

## ON THE ROLE OF ULTRAMODULARITY AND SCHUR CONCAVITY IN THE CONSTRUCTION OF BINARY COPULAS

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*Abstract.* We discuss and stress the role of ultramodularity and Schur concavity in special types of constructions of copulas. After recalling some known ultramodularity-based results, we focus on the so-called  $D$ -product of a copula and its dual. We show that for each copula  $D$  which is ultramodular and Schur concave on the left upper triangle of the unit square, this  $D$ -product of an arbitrary copula and its dual is again a copula. Several examples and counterexamples are given. Finally, some of our results are generalized to the case of semicopulas and quasi-copulas.

### 1. Introduction

Quasi-copulas [2, 24] and copulas [64, 59] are special binary aggregation functions. Copulas play a significant role in probability theory [30, 49], in particular in dependence modeling, as a consequence of Sklar's Theorem [19, 64]. Other related areas are generalized integration theory [36], decision theory [69], finance [9, 26], preference modeling [12, 15], but also fuzzy logics and the theory of fuzzy sets [13, 29, 56].

Several construction methods for (quasi-)copulas based on some a priori given (quasi-)copulas can be found in the literature, some of which are:

- (i) Archimedean copulas which are isomorphic transformations of the product copula  $\Pi$  (*strict Archimedean copulas*) or the Fréchet-Hoeffding lower bound  $W$  (*non-strict Archimedean copulas*) [1, 29, 41, 47, 49, 59];
- (ii) several types of ordinal sums, such as  $M$ -ordinal sums,  $W$ -ordinal sums and  $\Pi$ -ordinal sums [1, 20, 32, 46, 49, 58, 59];
- (iii) the product of *distorted copulas* [31, 40].

We start here with a construction where a copula and its dual are combined by means of another copula  $D$  (the so-called  $D$ -product, see Definition 3.1 below) which recently was shown to yield always a copula if, e.g., for  $D$  we take the product copula  $\Pi$  [16, 38].

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Our main aim is to identify copulas  $D$  such that, for an arbitrary copula, the  $D$ -product is again a copula. It turns out that in these investigations two distinguished inequalities for real functions play a key role: ultramodularity [42, 43] and Schur concavity [57] (see also [45, 55]).

Ultramodular real functions are applied in several areas, and they are also known under different names. In the case of an  $n$ -dimensional real domain, ultramodularity can be seen as a version of convexity. Under mild regularity assumptions, the set of ultramodular functions equals the intersection of the set of all supermodular functions and the set of all functions which are convex in each variable. Ultramodular functions have been used in economics, in particular in game theory when dealing with convex measure games [3], but they also have applications in multicriteria decision support systems [6]. In mathematical analysis, to the best of our knowledge ultramodular functions first appeared in [68] where they just were called convex functions, and some authors use the term Wright convexity for them [54]. In statistics, ultramodular functions play an important role in modelling stochastic orders and positive dependence among random vectors (see [48, 60]), and they are known there also as directional convex functions. For more details about ultramodular real functions we recommend [42]. Ultramodular binary copulas, characterized by the convexity of all of their horizontal and vertical sections, were studied recently in [33, 34]. An ultramodular copula describes the dependence structure of stochastically decreasing random vectors (see [49]), and thus each ultramodular copula is negative quadrant dependent (NQD).

The concepts of Schur convex functions (and Schur concave functions as their duals) were introduced in [57] as variants of convexity and concavity of real functions (see also [55]). In fact, each symmetric convex function is Schur convex (and each symmetric concave function is Schur concave). Within the class of binary copulas, each Schur concave copula is necessarily symmetric, and each associative copula is Schur concave [21]. Schur convex functions are used in the study of majorization [44], a preorder on vectors of real numbers, and inequalities related to it [61]. One of the early origins of majorization is in comparisons of income inequality, and it has applications also in physics, chemistry, political science, engineering, and economics [45]. An example of a Schur convex function is the maximum. The minimum and the product (the latter only in the case of strictly positive factors) are Schur concave, as well as all elementary symmetric functions (again only if all components are strictly positive) [62, 63]. In the framework of stochastics and aggregation functions [28], the variance and the standard deviation are Schur convex, whereas the Shannon entropy function, the Rényi entropy function and the Gini coefficient are Schur concave [4, 27, 39, 52, 53].

The paper is organized as follows. In the following section, some basic notions from the theory of aggregation, quasi-copulas and copulas are given, including ultramodularity and Schur concavity. Moreover, a copula construction method based on ultramodular copulas as introduced in [34] is recalled. Our main results in Section 3 are devoted to the so-called  $D$ -product of a copula and its dual, where  $D$  is a copula which is ultramodular and Schur concave on a suitable subset of the unit square. In Section 4 several examples and counterexamples are collected. Some generalizations of our results to the case of semicopulas and quasi-copulas are given in Section 5.

### 2. Binary aggregation functions, quasi-copulas and copulas

A (binary) aggregation function [28] is a function  $A: [0, 1]^2 \rightarrow [0, 1]$  which is monotone non-decreasing (in each component) and satisfies  $A(0, 0) = 0$  and  $A(1, 1) = 1$ .

We often require an aggregation function  $A$  to be 1-Lipschitz, i.e., for all numbers  $x_1, x_2, y_1, y_2 \in [0, 1]$

$$|A(x_1, y_1) - A(x_2, y_2)| \leq |x_1 - x_2| + |y_1 - y_2|.$$

Given a binary 1-Lipschitz aggregation function  $A: [0, 1]^2 \rightarrow [0, 1]$ , its dual function [28]  $A^*: [0, 1]^2 \rightarrow [0, 1]$  is defined by

$$A^*(x, y) = x + y - A(x, y).$$

Each 1-Lipschitz aggregation function  $A$  satisfies

$$W \leq A \leq W^*, \tag{2.1}$$

where the Fréchet-Hoeffding lower bound  $W$  is given by  $W(x, y) = \max(x + y - 1, 0)$ , and its dual  $W^*$  by  $W^*(x, y) = \min(x + y, 1)$ .

A (binary) semicopula (see [5, 22]) is an aggregation function  $S: [0, 1]^2 \rightarrow [0, 1]$  which satisfies  $S(0, x) = S(x, 0) = 0$  and  $S(1, x) = S(x, 1) = x$  for all  $x \in [0, 1]$ .

