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ON TWO BIVARIATE ELLIPTIC MEANS

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Abstract. This paper deals with the inequalities involving the Schwab-Borchardt mean SB and a new mean N introduced recently by this author. In particular optimal bounds, for SB are obtained. Inequalities involving quotients N/SB, for the data satisfying certain monotonicity conditions, are derived.

1. Introduction

In recent years means of two variables and their inequalities have attracted attention of several researchers. A complete list of research papers which deal with this subject is too long to be included here. A portion of this list is included in References of this work. In this paper we study two particular bivariate means whose definitions are included below.

In what follows the letters a and b will always stand for positive and unequal numbers.

The first mean investigated in this paper is called the Schwab-Borchardt mean and is defined as follows:

$$SB(a,b) \equiv SB = \begin{cases} \dfrac{\sqrt{b^2-a^2}}{\cos^{-1}(a/b)} & \text{if } a<b, \\ \dfrac{\sqrt{a^2-b^2}}{\cosh^{-1}(a/b)} & \text{if } b<a \end{cases} \tag{1}$$

(see, e.g., [2], [3]). This mean has been studied extensively in [18], [19], and in [8]. It is well known that the mean SB is strict, nonsymmetric and homogeneous of degree one in its variables.

Mean SB can also be represented in terms of the degenerated completely symmetric elliptic integral of the first kind (see, e.g., [15]). It has been pointed out in [18] that some well known bivariate means such as logarithmic mean and two Seiffert means (see [24, 25]) can be represented by the Schwab Borchardt mean of two simpler means such as geometric and arithmetic means or as the Schwab-Borchardt mean of arithmetic

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and the square – mean root mean. This idea was used lately by this author and other researchers as well. For more details see [8, 10, 11, 13, 14, 27, 5, 7, 28, 22, 23]

Another bivariate mean studied in this paper is defined as follows:

$$N(a,b) \equiv N = \frac{1}{2}\left(a + \frac{b^2}{SB(a,b)}\right) \tag{2}$$

(see [15]). It's easy to see that mean N is also strict, nonsymmetric and homogeneous of degree one in its variables. Some authors call this mean, Neuman mean of the second kind (see, e.g., [27, 5, 6, 7, 28, 22, 23, 4]). Mean N can be represented in terms of the degenerated completely symmetric elliptic integral of the second kind (see, e.g., [15]). By taking the N mean of two other means one can generate several new bivariate means. This idea was partially explored in [15].

This paper can be regarded as continuation of investigations initiated in author's earlier papers [8, 10, 12, 11, 15, 13, 14, 16, 17] and is organized as follows. Some preliminary results are given in Section 2. Optimal bounds for the Schwab-Borchardt mean are derived in Section 3. Inequalities involving quotients of means SB and N are established in Section 4.

2. Preliminaries

First of all we will give new formulas for means SB and N. It follows from (1) that

$$SB(a,b) \equiv SB = \begin{cases} b\dfrac{\sin r}{r} = a\dfrac{\tan r}{r} & \text{if } a < b, \\ b\dfrac{\sinh s}{s} = a\dfrac{\tanh s}{s} & \text{if } b < a, \end{cases} \tag{3}$$

where

$$\cos r = a/b \quad \text{if} \quad a < b \quad \text{and} \quad \cosh s = a/b \quad \text{if} \quad a > b. \tag{4}$$

Clearly

$$0 < r \leqslant r_0, \quad \text{where} \quad r_0 = \max\{\cos^{-1}(a/b) : 0 < a < b\} \tag{5}$$

and

$$0 < s \leqslant s_0, \quad \text{where} \quad s_0 = \max\{\cosh^{-1}(a/b) : a > b > 0\} \tag{6}$$

For the later use let us record similar formulas for the mean N. Using (2) and (3) we get

$$N(a,b) \equiv N = \frac{1}{2}b\left(\cos r + \frac{r}{\sin r}\right) = \frac{1}{2}a\left(1 + \frac{r}{\sin r \cos r}\right) \tag{7}$$

provided $a < b$. Similarly, if $a > b$, then

$$N(a,b) \equiv N = \frac{1}{2}b\left(\cosh s + \frac{s}{\sinh s}\right) = \frac{1}{2}a\left(1 + \frac{s}{\sinh s \cosh s}\right). \tag{8}$$

Here the domains for r and s are the same as these in (5) and (6).

