,j}) - EI(X_i < x_{n,j})$. By Lemma 3.1, $\{\xi_i\}$ is still END random variables with $E\xi_i = 0$ and $|\xi_i| \leqslant 2$. Taking $t = \varepsilon\tau_n/4$, and by Lemma 3.2, we have

$$\begin{aligned} P\left(|F_n(x_{n,j})-F(x_{n,j})| > \varepsilon\tau_n/2\right) &= P\left(\left|\sum_{i=1}^{n}\xi_i\right| > \varepsilon\tau_n/2\right) \\ &\leqslant 2M\exp\left\{-t\varepsilon n\tau_n/2 + t^2\sum_{i=1}^{n}E\xi_i^2\right\} \\ &\leqslant 2M\exp\left\{-t\varepsilon n\tau_n/2 + nt^2\right\} \\ &\leqslant 2M\exp\left\{-\frac{\varepsilon^2 n\tau_n^2}{16}\right\} \\ &\leqslant 2Mn^{-(2+k_0)}, \ \text{ where } k_0 > 0. \end{aligned} \tag{3.3}$$

Moreover, with relations (3.2) and (3.3), we get

$$P\left(\sup_x |F_n(x)-F(x)| > \varepsilon\tau_n\right) \leqslant 2Mn^{-(1+k_0)}.$$

Therefore, we establish the lemma 3.5. $\square$

4. Proofs of main results

Proof of Theorem 2.1. Since $K(x)$ is a bounded variation function, then there exists two monotone increasing functions $K_1(x)$ and $K_2(x)$, such that $K(x) = K_1(x) - K_2(x)$. Note that $K_1(x)$ and $K_2(x)$ are also bounded variation functions, then we get

$$\begin{aligned}
& n\,(f_n(x) - Ef_n(x)) \\
&= \sum_{j=1}^{n} \frac{1}{h_j}\left[K\left(\frac{x-X_j}{h_j}\right) - EK\left(\frac{x-X_j}{h_j}\right)\right] \\
&= \sum_{j=1}^{n} \frac{1}{h_j}\left[K_1\left(\frac{x-X_j}{h_j}\right) - EK_1\left(\frac{x-X_j}{h_j}\right)\right] - \sum_{j=1}^{n} \frac{1}{h_j}\left[K_2\left(\frac{x-X_j}{h_j}\right) - EK_2\left(\frac{x-X_j}{h_j}\right)\right] \\
&= \sum_{j=1}^{n} Y_j - \sum_{j=1}^{n} Z_j =: S_{n1} - S_{n2}, \qquad (4.1)
\end{aligned}$$

where

$$Y_j = \frac{1}{h_j}\left[K_1\left(\frac{x-X_j}{h_j}\right) - EK_1\left(\frac{x-X_j}{h_j}\right)\right],\ Z_j = \frac{1}{h_j}\left[K_2\left(\frac{x-X_j}{h_j}\right) - EK_2\left(\frac{x-X_j}{h_j}\right)\right].$$

By Lemma 3.1, we see easily that $\{Y_1,\cdots,Y_n\}$ and $\{Z_1,\cdots,Z_n\}$ are still END random variables with $EY_j = EZ_j = 0$ for $j = 1,\cdots,n$. Since $K_1(x)$ and $K_2(x)$ are bounded functions, which implies that

$$EY_j^2 = E\frac{1}{h_j^2}K_1\left(\frac{x-X_j}{h_j}\right)^2 \leqslant C_1\frac{1}{h_j^2} < \infty,\ EZ_j^2 \leqslant C_2\frac{1}{h_j^2} < \infty,\ j = 1,\cdots,n.$$

By $h_n \downarrow 0$, we can obtain

$$\sum_{j=1}^{n} Var(Y_j) \leqslant C_1 \sum_{j=1}^{n} \frac{1}{h_j^2} \leqslant C_1\frac{n}{h_n^2},\ \sum_{j=1}^{n} Var(Z_j) \leqslant C_2 \sum_{j=1}^{n} \frac{1}{h_j^2} \leqslant C_2\frac{n}{h_n^2}.$$

Set $\lambda(n) = [nh_n^2/(\log n(\log\log n)^l)]^{1/2}$, and taking $t = \varepsilon h_n^2(2C_1\lambda(n))$, then by

Lemma 3.2 we get

$$
\begin{aligned}
P\left(\lambda(n)\left|\frac{1}{n}S_{n1}\right|>\varepsilon\right) &= P\left(|S_{n1}|>\frac{n\varepsilon}{\lambda(n)}\right) \\
&\leqslant 2M\exp\left\{-\frac{\varepsilon nt}{\lambda(n)}+\frac{C_1nt^2}{h_n^2}\right\} \\
&= 2M\exp\left\{-\frac{\varepsilon^2nh_n^2}{2C_1\lambda^2(n)}+\frac{\varepsilon^2nh_n^2}{4C_1\lambda^2(n)}\right\} \\
&= 2M\exp\left\{-\frac{\varepsilon^2nh_n^2}{4C_1\lambda^2(n)}\right\} \\
&= 2M\exp\left\{-\frac{\varepsilon^2\log n(\log\log n)^l}{4C_1}\right\}. \\
&\leqslant 2Mn^{-(1+k_0)}.
\end{aligned}
$$