A (binary) quasi-copula (see [2, 24])  $Q: [0, 1]^2 \rightarrow [0, 1]$  is a 1-Lipschitz semicopula.

A 1-Lipschitz aggregation function  $A: [0, 1]^2 \rightarrow [0, 1]$  is a quasi-copula if and only if  $A(0, 1) = A(1, 0) = 0$  (see [37]) or, equivalently, if and only if  $A \leq M$ , where the Fréchet-Hoeffding upper bound  $M$  is given by  $M(x, y) = \min(x, y)$ .

To simplify some formulas, we shall also use the infix notations  $x \wedge y$  for  $\min(x, y)$  and  $x \vee y$  for  $\max(x, y)$ .

A (binary) copula  $C: [0, 1]^2 \rightarrow [0, 1]$  (see [64, 49, 30]) is a supermodular semicopula, i.e., for all  $\mathbf{x}, \mathbf{y} \in [0, 1]^2$

$$C(\mathbf{x} \vee \mathbf{y}) + C(\mathbf{x} \wedge \mathbf{y}) \geq C(\mathbf{x}) + C(\mathbf{y}). \tag{2.2}$$

Equivalently, a function  $C: [0, 1]^2 \rightarrow [0, 1]$  is a binary copula if and only if we have  $C(0, x) = C(x, 0) = 0$  and  $C(1, x) = C(x, 1) = x$  for all  $x \in [0, 1]$ , and if  $C$  is 2-increasing, i.e., for all  $x_1, x_2, y_1, y_2 \in [0, 1]$  with  $x_1 \leq x_2$  and  $y_1 \leq y_2$  for the volume  $V_C$  of the rectangle  $[x_1, x_2] \times [y_1, y_2]$

$$V_C([x_1, x_2] \times [y_1, y_2]) = C(x_1, y_1) - C(x_1, y_2) + C(x_2, y_2) - C(x_2, y_1) \geq 0. \tag{2.3}$$

Obviously, each copula is a quasi-copula but not vice versa. Each quasi-copula  $Q$  satisfies

$$W \leq Q \leq M, \tag{2.4}$$

and the same holds for copulas.

Observe that, for each quasi-copula  $Q$ ,  $1$  is not a neutral element of  $Q^*$ , so the dual of a quasi-copula is never a quasi-copula (nor is the dual of a copula a copula).

For a copula  $C$ , also the *co-copula* [1, 59]  $\bar{C}: [0, 1]^2 \rightarrow [0, 1]$  given by

$$\bar{C}(x, y) = 1 - C(1 - x, 1 - y)$$

is considered. Note that  $\bar{C}$  is never a copula.

The well-known *Frank functional equation* [23, 7] can be formulated as follows: find all associative copulas  $F$  and  $G$  such that  $F = (\bar{G})^*$ . Observe that a related problem in the framework of triangular norms and conorms (see [35]) can be formulated as follows: find all 1-Lipschitz triangular norms (i.e., associative copulas)  $F$  and  $G$  such that  $F = (\bar{G})^*$ .

A copula  $C: [0, 1]^2 \rightarrow [0, 1]$  is called *Archimedean* if there is a continuous, strictly decreasing convex function  $t: [0, 1] \rightarrow [0, \infty]$  with  $t(1) = 0$  such that for all  $(x, y) \in [0, 1]^2$  (see [47])

$$C(x, y) = t^{-1}(\min(t(x) + t(y), t(0))). \tag{2.5}$$

The function  $t$  is called an *additive generator* of  $C$ , and it is unique up to a positive multiplicative constant. Note that an Archimedean copula is necessarily associative and satisfies  $C(x, x) < x$  for all  $x \in ]0, 1[$ .

### 2.1. Ultramodular functions and ultramodular copulas

Given  $A \subseteq \mathbb{R}^n$ , a function  $f: A \rightarrow \mathbb{R}$  is called *ultramodular* [42] if its increments are monotone non-decreasing, i.e., if for all  $\mathbf{x}, \mathbf{y} \in A$  with  $\mathbf{x} \leq \mathbf{y}$  and all  $\mathbf{h} \geq \mathbf{0}$  such that  $\mathbf{x} + \mathbf{h}, \mathbf{y} + \mathbf{h} \in A$  we have

$$f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) \leq f(\mathbf{y} + \mathbf{h}) - f(\mathbf{y}). \tag{2.6}$$

If  $n = 2$  and  $A = [0, 1]^2$ , then a copula  $C[0, 1]^2 \rightarrow [0, 1]$  is ultramodular [33, 34] if and only if for all  $\mathbf{x}, \mathbf{y}, \mathbf{z} \in [0, 1]^2$  satisfying  $\mathbf{x} + \mathbf{y} + \mathbf{z} \in [0, 1]^2$  we have

$$C(\mathbf{x} + \mathbf{y} + \mathbf{z}) + C(\mathbf{x}) \geq C(\mathbf{x} + \mathbf{y}) + C(\mathbf{x} + \mathbf{z}). \tag{2.7}$$

Note that ultramodular copulas are just copulas with convex horizontal and vertical sections.

Out of the three basic copulas  $W$ ,  $M$  and the product copula  $\Pi$  given by  $\Pi(x, y) = x \cdot y$ , only  $W$  and  $\Pi$  are ultramodular. However, the upper Fréchet-Hoeffding bound  $M$  is ultramodular on the upper left triangle

$$\Delta = \{(x, y) \in [0, 1]^2 \mid x \leq y\},$$

so Theorem 3.1 in Section 3 will apply to  $M$  as well. The following result is a consequence of [34, Theorem 3.1] (see also [8]).