To this end the letters a and b will stand for positive and unequal numbers. Also, the symbols G, A, and Q will be used to denote, respectively, the geometric, arithmetic, and the root-square means of a and b. Recall that

$$G=\sqrt{ab}, \quad A=\frac{a+b}{2}, \quad Q=\sqrt{\frac{a^2+b^2}{2}}.$$

For the sake of presentation let us recall definitions of certain means of a and b. Two Seiffert means P and T are defined as follows:

$$P=A\frac{v}{\sin^{-1}v}, \quad T=A\frac{v}{\tan^{-1}v} \tag{9}$$

(see [24] and) and [25]), where $v=\dfrac{a-b}{a+b}$. Clearly $0<|v|<1$. We shall also use the logarithmic mean L and the Neuman-Sándor mean M, introduced in [18] and studied in [21, 10, 12], and [11]. The last two means are defined as follows

$$L=\frac{a-b}{\log a-\log b}=A\frac{v}{\tanh^{-1}v}, \quad M=A\frac{v}{\sinh^{-1}v}. \tag{10}$$

It is known (see [18]) that

$$G<L<P<A<M<T<Q. \tag{11}$$

Thus the means listed in the last chain are comparable. Moreover, four means which appear in (8) and (9) are generated by the Schwab-Borchardt mean. The following result

$$\begin{aligned} L&=SB(A,G), \quad P=SB(G,A),\\ M&=SB(Q,A), \quad T=SB(A,Q) \end{aligned} \tag{12}$$

has been established in [18].

We will apply, on several occasions, the l′Hopital Monotonicity Rule [1]:

Let $c,d\in\mathbb{R}$ $(c<d)$ and let $f,g:[c,d]\to\mathbb{R}$ be continuous functions that are differentiable on (c,d). Assume that $g'(x)\neq 0$ for each $x\in(c,d)$. If f'/g' is increasing (decreasing) on (c,d), then so are $\dfrac{f(x)-f(c)}{g(x)-g(c)}$ and $\dfrac{f(x)-f(d)}{g(x)-g(d)}$.

If monotonicity of f'/g' is strict, then so is monotonicity of two functions represented by the above quotients.

3. Optimal bounds for mean *SB*

The first problem discussed in this paper is formulated as follows:

PROBLEM 1. Find all numbers α and β such that the two-sided inequality

$$b\frac{1+\alpha v}{1-\alpha v}<SB(a,b)<b\frac{1+\beta v}{1-\beta v} \tag{13}$$

is satisfied for all numbers a and b. Here

$$v = \frac{a-b}{a+b}. \tag{14}$$

In order to prove the first result of this section we need two auxiliary functions

$$f_1(r) = \frac{(r-\sin r)(1+\cos r)}{(r+\sin r)(1-\cos r)} =: \frac{n_1(r)}{d_1(r)} \qquad \left(0 < r < \frac{\pi}{2}\right) \tag{15}$$

and

$$f_2(s) = \frac{(\sinh s - s)(\cosh s + 1)}{(\sinh s + s)(\cosh s - 1)} =: \frac{n_2(s)}{d_2(s)} \qquad (s > 0). \tag{16}$$

We have the following:

THEOREM 1. *If $a < b$, then the optimal values α and β must to satisfy*

$$\alpha > \frac{1}{3} \qquad \text{and} \qquad \beta < \lambda := f_1(r_0^-), \tag{17}$$

where r_0 is defined in (5). Otherwise, if $a > b$, then

$$\alpha < \frac{1}{3} \qquad \text{and} \qquad \beta > \mu := f_2(s_0^-), \tag{18}$$

where s_0 is defined in (6).

