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EXTENSION OF DETERMINANTAL INEQUALITIES OF POSITIVE DEFINITE MATRICES

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Abstract. In this short note, we extend some known determinantal inequalities of positive definite matrices to a larger class of matrices, namely, matrices whose numerical range is contained in a sector.

1. Introduction

Let $\mathbb{M}_n(\mathbb{C})$ denote the set of $n\times n$ complex matrices. For $A\in\mathbb{M}_n(\mathbb{C})$, the conjugate transpose of A is denoted by A^*, and recall the Cartesian decomposition ([8, p. 6]) $A=\Re A+i\Im A$, where $\Re A=\frac{1}{2}(A+A^*)$ and $\Im A=\frac{1}{2i}(A-A^*)$. For two Hermitian matrices $A,B\in\mathbb{M}_n(\mathbb{C})$, $A\prec B$ $(A\preceq B)$ means that $B-A$ is positive definite (semidefinite). In particular, a positive definite (positive semidefinite) matrix A can be expressed as $A\succ 0$ $(A\succeq 0)$. We also consider $A\in\mathbb{M}_n(\mathbb{C})$ to be partitioned as

$$A=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{bmatrix}, \tag{1.1}$$

where diagonal blocks are square matrices. $\operatorname{diag}(D_1,\ldots,D_k)$ denotes the block diagonal matrix whose diagonal blocks are $D_1,\ldots,D_k$. In (1.1), if A_{11} is nonsingular, the Schur complement of A_{11} in A is defined by $A/A_{11}=A_{22}-A_{21}A_{11}^{-1}A_{12}$.

The numerical range of $A\in\mathbb{M}_n(\mathbb{C})$ is defined by

$$W(A)=\{x^*Ax|x\in\mathbb{C}^n,x^*x=1\}.$$

For $\alpha\in[0,\frac{\pi}{2})$, let S_α be the sector in the complex plane given by

$$S_\alpha=\{z\in\mathbb{C}|\Re z>0,|\Im z|\preceq(\Re z)\tan\alpha\}=\{re^{i\theta}|r>0,|\theta|\leqslant\alpha\}.$$

Clearly, if A is positive definite, $W(A)\subset S_0$.

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Paper JMI-11-31

For fundamentals of numerical range, the readers can refer to [9, 10]. As $0 \notin S_\alpha$, if $W(A) \subset S_\alpha$, A is necessarily nonsingular.

Based on a recent result of Lin [3], Choi [1] proved the following determinantal inequality for positive definite matrices.

THEOREM 1.1. [1] *Let A_i, $i = 1, \ldots, m$, be positive definite matrices whose diagonal blocks are n_j-square matrices $A_i^{(j)}$ for $j = 1, \ldots, k$. Then*

$$\det\Big(\sum_{i=1}^{m} A_i^{-1}\Big) \geqslant \det\Big(\sum_{i=1}^{m} (A_i^{(1)})^{-1}\Big) \cdots \det\Big(\sum_{i=1}^{m} (A_i^{(k)})^{-1}\Big). \tag{1.2}$$

In [7], Haynsworth gave the result for the Schur complement of 2×2 block matrices.

THEOREM 1.2. [7] *Suppose $A, B \in \mathbb{M}_n(\mathbb{C})$ are Hermitian matrices, partitioned as in (1.1), $A = (A_{ij})$, $B = (B_{ij})$, $i, j = 1, 2$, where A_{11} and B_{11} are square of order m. If $A \succeq 0$, $B \succeq 0$, $A_{11} \succ 0$, $B_{11} \succ 0$, then*

$$\det((A+B)/(A_{11}+B_{11})) \geqslant \det A/\det A_{11} + \det B/\det B_{11}. \tag{1.3}$$

This result (1.3) has been extended in [4]. In this paper, we extend the results (1.2) and (1.3) to the case of matrices whose numerical ranges are contained in a sector.

2. Main results

The following lemmas are useful for proving the above theorems.