**THEOREM 2.1.** *Let  $C: [0, 1]^2 \rightarrow [0, 1]$  be an Archimedean copula with a two times differentiable additive generator  $t: [0, 1] \rightarrow [0, \infty]$ . Then  $C$  is ultramodular if and only if  $t'$  is constant or  $\frac{1}{t}$  is a convex function.*

If  $(C_i)_{i \in I}$  is a family of copulas and  $(]a_i, e_i[)_{i \in I}$  a family of non-empty, pairwise disjoint open subintervals of  $[0, 1]$ , the *ordinal sum*  $M-\langle (a_i, e_i, C_i) \rangle_{i \in I}$  of the *summands*  $\langle a_i, e_i, C_i \rangle$ ,  $i \in I$  (which we shall call here *M-ordinal sum*) is well-known from the literature [58, 41, 23, 35]), and it is based on a result in the theory of abstract semi-groups [11]. In this construction, the “gaps” between the squares  $[a_i, e_i]^2$  are filled by the upper Fréchet-Hoeffding bound  $M$ , and the result is always a copula. Another ordinal sum construction based on the lower Fréchet-Hoeffding bound  $W$  was considered more recently in [14, 20, 32, 46].

Let  $(C_i)_{i \in I}$  be a family of copulas and  $(]a_i, e_i[)_{i \in I}$  be a family of non-empty, pairwise disjoint open subintervals of  $[0, 1]$ . The *W-ordinal sum* of the *summands*  $\langle a_i, e_i, C_i \rangle$ ,  $i \in I$ , is the copula  $C = W-\langle (a_i, e_i, C_i) \rangle_{i \in I}$  defined by

$$C(x, y) = \begin{cases} a_i + (e_i - a_i) \cdot C_i\left(\frac{x - a_i}{e_i - a_i}, \frac{y - 1 + e_i}{e_i - a_i}\right) & \text{if } (x, y) \in [a_i, e_i[ \times [1 - e_i, 1 - a_i[, \\ W(x, y) & \text{otherwise.} \end{cases} \tag{2.8}$$

**PROPOSITION 2.2.** *Let  $(C_i)_{i \in I}$  be a family of copulas and  $(]a_i, e_i[)_{i \in I}$  be a family of non-empty, pairwise disjoint open subintervals of  $[0, 1]$ . The W-ordinal sum  $W-\langle (a_i, e_i, C_i) \rangle_{i \in I}$  is ultramodular if and only if, for each  $i \in I$ , the copula  $C_i$  is ultramodular.*

As a consequence of [18, Theorem 2] and [34, Theorem 4.1] we get:

**THEOREM 2.3.** *Let  $C_1, C_2, D: [0, 1]^2 \rightarrow [0, 1]$  be copulas and assume that  $D$  is ultramodular. Then, for all monotone non-decreasing functions  $f_1, f_2, g_1, g_2: [0, 1] \rightarrow [0, 1]$  satisfying*

$$D(f_1(x), f_2(x)) = D(g_1(x), g_2(x)) = x$$

for all  $x \in [0, 1]$ , also the function  $E: [0, 1]^2 \rightarrow [0, 1]$  given by

$$E(x, y) = D(C_1(f_1(x), g_1(y)), C_2(f_2(x), g_2(y))) \tag{2.9}$$

is a copula.

**EXAMPLE 2.4.** Here are two examples of the construction in Theorem 2.3:

- (a) For each copula  $C$  and all  $\alpha, \beta \in [0, 1]$  the function  $E: [0, 1]^2 \rightarrow [0, 1]$  given by  $E(x, y) = C(x^\alpha, y^\beta) \cdot C(x^{1-\alpha}, y^{1-\beta})$  is a copula (this result was obtained independently in [31], see also [40]). Putting  $C = W$  and  $\alpha = \beta = 0.5$ , we obtain the *Clayton copula* with parameter  $-0.5$  (see [10, 25, 49]) given by  $C_{-0.5}(x, y) = (\max(\sqrt{x} + \sqrt{y} - 1, 0))^2$ .
- (b) If  $f, g: [0, 1] \rightarrow [0, 1]$  are monotone non-decreasing 1-Lipschitz functions satisfying  $f, g \geq \text{id}_{[0,1]}$  and if  $C_1, C_2$  are copulas, then also the function  $E: [0, 1]^2 \rightarrow [0, 1]$  given by

$$E(x, y) = W(C_1((f(x), g(y))), C_2((1 + x - f(x), 1 + y - g(y))))$$

is a copula. In particular, if we put  $C_1 = C_2 = \Pi$  and  $f(x) = g(x) = \frac{x+1}{2}$  then we obtain the *Sugeno-Weber copula* with parameter 0.5 (see [66, 67, 35]) given by  $E(x, y) = \max(\frac{xy+x+y-1}{2}, 0)$ .

However, if the copula  $D$  in Theorem 2.3 fails to be ultramodular, then the function constructed via (2.9) may not even be a quasi-copula:

EXAMPLE 2.5. Put  $C = \Pi$  and  $D = M$  and recall that  $M$  is not ultramodular. Define the functions  $f_1, f_2: [0, 1] \rightarrow [0, 1]$  by

$$f_1(x) = \begin{cases} x & \text{if } x \in [0, \frac{1}{2}], \\ 5x - 2 & \text{if } x \in ]\frac{1}{2}, \frac{3}{5}], \\ 1 & \text{otherwise,} \end{cases} \quad f_2(x) = \max(x, \frac{1}{2}),$$

and put  $g_1 = f_1$  and  $g_2 = f_2$ . Clearly,  $M(f_1(x), f_2(x)) = M(g_1(x), g_2(x)) = x$  for all  $x \in [0, 1]$ . But for the function  $E: [0, 1]^2 \rightarrow [0, 1]$  given by

$$E(x, y) = M(\Pi(f_1(x), g_1(y)), \Pi(f_2(x), g_2(y)))$$

we have  $|E(\frac{3}{5}, \frac{3}{10}) - E(\frac{1}{2}, \frac{3}{10})| = \frac{3}{20} > \frac{1}{10} = |\frac{3}{5} - \frac{1}{2}| + |\frac{3}{10} - \frac{3}{10}|$ , i.e.,  $E$  is not 1-Lipschitz and, therefore neither a copula nor a quasi-copula. Note, however, that  $E$  is a semicopula (see Corollary 5.1).