Thus, by Borel-Cantelli lemma, we obtain

$$\frac{\lambda(n)}{n}S_{n1}\to 0,\quad a.s. \tag{4.2}$$

Similarly to the proof of (4.2), we have

$$\frac{\lambda(n)}{n}S_{n2}\to 0,\quad a.s. \tag{4.3}$$

Then, from (4.1), (4.2) and (4.3), it follows that

$$\lambda(n)\left(f_n(x)-Ef_n(x)\right)\to 0,\quad a.s. \tag{4.4}$$

On the other hand, by $h_n=O(n^{-\nu})$ and $\nu\geqslant 1/6$, we get

$$
\begin{aligned}
\frac{\lambda(n)}{n}\sum_{j=1}^{n}h_j^2 &= \frac{h_n}{\sqrt{n\log n(\log\log n)^l}}\sum_{j=1}^{n}h_j^2 \\
&\leqslant C\frac{n^{-\nu}}{\sqrt{n\log n(\log\log n)^l}}\sum_{j=1}^{n}j^{-2\nu} \\
&\leqslant C\frac{n^{-\nu}}{\sqrt{n\log n(\log\log n)^l}}n^{(1-2\nu)^+} \\
&\longrightarrow 0.
\end{aligned}
$$

This, together with Lemma 3.3, we obtain

$$\lambda(n)(Ef_n(x)-f(x))=\frac{\lambda(n)}{n}\sum_{j=1}^{n}h_j^2\cdot\left(\frac{1}{n}\sum_{i=1}^{n}h_i^2\right)^{-1}(Ef_n(x)-f(x))\to 0\ \ a.s. \tag{4.5}$$

Therefore, combining (4.4) and (4.5), we get the result. $\square$

Proof of Theorem 4.2. Set $\xi_j(x) = h_j^{-1}\left[K\left(\frac{x-X_j}{h_j}\right) - EK\left(\frac{x-X_j}{h_j}\right)\right]$, then we can write

$$n(f_n(x) - Ef_n(x)) = \sum_{j=1}^{n} \frac{1}{h_j}\left[K\left(\frac{x-X_j}{h_j}\right) - EK\left(\frac{x-X_j}{h_j}\right)\right] =: S_n.$$

By $K(\cdot)$ is monotone and bounded functions, and using Lemma 3.1, $\{\xi_j(x)\}$ are still END random variables with $E\xi_j(x) = 0$. Then, by Lemma 3.3, we obtain

$$\begin{aligned} EK^2(\frac{x-X_j}{h_j}) &= \int_R K^2(\frac{x-u}{h_j})f(u)du \\ &= h_j\int_R K^2(u)f(x-h_ju)du \leqslant Ch_j. \end{aligned}$$

Hence

$$E\xi_j^2(x) = h_j^{-2}E\left[K\left(\frac{x-X_j}{h_j}\right) - EK\left(\frac{x-X_j}{h_j}\right)\right]^2 \leqslant h_j^{-2}EK^2\left(\frac{x-X_j}{h_j}\right) \leqslant \frac{C}{h_j} < \infty.$$

Note that $h_n \downarrow 0$, then we get

$$\sum_{j=1}^{n} E\xi_j^2(x) \leqslant C\sum_{j=1}^{n}\frac{1}{h_j} \leqslant \frac{C_3 n}{h_n}.$$

Set $\gamma(n) = [nh_n/(\log n(\log\log n)^l)]^{1/2}$. Choosing $t = \varepsilon h_n/(2C_3\gamma(n))$, by Lemma 3.2, then

$$\begin{aligned} P\left(\gamma(n)\left|f_n(x) - Ef_n(x)\right| > \varepsilon\right) &= P(|S_n| > n\varepsilon/\gamma(n)) \\ &\leqslant 2M\exp\left\{-\frac{\varepsilon nt}{\gamma(n)} + \frac{C_3nt^2}{h_n}\right\} \\ &= 2M\exp\left\{-\frac{\varepsilon^2 nh_n}{2C_3\gamma^2(n)} + \frac{\varepsilon^2 nh_n}{4C_3\gamma^2(n)}\right\} \\ &= 2M\exp\left\{-\frac{\varepsilon^2 nh_n}{4C_3\gamma^2(n)}\right\} \\ &= 2M\exp\left\{-\frac{\varepsilon^2\log n(\log\log n)^l}{4C_3}\right\} \\ &\leqslant 2Mn^{-(1+k_0)}. \end{aligned}$$