Proof. Utilizing (3) we rewrite the inequality (13) as follows

$$\frac{1+\alpha v}{1-\alpha v} < \frac{\sin r}{r} < \frac{1+\beta v}{1-\beta v}. \tag{19}$$

Since $a = b\cos r$ (see (4)),

$$v = \frac{\cos r - 1}{\cos r + 1}.$$

This in conjunction with (19) gives

$$\frac{(1+\cos r) - \alpha(1-\cos r)}{(1+\cos r) + \alpha(1-\cos r)} < \frac{\sin r}{r} < \frac{(1+\cos r) - \beta(1-\cos r)}{(1+\cos r) + \beta(1-\cos r)}$$

or what is the same that

$$\beta < f_1(r) < \alpha. \tag{20}$$

Making use of (15) we obtain

$$\frac{n_1'(r)}{d_1'(r)} = \frac{2\sin r - r}{2\sin r + r} =: g_1(r). \tag{21}$$

Differentiation gives

$$g_1'(r) = 4\cos t \frac{r - \tan r}{(2\sin r + r)^2}.$$

Using the well-known inequality [26, 4.18.2]: $x < \tan x$ $(0 < x < \pi/2)$ we conclude that $g_1'(r) < 0$. Thus the function $\dfrac{n_1'(r)}{d_1'(r)}$ is strictly decreasing on its domain. We invoke now l′lHopital Monotonicity Rule to conclude that the function

$$\frac{n_1(r)}{d_1(r)} = f_1(r)$$

is also strictly decreasing. It is easy to verify that $f_1(0^+) = \frac{1}{3}$. This in conjunction with (20) gives

$$\beta < \lambda \leqslant f_1(r) \leqslant \frac{1}{3} < \alpha.$$

Hence (17) follows.

Assume now that $a > b$. It follows from (13) using (3) that

$$\frac{(1+\cosh s) - \alpha(1-\cosh s)}{(1+\cosh s) + \alpha(1-\cosh s)} < \frac{\sinh s}{s} < \frac{(1+\cosh s) - \beta(1-\cosh s)}{(1+\cosh s) + \beta(1-\cosh s)}.$$

A simple algebra yields

$$\alpha < f_2(s) < \beta, \tag{22}$$

where $f_2(s)$ is defined in (16). Differentiation gives

$$\frac{n_2'(s)}{d_2'(s)} = \frac{2\sinh s - s}{2\sinh s + s} =: g_2(s).$$

Differentiating again we obtain

$$g_2'(s) = 4\cosh s \frac{s - \tanh s}{(2\sinh s + s)^2} > 0,$$

where the last inequality is immediate consequence of the well-known one $x > \tanh x$, $(x > 0)$. See, e.g., [26, 4.32.2]. Thus the function $g_2(s)$ is strictly increasing. This in turn implies that the functions $\frac{n_2'(s)}{d_2'(s)}$ and $f_2(s)$ are also strictly increasing for all $s > 0$. Taking into account that $f_2(0^+) = \frac{1}{3}$ we obtain

$$\alpha < \frac{1}{3} \leqslant f_2(s) \leqslant \mu < \beta.$$

The proof is complete. □

Applying Theorem 1 to (2) we obtain

$$\frac{1}{2}\left(a + b\frac{1-\beta v}{1+\beta v}\right) < N(a,b) < \frac{1}{2}\left(a + b\frac{1-\alpha v}{1+\alpha v}\right), \tag{23}$$

where α and β must to satisfy either conditions (17) or (18).

The following corollaries involving double inequalities for bivariate means follow easily from Theorem 1.

COROLLARY 1. *Let G, A, P, Q, and T be the bivariate means of two positive and unequal numbers. (See Section 2). Then the following two-sided inequalities*

$$\frac{A+2G}{2A+G} < \frac{P}{A} < \frac{2A+\pi A}{\pi A+2G} \tag{24}$$

and

$$\frac{Q+2A}{2Q+A} < \frac{T}{Q} < \frac{(1+\lambda)Q+(1-\lambda)A}{(1-\lambda)Q+(1+\lambda)A} \tag{25}$$

hold true. Here

$$\lambda = f_1(\frac{\pi}{4}) = \frac{(\pi-2\sqrt{2})(2+\sqrt{2})}{(\pi+2\sqrt{2})(2-\sqrt{2})} = 0.3057\ldots. \tag{26}$$