LEMMA 2.1. [2, p. 68] *Let $A \in \mathbb{M}_n(\mathbb{C})$. If $\Re A$ is positive definite, then*

$$\det(\Re A) \leqslant |\det(A)|. \tag{2.1}$$

LEMMA 2.2. [5, 6] *Let $A \in \mathbb{M}_n(\mathbb{C})$ with $W(A) \subset S_\alpha$. Then A can be decomposed as $A = XZX^*$ for some invertible matrix $X \in \mathbb{M}_n(\mathbb{C})$ and $Z = \operatorname{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n})$ with $|\theta_j| \leqslant \alpha$ for all j.*

LEMMA 2.3. [4, Lemma 2.6] *Let $A \in \mathbb{M}_n(\mathbb{C})$ with $W(A) \subset S_\alpha$. Then*

$$\sec^n(\alpha)\det(\Re A) \geqslant |\det(A)|. \tag{2.2}$$

Proof. By Lemma 2.2, we have $A = XZX^*$ for some invertible matrix $X \in \mathbb{M}_n(\mathbb{C})$ and $Z = \operatorname{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n})$ with $|\theta_j| \leqslant \alpha$ for all j. Thus, $|\det(A)| = |\det(XZX^*)|$. After dividing by $|\det X|^2$, the inequality holds: $|\det A|/|\det X|^2 = |\det Z| = 1$. Furthermore, $\sec^n \alpha \det(\Re Z) \geqslant 1$. Therefore, $\sec^n \alpha \det(\Re Z) \geqslant |\det A|/|\det X|^2$. $\square$

LEMMA 2.4. [4, Lemma 2.4] *Let $A \in \mathbb{M}_n(\mathbb{C})$ with $\mathfrak{R}A$ positive definite. Then*

$$(\mathfrak{R}A)^{-1} \succeq \mathfrak{R}(A^{-1}). \tag{2.3}$$

Now we present some generalizations of Theorem 1.1 and Theorem 1.2. First of all, Theorem 1.1 is extended to the class of matrices whose numerical ranges are contained in a sector as follows:

THEOREM 2.5. *Let $A_i,\ i=1,\dots,m,$ be a sequence of $n\times n$ matrices whose diagonal blocks are n_j-square matrices $A_i^{(j)}$ for $j=1,\dots,k$. Assume that $W(A_i)\subset S_\alpha$, $i=1,\dots,m,\ \alpha\in[0,\frac{\pi}{2})$. Then*

$$|\det(\sum_{i=1}^m A_i^{-1})| \geqslant \cos^{3n}(\alpha)|\det(\sum_{i=1}^m (A_i^{(1)})^{-1})|\cdots|\det(\sum_{i=1}^m (A_i^{(k)})^{-1})|. \tag{2.4}$$

Proof. Compute

$$\begin{aligned}
|\det(\sum_{i=1}^m A_i^{-1})| &\geqslant \det(\mathfrak{R}(\sum_{i=1}^m A_i^{-1})) = \det(\mathfrak{R}(A_1^{-1}+\cdots+A_m^{-1})) && \text{(by (2.1))}\\
&= \det\mathfrak{R}((X_1^{-1})^*Z_1^{-1}X_1^{-1}+\cdots+(X_m^{-1})^*Z_m^{-1}X_m^{-1}) && \text{(by Lemma 2.2)}\\
&= \det((X_1^{-1})^*\mathfrak{R}(Z_1^{-1})X_1^{-1}+\cdots+(X_m^{-1})^*\mathfrak{R}(Z_m^{-1})X_m^{-1})\\
&\geqslant \det(\cos^2(\alpha)(X_1^{-1})^*(\mathfrak{R}Z_1)^{-1}X_1^{-1}+\cdots+\cos^2(\alpha)(X_m^{-1})^*(\mathfrak{R}Z_m)^{-1}X_m^{-1})\\
&\qquad\qquad \text{(by } \mathfrak{R}(Z^{-1})\succeq\cos^2(\alpha)(\mathfrak{R}Z)^{-1})\\
&= \cos^{2n}(\alpha)\det((X_1^{-1})^*(\mathfrak{R}Z_1)^{-1}X_1^{-1}+\cdots+(X_m^{-1})^*(\mathfrak{R}Z_m)^{-1}X_m^{-1})\\
&= \cos^{2n}(\alpha)\det((X_1\mathfrak{R}Z_1X_1^*)^{-1}+\cdots+(X_m\mathfrak{R}Z_mX_m^*)^{-1})\\
&\geqslant \cos^{2n}(\alpha)\det(\sum_{i=1}^m(\mathfrak{R}A_i^{(1)})^{-1})\cdots\det(\sum_{i=1}^m(\mathfrak{R}A_i^{(k)})^{-1}) && \text{(by (1.2))}\\
&\geqslant \cos^{2n}(\alpha)\det(\sum_{i=1}^m\mathfrak{R}((A_i^{(1)})^{-1}))\cdots\det(\sum_{i=1}^m\mathfrak{R}((A_i^{(k)})^{-1})) && \text{(by (2.3))}\\
&\geqslant \cos^{3n}(\alpha)|\det(\sum_{i=1}^m(A_i^{(1)})^{-1})|\cdots|\det(\sum_{i=1}^m(A_i^{(k)})^{-1})|, && \text{(by (2.2))}
\end{aligned}$$

where $X_i\ (i=1,\dots,m)$ and $Z_i\ (i=1,\dots,m)$ correspond to the invertible matrices and diagonal matrices in Lemma 2.2, respectively. □