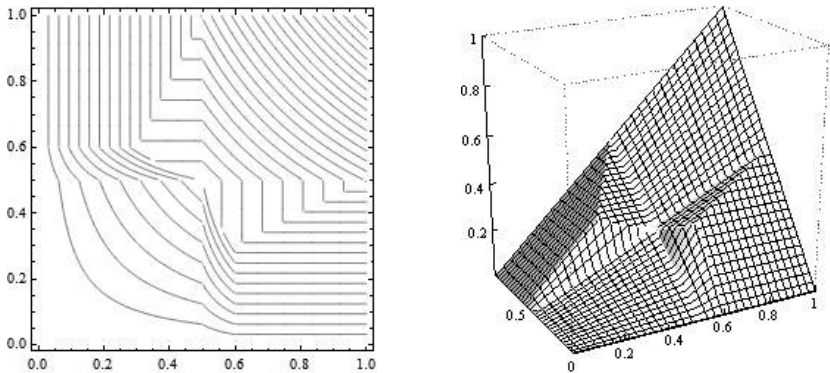


Figure 1: The function  $E$  in Example 2.5

### 2.2. Schur concave functions and Schur concave copulas

The third property of real functions we need in this paper is a special type of monotonicity, namely, the Schur concavity which reverses majorization [44].

Given a vector  $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ , we denote by  $\mathbf{x}^\downarrow = (x_1^\downarrow, x_2^\downarrow, \dots, x_n^\downarrow) \in \mathbb{R}^n$  the vector with the same components, but sorted in descending order.

Then a vector  $\mathbf{y} = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$  *majorizes* a vector  $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$  (in symbols  $\mathbf{y} \succ \mathbf{x}$ ) if  $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$  and, for all  $k \in \{1, 2, \dots, n\}$ , we have

$$\sum_{i=k}^n x_i^\downarrow \leq \sum_{i=k}^n y_i^\downarrow. \tag{2.10}$$

Note that  $\succ$  is not a partial order on  $\mathbb{R}^n$ : from  $\mathbf{y} \succ \mathbf{x}$  and  $\mathbf{x} \succ \mathbf{y}$  we only can conclude that  $\mathbf{x}$  and  $\mathbf{y}$  have the same components, but not necessarily in the same order.

If  $A \subseteq \mathbb{R}^n$  then a function  $f: A \rightarrow \mathbb{R}$  is said to be *Schur convex* [57] if for all  $\mathbf{x}, \mathbf{y} \in A$  with  $\mathbf{y} \succ \mathbf{x}$  we have  $f(\mathbf{y}) \geq f(\mathbf{x})$ . A function  $f: A \rightarrow \mathbb{R}$  is said to be *Schur concave* if its negation  $-f$  is Schur convex.

In the case  $n = 2$  and  $A = [0, 1]^2$ , a function  $f: [0, 1]^2 \rightarrow [0, 1]$  is Schur concave if and only if, for all  $(x, y), (u, v) \in [0, 1]^2$  with  $x + y = u + v$  and  $\min(x, y) \leq \min(u, v)$ , we have  $f(x, y) \leq f(u, v)$ .

Equivalently, the Schur concavity of  $f$  means that, for all  $(x, y) \in [0, 1]^2$  and all  $\lambda \in [0, 1]$ ,

$$f(x, y) \leq f(\lambda \cdot x + (1 - \lambda) \cdot y, (1 - \lambda) \cdot x + \lambda \cdot y).$$

We will need the Schur concavity of a copula  $D: [0, 1]^2 \rightarrow [0, 1]$  on the upper left triangle  $\Delta = \{(x, y) \in [0, 1]^2 \mid x \leq y\}$  only, which means that for all  $(x, y) \in \Delta$  and for all  $\varepsilon > 0$  with  $(x + \varepsilon, y - \varepsilon) \in \Delta$  we have

$$D(x, y) \leq D(x + \varepsilon, y - \varepsilon).$$

Observe that each Schur concave copula is symmetric, and that symmetric copulas which are Schur concave on  $\Delta$  are just Schur concave copulas. However, in general, a copula which is Schur concave on  $\Delta$  need not be symmetric. Clearly, each of the three basic copulas  $W$ ,  $\Pi$  and  $M$  is Schur concave, as well as each associative copula.

### 3. *D*-Product of a copula and its dual

Observe that the dual of a copula  $C$  can be characterized as a binary aggregation function  $C^*: [0, 1]^2 \rightarrow [0, 1]$  having 0 as neutral element, 1 as annihilator and being *submodular*, i.e., for all  $\mathbf{x}, \mathbf{y} \in [0, 1]^2$  we have  $C^*(\mathbf{x} \vee \mathbf{y}) + C^*(\mathbf{x} \wedge \mathbf{y}) \leq C^*(\mathbf{x}) + C^*(\mathbf{y})$ .

Given an aggregation function  $D$  and a 1-Lipschitz aggregation function  $A$ , consider the function  $D(A, A^*): [0, 1]^2 \rightarrow [0, 1]$  defined by

$$D(A, A^*)(x, y) = D(A(x, y), A^*(x, y)) \tag{3.1}$$

and call it the *D-product of A and its dual A\**.

In many cases involving the three basic copulas  $W$ ,  $\Pi$  and  $M$  this construction always yields a copula:

- (i) for each copula  $C$  we trivially get  $W(C, C^*) = W$  and  $M(C, C^*) = C$ , and for each copula  $D$  we have  $D(W, W^*) = W$ ;

(ii)  $D(M, M^*)$  is a copula if and only if, for the copula  $C_{\delta_D} : [0, 1]^2 \rightarrow [0, 1]$  given by

$$C_{\delta_D}(x, y) = \min\left(x, y, \frac{D(x, x) + D(y, y)}{2}\right)$$

and for the restriction  $D \upharpoonright_{\Delta}$  of  $D$  to  $\Delta$  we have  $D \upharpoonright_{\Delta} \leq C_{\delta_D} \upharpoonright_{\Delta}$  — this condition is always satisfied for a symmetric and, a fortiori, for an associative copula  $D$  in which case we have  $D(M, M^*) = D$  (note that the copula  $C_{\delta_D}$  is the *diagonal copula* (see [50]) whose diagonal section coincides with the diagonal section  $\delta_D$  of  $D$ );

(iii) in a recent paper [38] it was shown that, for each copula  $C$ , also  $\Pi(C, C^*)$  is a copula.

In Example 4.1 we will show that there are copulas  $C$  and  $D$  such that the  $D$ -product  $D(C, C^*)$  is not a copula.