Hence, applying the Borel-Cantelli Lemma, we have

$$\gamma(n)\left(f_n(x) - Ef_n(x)\right) \to 0, \quad a.s. \tag{4.6}$$

Therefore, from $h_n = O(n^{-\nu})$ for $\nu \geqslant 1/5$,

$$\begin{aligned}
\frac{\gamma(n)}{n}\sum_{j=1}^{n} h_j^2 &= \sqrt{\frac{h_n}{n\log n(\log\log n)^l}}\sum_{j=1}^{n} h_j^2 \\
&\leqslant C\sqrt{\frac{n^{-\nu}}{n\log n(\log\log n)^l}}\sum_{j=1}^{n} j^{-2\nu} \\
&\leqslant \frac{n^{(1-2\nu)^+}}{\sqrt{n^{1+\nu}\log n(\log\log n)^l}} \\
&\longrightarrow 0.
\end{aligned}$$

Thus, from lemma 3.2 we obtain

$$\gamma(n)(Ef_n(x) - f(x)) = \frac{\gamma(n)}{n}\sum_{j=1}^{n} h_j^2 \cdot \left(\frac{1}{n}\sum_{i=1}^{n} h_i^2\right)^{-1}(Ef_n(x) - f(x)) \to 0 \;\; a.s. \qquad (4.7)$$

Therefore, the conclusion follows from (4.6) and (4.7). $\square$

Proof of Theorem 2.3. Set $\overline{F}(x) = 1 - F(x)$, $\overline{F}_n(x) = 1 - F_n(x)$, by (1.2), we get

$$|r_n(x) - r(x)| \leqslant \frac{\overline{F}(x)|f_n(x) - f(x)| + f(x)|\overline{F}_n(x) - \overline{F}(x)|}{\overline{F}_n(x)\overline{F}(x)}. \qquad (4.8)$$

From $0 < \overline{F}(x_0) \leqslant \overline{F}(x) \leqslant 1$ for all $x \leqslant x_0$, $\sup_x f(x) \leqslant C < \infty$, by using Theorem 2.1 and Lemma 3.5, it is easy to see that

$$[nh_n^2/(\log n(\log\log n)^2)]^{1/2}(f_n(x) - f(x)) \to 0, \;\; a.s., \qquad (4.9)$$

and

$$n^{1/2}/((\log n)^{1/2}\log\log n)\sup_{x\leqslant x_0}|F_n(x) - F(x)| \to 0, \;\; a.s.. \qquad (4.10)$$

On the other hand, as n large enough, for $x \leqslant x_0$, we have

$$\overline{F}_n(x) > \overline{F}(x) - \frac{\overline{F}(x_0)}{2} > \frac{\overline{F}(x_0)}{2} > 0.$$

Therefore, the conclusion follows from (4.8), (4.9) and (4.10). $\square$

Proof of Theorem 2.4. Similarly to the proof of Theorem 2.3, one can verify the conclusion. $\square$

REFERENCES

[1] L. LIU, *Precise large deviations for dependent random variables with heavy tails*, Stat. Probabil. Lett, 2009, **79** (9): 1290–1298.

[2] Y. Q. CHEN, A. Y. CHEN, W. NG KAI, *The strong law of large numbers for extended negatively dependent random variables*, J. Appl. Prob., 2010, **47** (4): 908–922.

[3] L. LIU, *Necessary and sufficient conditions for moderate deviations of dependent random variables with heavy tails*, Sci. China Ser. A: Math. 2010, **53** (6): 1421–1434.

[4] A. T. SHEN, *Probability inequalities for END sequence and their applications*, J. Inequal. Appl., 2011, **2011** (98): 1–13.

[5] Y. F. WU, M. GUAN, *Convergence properties of the partial sums for sequences of end random variables*, J. Korean Math. Soc. 2012, **49** (6): 1097–1110.

[6] X. J. WANG, T. C. HU, A. VOLODIN, S. H. HU, *Complete convergence for weighted sums and arrays of rowwise extended negatively dependent random variables*, Communications in Statistics-Theory and Methods, 2013, **42** (13): 2391–2401.

[7] X. J. WANG, L. L. ZHENG, C. XU, S. H. HU, *Complete consistency for the estimator of nonparametric regression models based on extended negatively dependent errors*, Statistics: A Journal of Theoretical and Applied Statistics, 2015, **49** (2): 396–407.

[8] C. T. WOLVERTON, T. J. WAGNER, *Asymptotically optimal discriminant functions for pattern classification*, IEEE Trans Inform Theory, 1969, **15**: 258–265.

[9] H. Y. LIANG, J. BAEK, *Asymptotic normality of recursive density estimates under some dependence assumptions*, Metrika, 2004, **60**: 155–166.

[10] Y. M. LI, S. C. YANG, *Strong convergence rate of recursive probability density estimator for NA sequences*, Chinese J. Engineering Mathematics, 2005, **22** (4): 659–665.

[11] Y. M. LI, C. D. WEI, S. C. YANG, *The recursive kernel distribution function estimator based on negatively and positively associated sequences*, Communications in Statistics-Theory and Methods, 2010, **39** (20): 3585–3595.

[12] S. C. YANG, *Consistency of nearest neighbor estimator of density function for negative associated samples*, Acta Mathematice applicatae Sinica, 2003, **26** (3): 385–395.

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