REMARK 2. When $\alpha=0$, our Theorem 2.5 reduces to Theorem 1.1.

Next, Theorem 1.2 is extended to the class of matrices whose numerical ranges are contained in a sector as follows.

THEOREM 2.6. *Let* $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \in \mathbb{M}_n(\mathbb{C})$ *with* $A_{11}, B_{11} \in \mathbb{M}_m(\mathbb{C})$ *and* $W(A), W(B) \subset S_\alpha$, $\alpha \in [0, \frac{\pi}{2})$, *then*

$$\frac{|\det(A+B)|}{|\det(A_{11}+B_{11})|} \geqslant \cos^{3(n-m)} \alpha \left(\frac{|\det A|}{|\det A_{11}|} + \frac{|\det B|}{|\det B_{11}|} \right). \tag{2.5}$$

Proof. Compute

$$\begin{aligned}
\frac{|\det(A+B)|}{|\det(A_{11}+B_{11})|} &\geqslant \frac{\det \Re(A+B)}{|\det(A_{11}+B_{11})|} && \text{(by (2.1))} \\
&\geqslant \cos^{2(n-m)} \alpha \det(\Re(A/A_{11}) + \Re(B/B_{11})) && \text{(by [4, Theorem 3.1])} \\
&\geqslant \cos^{2(n-m)} \alpha (\det(\Re(A/A_{11})) + \det(\Re(B/B_{11}))) \\
&\geqslant \cos^{3(n-m)} \alpha \left(|\det(A/A_{11})| + |\det(B/B_{11})|\right). && \text{(by (2.2))} \quad \square
\end{aligned}$$

REMARK 3. When $\alpha = 0$, our Theorem 2.6 reduces to Haynsworth's result (Theorem 1.2). In Theorem 2.6, if $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in \mathbb{M}_{k+1}(\mathbb{C})$ with $A_{11} \in \mathbb{M}_k(\mathbb{C})$, then

$$\sec^3 \alpha \left| \frac{\det(A_{k+1}+B_{k+1})}{\det(A_k+B_k)} \right| \geqslant \left| \frac{\det A_{k+1}}{\det A_k} \right| + \left| \frac{\det B_{k+1}}{\det B_k} \right|,$$

which is the result in [4, (4.1)]. Thus, our result is a generalization of (4.1) in [4].

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REFERENCES

[1] D. CHOI, *Determinantal inequalities of positive definite matrices*, Math. Inequal. Appl., Preprint.

[2] S. G. WANG, M. X. WU, Z. Z. JIA, *Matrix Inequalities*, 2nd ed., Science Press, Beijing, 2004 (in Chinese).

[3] M. LIN, *Determinantal inequalities for block triangular matrices*, Math. Inequal. Appl. **18** (2015) 1079–1086.

[4] M. LIN, *Extension of a result of Haynsworth and Hartfiel*, Arch. Math. **104** (2015) 93–100.

[5] S. W. DRURY, *Fischer determinantal inequalities and Higham's Conjecture*, Linear Algebra Appl. **439** (2013) 3129–3133.

[6] F. ZHANG, *A matrix decomposition and its applications*, Linear Multilinear Algebra **63** (2015), no. 10, 2038–2042.

[7] E. V. HAYNSWORTH, *Applications of an inequality for the Schur complement*, Proc. Amer. Math. Soc. **24** (1970) 512–516.

[8] R. BHATIA, *Matrix Analysis*, Grad. Texts in Math., vol. **169**, Springer-Verlag, New York, 1997.

[9] K. E. GUSTAFSON, D. K. M. RAO, *Numerical Range: The Field of Values of Linear Operators and Matrices*, Springer, New York, 1997.
[10] R. A. HORN, C. R. JOHNSON, *Topics in Matrix Analysis*, Cambridge University Press, 1997.

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