Looking at the result of [38], a natural question is therefore under which conditions for a copula  $D$  the  $D$ -product  $D(C, C^*)$  is a copula for each copula  $C$ . To answer this question, we need a special type of monotonicity of the copula  $D$ .

In our main result below it suffices to require the ultramodularity (2.7) and the Schur concavity of copulas to hold on the upper left triangle  $\Delta \subseteq [0, 1]^2$  only, i.e., only for arguments in  $\Delta$  rather than in  $[0, 1]^2$ .

**THEOREM 3.1.** *Let  $C$  be a binary copula and let  $D$  be a binary copula which is ultramodular and Schur concave on the upper left triangle  $\Delta$ . Then the function  $D(C, C^*)$  is a copula.*

*Proof.* For the sake of brevity, put  $D_C = D(C, C^*)$ . Obviously, we have  $D_C(x, 0) = D_C(0, x) = 0$  and  $D_C(x, 1) = D_C(1, x) = x$  for all  $x \in [0, 1]$  and for all copulas  $C$  and  $D$ , so only the supermodularity (2.2) of  $D_C$  remains to be shown. Observe that, for a function defined on  $[0, 1]^2$ , the validity of (2.2) is equivalent to its 2-increasingness, i.e., we have to show that

$$D_C(x_1, y_1) + D_C(x_2, y_2) - D_C(x_1, y_2) - D_C(x_2, y_1) \geq 0 \tag{3.2}$$

for all  $(x_1, x_2), (y_1, y_2) \in [0, 1]^2$  with  $x_1 \leq x_2$  and  $y_1 \leq y_2$ .

Now fix an arbitrary copula  $C$ , a copula  $D$  which is ultramodular and Schur concave on  $\Delta$  and points  $(x_1, x_2), (y_1, y_2) \in [0, 1]^2$  with  $x_1 \leq x_2$  and  $y_1 \leq y_2$ . Put

$$\begin{aligned} C(x_1, y_1) &= u, & C(x_2, y_1) &= u + \alpha, \\ C(x_1, y_2) &= u + \beta, & C(x_2, y_2) &= u + \gamma. \end{aligned}$$

Evidently,  $\alpha \in [0, x_2 - x_1]$ ,  $\beta \in [0, y_2 - y_1]$  and  $\gamma \in [\alpha + \beta, x_2 - x_1 + y_2 - y_1]$ . Identifying two-dimensional vectors in  $[0, 1]^2$  with points in  $[0, 1]^2$ , we see that

$$P_1 = (C(x_1, y_1), C^*(x_1, y_1)) = (u, x_1 + y_1 - u) \in \Delta.$$



Similarly, the points  $P_2 = (u + \alpha, x_2 + y_1 - u - \alpha)$ ,  $P_3 = (u + \beta, x_1 + y_2 - u - \beta)$  and  $P_4 = (u + \gamma, x_1 + y_2 - u - \gamma)$  belong to  $\Delta$ . Put

$$P_5 = P_2 + P_3 - P_1 = (u + \alpha + \beta, x_2 + y_2 - u - \alpha - \beta).$$

If  $P_5 \in \Delta$ , then because of the Schur concavity of  $D$  on  $\Delta$  we obtain  $D_C(x_2, y_2) = D(P_4) \geq D(P_5)$ , and the ultramodularity of  $D$  on  $\Delta$  implies  $D(P_5) + D(P_1) \geq D(P_2) + D(P_3)$ , i.e.,  $D_C$  is 2-increasing in this case.

If  $P_5 \notin \Delta$ , we define the points  $P_6 - P_{10}$  as follows:  $P_6$  is the intersection of the lines  $l_1$ , passing through  $(0, 1)$  and  $(1, 1)$ , and  $l_2$  connecting  $P_2$  and  $P_5$ ;  $P_7$  is the common point of the lines  $l_1$  and  $l_3$ , passing through  $P_3$  and  $P_5$ ;  $P_8 = P_6 - (P_5 - P_7)$ ;  $P_9$  is the intersection of the lines  $l_1$  and  $l_4$ , passing through  $P_4$  and  $P_5$ , and  $P_{10} = P_1 + P_6 - P_2$  (for an illustration of this case see Figure 2).

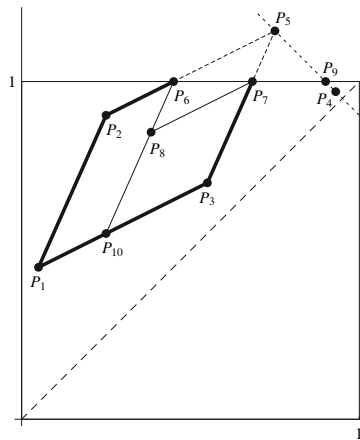


Figure 2: The case  $P_5 \notin \Delta$

Then the inequality (3.2) turns into

$$\begin{aligned} & (D(P_1) + D(P_4) - D(P_2) - D(P_3)) \\ &= \left( D(P_1) + D(P_6) - D(P_2) - D(P_{10}) \right) + \left( D(P_{10}) + D(P_7) - D(P_3) - D(P_8) \right) \\ & \quad + \left( D(P_4) - D(P_9) \right) + \left( D(P_9) - D(P_7) - D(P_6) + D(P_8) \right) \geq 0. \end{aligned}$$

This inequality can be verified by the combination of the following four arguments:

- (a)  $D(P_1) + D(P_6) - D(P_2) - D(P_{10}) \geq 0$  holds because of the ultramodularity of  $D$  on  $\Delta$ ;
- (b)  $D(P_{10}) + D(P_7) - D(P_3) - D(P_8) \geq 0$  holds because of the ultramodularity of  $D$  on  $\Delta$ ;

- (c)  $D(P_4) - D(P_9) \geq 0$  holds because of the Schur concavity of  $D$  on  $\Delta$ ;
- (d) if we write  $P_8 = (v, w)$ ,  $P_6 = (v + \delta, 1)$  and  $P_7 = (v + \varepsilon, 1)$  then this implies  $P_5 = (v + \delta + \varepsilon, 2 - w)$  and  $P_9 = (v + \delta + \varepsilon + 1 - w, 1)$ . Then, because of  $D \geq W$ , we obtain

$$\begin{aligned} D(P_9) - D(P_7) - D(P_6) + D(P_8) &= v + \delta + \varepsilon + 1 - w - (v + \varepsilon) - (v + \delta) + D(v, w) \\ &\geq 1 - v - w + \max(v + w - 1, 0) \\ &\geq 0. \end{aligned}$$

Summarizing,  $D_C$  is a copula.  $\square$

Clearly, if the copula  $C$  in Theorem 3.1 is also symmetric, so is  $D(C, C^*)$ .