Proof. For the proof of (24) we apply Theorem 1 with $a := G$ and $b := A$. Taking into account that $G = A\sqrt{1-v^2}$ we have $\cos r = G/A = \sqrt{1-v^2}$. Since $0 < |v| < 1$, $0 < \cos r < 1$. This yields $0 < r < \pi/2$. Thus $r_0 = \pi/2$. Taking into account that $f_1(0^+) = 1/3 = \alpha$ and also that $f_1(\pi/2) = (\pi-2)/(\pi+2) = \beta$ we obtain the asserted result utilizing formulas (13) and (14) and $SB(G,A) = P$, where the first Seiffert mean P satisfies a second equation of (12). In the proof of (25) we follow the lines introduced above. We let $a := A$ and $b := Q$. Making use of $Q = A\sqrt{1+v^2}$ we see that $1/\sqrt{2} < \cos r < 1$ which yields $0 < r < \pi/4$. Thus $r_0 = \pi/4$. Making use of (13) and (14) with $\alpha = 1/3$, $\beta = \lambda$ we obtain the desired result utilizing a formula $SB(A,Q) = T$ (see (12)). □

COROLLARY 2. *Let G L, A, M, Q be the bivariate means of two positive and unequal numbers. (See Section 2). Then the following two-sided inequalities*

$$\frac{A+2G}{2A+G} < \frac{L}{G} < \frac{A}{G} \tag{27}$$

and

$$\frac{2Q+A}{Q+2A} < \frac{M}{A} < \frac{(1+\mu)Q+(1-\mu)A}{(1-\mu)Q+(1+\mu)A} \tag{28}$$

hold true. Here

$$\mu = \frac{(1-\sinh^{-1}(1))(\sqrt{2}+1)}{(1+\sinh^{-1}(1))(\sqrt{2}-1)} = 0.3675\ldots. \tag{29}$$

Proof. We provide only a sketchy proof of inequalities (27) and (28). In the first case we let $a := A$ and $b := G$. Then $\cosh(s) = a/b = A/G = 1/\sqrt{1-v^2}$. This implies that $0 < s < \infty$. Thus $\alpha = f_2(0^+) = 1/3$ and $\beta = f_2(\infty^-) = 1$. We leave the completion of this proof to the interested reader. Finally for the proof of (28) we let $a := Q$ and $b := A$. Then $\cosh(s) = Q/A = \sqrt{1+v^2}$. This yields $0 < s < \cosh^{-1}(\sqrt{2}) = \sinh^{-1}(1)$. Easy computations yield $\alpha = f_2(0^+) = 1/3$ and $\beta = \mu = f_2(\sinh^{-1}(1))$ where μ is defined in (29). We omit further details. □

4. Inequalities involving quotients of means SB and N

In order to formulate problem discussed in this section let us introduce more notation. Let Φ and Ψ be bivariate means which are homogeneous of degree 1 in both variables. Further let $a = (a_1, a_2)$ and $b = (b_1, b_2)$ be ordered vectors of positive numbers. Assuming that

$$\frac{a_1}{a_2} > \frac{b_1}{b_2} > 1 \tag{30}$$

we ask for which pairs (Φ, Ψ) the following inequality

$$\frac{\Phi(a)}{\Phi(b)} > \frac{\Psi(a)}{\Psi(b)} \tag{31}$$

holds true for all vectors a and b?

It is known that this monotonicity property is satisfied by pairs of Stolarsky means, Gini means and other pairs of means. For more details see [20] and the references therein.

The goal of this section to demonstrate that the means SB and N satisfy (31). In the proof of the main result of this section the following result plays a crucial role. We have

PROPOSITION 1. *If $0 < x < 1$, then the function*

$$f(x) = \frac{N(x,1)}{SB(x,1)} \tag{32}$$

is strictly decreasing on its domain and is strictly increasing for all $x > 1$.

Proof. Let $0 < x < 1$. Using (2) we obtain

$$f(x) = \frac{\frac{1}{2}\left(x + \frac{1}{SB(x,1)}\right)}{SB(x,1)} = \frac{1}{2}\frac{xSB(x,1)+1}{SB^2(x,1)}.$$

Making use of (3) we write an expression for f as

$$f = \frac{1}{2}\frac{r(\sin r\cos r + r)}{\sin^2 r} =: \frac{1}{2}g(r) =: \frac{1}{2}\frac{n(r)}{d(r)}, \tag{33}$$

where $\cos r = x$. Differentiation yields

$$(\sin 2r)^2\left(\frac{n'(r)}{d'(r)}\right)' = 2r\Big[\frac{\sin 2r}{2r}(2+\cos 2r) - (1+2\cos 2r)\Big] =: h(r). \tag{34}$$

Using the inequality $\frac{\sin 2r}{2r} > \left(\frac{1+2\cos 2r}{3}\right)^{1/2}$ (see [21, 9]) we obtain

$$h(r) > 2r(1+2\cos 2r)\lambda(r), \tag{35}$$

where

$$\lambda(r) = \frac{2+\cos 2r}{\sqrt{3(1+2\cos 2r)}} - 1. \tag{36}$$

To prove that $\lambda(r) > 0$ it suffices to show that

$$(2+\cos 2r)^2 > 3(1+2\cos 2r).$$

It is easy to see that the last inequality can be written as $(1-\cos 2r)^2 > 0$. Using (36) and (35) we obtain $h(r) > 0$. This and (34) implies that $h(r) > 0$. This in turn implies that the function $n'(r)/d'(r)$ is strictly increasing. Utilizing l$'$Hopital Monotonicity Rule and (33) we arrive at the conclusion that the function $g(r)$ is strictly increasing. To obtain the asserted result we use (33) and the fact that $g'(r) > 0$ to obtain $f'(x) = \frac{1}{2}g'(r)(-\sin r) < 0$. Thus the function $f(x)$ is strictly decreasing if $0 < x < 1$.

Assume now that $x > 1$. With $f(x)$ as defined in (32) we have

$$f(x) = \frac{1}{2}\frac{xSB(x,1)+1}{SB^2(x,1)}.$$

Application of (3) gives

$$f = \frac{1}{2}\frac{s(\sinh s\cosh s + s)}{\sinh^2 s} =: \frac{1}{2}g(s) =: \frac{n(s)}{d(s)}. \tag{37}$$

Differentiation yields

$$\left(\frac{n'(s)}{d'(s)}\right)' = \frac{\mu(s)}{2(\sinh s\cosh s)^2}, \tag{38}$$

where

$$\mu(s) = 2\sinh s\cosh^3 s - 4s\cosh^2 s + \sinh s\cosh s + s.$$

Hence

$$\mu'(s) = 8(s\cosh s)^2\left[\left(\frac{\sinh s}{s}\right)^2 - \frac{\tanh s}{s}\right].$$

Using the well known inequality

$$\frac{\tanh s}{s} < 1 < \frac{\sinh s}{s}$$

$(s \neq 0)$ we conclude that $\mu'(s) > 0$. Taking into account that $\mu(0) = 0$ and also using the fact that $\mu(s)$ is strictly increasing we obtain $\mu(s) > 0$ on its domain. This in turn implies (see (38)) that the function $\frac{n'(s)}{d'(s)}$ is also strictly increasing. Making use of l$'$Hopital Monotonicity Rule and (37) we see that the function $g(s)$ is also strictly increasing. Utilizing (37) we obtain

$$f'(x) = \frac{1}{2}g'(s)\sinh s > 0$$

$(s > 0)$. This completes the proof. $\square$

We are in a position to prove the main result of this section.

THEOREM 2. *Let*

$$a = (a_1, a_2) \qquad \text{and} \qquad b = (b_1, b_2)$$

satisfy inequalities (30). Then

$$\frac{N(a)}{SB(a)} > \frac{N(b)}{SB(b)}. \tag{39}$$

Proof. Let

$$x = \frac{a_1}{a_2} \qquad \text{and} \qquad y = \frac{b_1}{b_2}. \tag{40}$$

It follows from (30) that $x > y > 1$. Monotonicity of the function N/SB yields

$$\frac{N(x,1)}{SB(x,1)} > \frac{N(y,1)}{SB(y,1)}.$$

Using x and y as defined in (40) and next multiplying numerator and denominator of the first quotient by a_2 and also multiplying numerator and denominator of the second quotient by b_2, we obtain the assertion utilizing the fact that all means are homogeneous of degree 1. $\square$

It is worth mentioning that inequality (39) is also satisfied if components of a and b are permuted, i.e., if $a = (a_2, a_1)$ and $b = (b_2, b_1)$. We omit further details.

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