It is remarkable that  $D(C, C^*)$  in Theorem 3.1 preserves the ultramodularity and the Schur concavity on  $\Delta$  of the copulas  $C$  and  $D$ :

**PROPOSITION 3.2.** *Let  $C, D$  be binary copulas which are ultramodular and Schur concave on the upper left triangle  $\Delta$ . Then also the copula  $D(C, C^*)$  is ultramodular and Schur concave on  $\Delta$ .*

*Proof.* For all  $(x, y) \in \Delta$  and for all  $\alpha \in [0, \min(1 - x, y, \frac{y-x}{2})]$  the Schur concavity of  $C$  implies  $C(x, y) \leq C(x + \alpha, y - \alpha)$ , i.e.,  $C(x + \alpha, y - \alpha) = C(x, y) + \beta$  for some  $\beta \geq 0$ . Then, by the Schur concavity of  $D$ , we obtain (writing again  $D_C = D(C, C^*)$ )

$$\begin{aligned} D_C(x + \alpha, y - \alpha) &= D(C(x, y) + \beta, x + y - C(x, y) - \beta) \\ &\geq D(C(x, y), x + y - C(x, y)) \\ &= D_C(x, y), \end{aligned}$$

i.e.,  $D_C$  is Schur concave on  $\Delta$ .

To show the ultramodularity of  $D_C$  on  $\Delta$ , it suffices to prove that all horizontal and vertical sections of  $D_C$  are convex in  $\Delta$  (see [34]). Fix  $y_0 \in [0, 1]$ . The convexity of  $D_C(\cdot, y_0)$  on  $[0, y_0]$ , i.e., its ultramodularity as a function in one variable, is equivalent to the validity of the inequality

$$D_C(x + \alpha + \beta, y_0) + D_C(x, y_0) \geq D_C(x + \alpha, y_0) + D_C(x + \beta, y_0)$$

for all  $\alpha, \beta \geq 0$  and all  $x \in [0, y_0 - \alpha - \beta]$ . Let us write briefly  $C(x, y_0) = u$  and  $C^*(x, y_0) = v$ . Then we have

$$\begin{aligned} C(x + \alpha, y_0) &= u + \gamma, & C^*(x + \alpha, y_0) &= v + \alpha - \gamma, \\ C(x + \beta, y_0) &= u + \delta, & C^*(x + \beta, y_0) &= v + \beta - \delta, \\ C(x + \alpha + \beta, y_0) &= u + \varepsilon, & C^*(x + \alpha + \beta, y_0) &= v + \alpha + \beta - \varepsilon \end{aligned}$$

for some  $\gamma \in [0, \alpha]$ , some  $\delta \in [0, \beta]$  and some  $\varepsilon \in [0, \alpha + \beta]$ . Note that the ultramodularity of  $C$  on  $\Delta$  implies  $\varepsilon \geq \gamma + \delta$ . Now we have

$$\begin{aligned} D_C(x + \alpha + \beta, y_0) + D_C(x, y_0) &= D(u + \varepsilon, v + \alpha + \beta - \varepsilon) + D(u, v) \\ &\geq D(u + \gamma + \delta, v + \alpha - \gamma + \beta - \delta) + D(u, v) \\ &\geq D(u + \gamma, v + \alpha - \gamma) + D(u + \delta, v + \beta - \delta) \\ &= D_C(x + \alpha, y_0) + D_C(x + \beta, y_0), \end{aligned}$$

where the first inequality follows from the Schur concavity of  $D$  on  $\Delta$ , and the second inequality from the ultramodularity of  $D$  on  $\Delta$ . Therefore,  $D_C(\cdot, y_0)$  is convex on  $[0, y_0]$ .

In complete analogy, the convexity of  $D_C(x_0, \cdot)$  on  $[x_0, 1]$  can be shown for each fixed  $x_0 \in [0, 1]$ .

Summarizing, the copula  $D_C$  is Schur concave and ultramodular on  $\Delta$ .  $\square$

It turns out that the ultramodularity of  $D$  is a necessary condition if we want  $D(C, C^*)$  to be a copula for each copula  $C$ :

**THEOREM 3.3.** *Let  $D$  be a binary copula such that for each binary copula  $C$  the function  $D(C, C^*)$  is a copula. Then  $D$  is ultramodular on the upper left triangle  $\Delta$ .*

*Proof.* Let  $D$  be a copula such that for each copula  $C$  the function  $D(C, C^*)$  is a copula. Based on [34, 42], the ultramodularity of  $D$  on  $\Delta$  is equivalent to the convexity of the restrictions of all horizontal and vertical sections of  $D$  to  $\Delta$ . Fix  $v \in ]0, 1[$  and consider the function  $f_v: [0, v] \rightarrow [0, 1]$  given by  $f_v(u) = D(u, v)$  (observe that  $f_0$  is defined on a single point and thus convex, and that  $f_1 = \text{id}_{[0,1]}$  is also convex). The convexity of  $f_v$  is equivalent to its Jensen convexity, i.e., for all  $(u, w) \in [0, 1]^2$  with  $w \in ]u, v]$

$$f_v(u) + f_v(w) \geq 2 \cdot f_v\left(\frac{u+w}{2}\right). \tag{3.3}$$

Consider the  $M$ -ordinal sum  $C_v = M-\langle(0, v, W)\rangle$  given by

$$C_v(x, y) = \begin{cases} \max(x + y - v, 0) & \text{if } (x, y) \in [0, v]^2, \\ M(x, y) & \text{otherwise.} \end{cases}$$

Fix the points  $\mathbf{x} = (\frac{u+v}{2}, \frac{w+v}{2})$  and  $\mathbf{y} = (\frac{w+v}{2}, \frac{u+v}{2})$  in  $[0, v]^2$ . Observe that then  $C_v(\mathbf{x}) = C_v(\mathbf{y}) = \frac{u+w}{2}$ ,  $C_v(\mathbf{x} \vee \mathbf{y}) = w$  and  $C_v(\mathbf{x} \wedge \mathbf{y}) = u$ , while  $(C_v)^*$  in all these points has the value  $v$ . Since  $D(C_v, (C_v)^*)$  is a copula, it is supermodular which implies (3.3).

Similarly, the convexity of the restrictions of the vertical sections  $g_u$  of  $D$  to  $\Delta$  for all  $u \in ]0, 1[$  is shown, considering the  $M$ -ordinal sum  $C^u = M-\langle(u, 1, W)\rangle$ . As a consequence,  $D$  is a copula which is ultramodular on  $\Delta$ .  $\square$

REMARK 3.4.

(i) Observe that for a copula  $D \neq M$  which is ultramodular on  $\Delta$  we necessarily have  $D(x, y) < x$  for all  $(x, y) \in \Delta \cap ]0, 1[^2$ . Therefore, if the copula  $D$  is ultramodular and Schur concave on  $\Delta$  then for each copula  $C$  we have  $D(C, C^*) < C$ , and we can define a sequence of copulas  $(C_n)_{n \in \mathbb{N}}$  putting

- (a)  $C_1 = C$ ,
- (b)  $C_{n+1} = D(C_n, (C_n)^*)$  for each  $n \in \mathbb{N}$ .

Then, if  $D \neq M$ , we obtain  $\lim_{n \rightarrow \infty} C_n = W$  for each copula  $C$ .

(ii) Keeping the notations of (i), define the ternary function  $f: \{(x, y, z) \in [0, 1]^3 \mid x + y - z \in [0, 1]\} \rightarrow [0, 1]$  by  $f(x, y, z) = D(z, x + y - z)$ . For an arbitrary but fixed point  $(x, y) \in [0, 1]^2$ , denote  $z_n = C_n(x, y)$ . This means that  $z_1 = C(x, y)$ , and, for  $n \in \mathbb{N}$ ,  $z_{n+1} = f(x, y, z_n)$ . As a consequence of (i),  $\lim_{n \rightarrow \infty} z_n = \max(x + y - 1, 0)$ .

(iii) Observe that Theorem 3.1 cannot be modified replacing the dual copula  $C^*$  by the co-copula  $\bar{C}$ . Indeed, for the  $M$ -ordinal sum  $C = M\text{-}(\langle 0, \frac{1}{2}, \Pi \rangle)$  given by

$$C(x, y) = \begin{cases} 2xy & \text{if } (x, y) \in [0, \frac{1}{2}]^2, \\ M(x, y) & \text{otherwise} \end{cases}$$

we obtain

$$V_{\Pi(C, \bar{C})}([\frac{3}{10}, \frac{2}{5}]^2) = -\frac{1}{100} < 0,$$

i.e.,  $\Pi(C, \bar{C})$  is not a copula.

Taking into account the flipping method for constructing new copulas [49] which transforms ultramodular copulas into copulas with concave horizontal and vertical sections [43], Theorem 3.1 can be modified as follows:

COROLLARY 3.5. *If  $E: [0, 1]^2 \rightarrow [0, 1]$  is a binary copula*

- (i) *which is concave on the horizontal and vertical sections which are contained in the triangle  $\Delta_*$  determined by the points  $(0, 0)$ ,  $(0, 1)$  and  $(1, 0)$ ,*
- (ii) *and which is 1-Lipschitz with respect to the Chebyshev norm on the affine sections which are parallel to the main diagonal and contained in  $\Delta_*$ , i.e., we have  $E(x + \varepsilon, y + \varepsilon) - E(x, y) \leq \varepsilon$  for all  $x, y \geq 0$  and all  $\varepsilon > 0$  with  $x + y + 2\varepsilon \leq 1$ ,*

*then, for each binary copula  $C: [0, 1]^2 \rightarrow [0, 1]$ , the function  $E_C: [0, 1]^2 \rightarrow [0, 1]$  given by*

$$E_C(x, y) = C(x, y) - E(C(x, y), 1 - C^*(x, y))$$

*is a copula.*

Since the product copula  $\Pi$  is invariant under flipping, the constructions in Theorem 3.1 and Corollary 3.5 yield the same result in the case  $D = E = \Pi$ , i.e.,  $\Pi(C, C^*) = \Pi_C$  for each copula  $C$ . For the extremal cases  $E = W$  (which vanishes on  $\Delta_*$ ) and  $E = M$  we obtain  $W_C = C$  and  $M_C = W$  for each copula  $C$ .

### 4. Examples and counterexamples

EXAMPLE 4.1. These examples illustrate the importance of the hypotheses in Theorem 3.1:

- (i) Consider the  $M$ -ordinal sum  $C = M-\langle(\frac{1}{4}, \frac{13}{24}, W)\rangle$  (see Figure 3 left) and the  $W$ -ordinal sum  $D = W-\langle(0, \frac{1}{2}, \Pi)\rangle$  (see Example 3.4(ii) in [34] and Figure 3 right) given by, respectively,

$$C(x, y) = \begin{cases} \max(x + y - \frac{13}{24}, \frac{1}{4}) & \text{if } (x, y) \in [\frac{1}{4}, \frac{13}{24}]^2, \\ M(x, y) & \text{otherwise,} \end{cases}$$

$$D(x, y) = \begin{cases} x(2y - 1) & \text{if } (x, y) \in [0, \frac{1}{2}] \times [\frac{1}{2}, 1], \\ W(x, y) & \text{otherwise.} \end{cases}$$

Note that  $D$  is ultramodular, but not Schur concave on the upper left triangle  $\Delta$ . Then we have

$$V_{D(C, C^*)}([\frac{3}{8}, \frac{1}{2}]^2) = -\frac{5}{288} < 0,$$

i.e.,  $D(C, C^*)$  is not a copula.

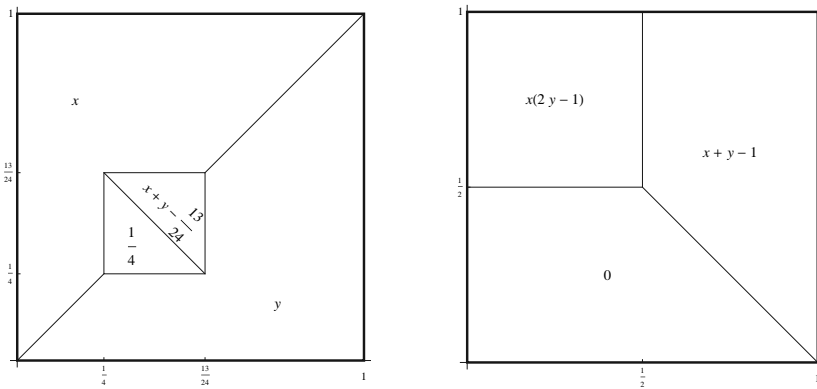


Figure 3: The copulas  $C$  (left) and  $D$  considered in Example 4.1 (i)

- (ii) Consider the  $M$ -ordinal sum  $D = M-\langle(\frac{1}{2}, 1, W)\rangle$  (see Figure 4 left) given by

$$D(x, y) = \begin{cases} \max(x + y - 1, \frac{1}{2}) & \text{if } (x, y) \in [\frac{1}{2}, 1]^2, \\ M(x, y) & \text{otherwise.} \end{cases}$$

Note that  $D$  is Schur concave, but not ultramodular on the upper left triangle  $\Delta$ . Then we have

$$V_{D(\Pi, \Pi^*)}([\frac{2}{3}, \frac{3}{4}]^2) = -\frac{1}{18} < 0,$$

i.e.,  $D(\Pi, \Pi^*)$  is not a copula.

(iii) If  $D$  is a symmetric copula then we know that  $D(M, M^*) = D$ . Consider the (non-symmetric) copula  $D = W-\langle\langle 0, \frac{1}{2}, \Pi \rangle\rangle$  as in (i). Then  $D(M, M^*)$  is a copula different from  $D$  — more precisely

$$D(M, M^*) = W-\langle\langle 0, \frac{1}{2}, \Pi \rangle, \langle \frac{1}{2}, 1, \Pi \rangle \rangle \neq D.$$

(iv) Consider the  $W$ -ordinal sum  $D = W-\langle\langle 0, \frac{4}{5}, M \rangle\rangle$  (see Figure 4 right) given by

$$D(x, y) = \begin{cases} \min(x, y - \frac{1}{5}) & \text{if } (x, y) \in [0, \frac{4}{5}] \times [\frac{1}{5}, 1], \\ W(x, y) & \text{otherwise.} \end{cases}$$

Note that  $D$  is neither Schur concave nor ultramodular on the upper left triangle  $\Delta$  nor symmetric. Then we have

$$V_{D(M, M^*)}(\left[\frac{2}{5}, \frac{3}{5}\right]^2) = -\frac{1}{5} < 0,$$

i.e.,  $D(M, M^*)$  is not a copula.

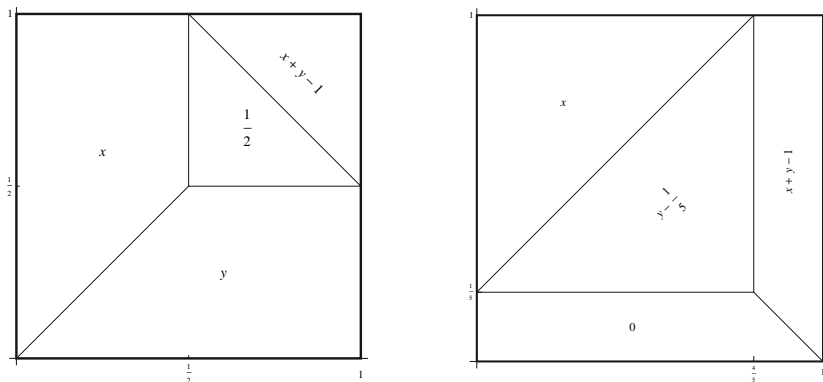


Figure 4: The copulas considered in Example 4.1 (ii) left and Example 4.1 (iv)

### 5. Generalization to quasi-copulas and semicopulas

Observe first that the construction (2.9) given in Theorem 2.3 can be applied also to the case of semicopulas.

COROLLARY 5.1. *Let  $S_1, S_2, T: [0, 1]^2 \rightarrow [0, 1]$  be semicopulas. Then, for all monotone non-decreasing functions  $f_1, f_2, g_1, g_2: [0, 1] \rightarrow [0, 1]$  satisfying*

$$T(f_1(x), f_2(x)) = T(g_1(x), g_2(x)) = x$$

for all  $x \in [0, 1]$ , also the function  $U: [0, 1]^2 \rightarrow [0, 1]$  given by

$$U(x, y) = T(S_1(f_1(x), g_1(y)), S_2(f_2(x), g_2(y))) \tag{5.1}$$

is a semicopula.

*Proof.* The monotonicity of  $U$  follows from the monotonicity of the semicopulas  $S_1, S_2$  and  $T$  and of each of the functions  $f_1, f_2, g_1,$  and  $g_2$  involved in (5.1). Moreover,  $T(f_1(1), f_2(1)) = 1$  implies  $f_1(1) = f_2(1) = 1$ , and in analogy we get  $g_1(1) = g_2(1) = 1$ . Then for all  $x, y \in [0, 1]$

$$U(x, 1) = T(S_1(f_1(x), 1), S_2(f_2(x), 1)) = T(f_1(x), f_2(x)) = x,$$

and, similarly,  $U(1, y) = y$ , showing that  $U$  is a semicopula.  $\square$

This shows that the function  $E$  considered in Example 2.5 is indeed a semicopula.

**COROLLARY 5.2.** *Let  $Q_1, Q_2, R: [0, 1]^2 \rightarrow [0, 1]$  be quasi-copulas and assume that  $R: [0, 1]^2 \rightarrow [0, 1]$  is ultramodular. Then, for all continuous, monotone non-decreasing functions  $f_1, f_2, g_1, g_2: [0, 1] \rightarrow [0, 1]$  satisfying*

$$R(f_1(x), f_2(x)) = R(g_1(x), g_2(x)) = x$$

for all  $x \in [0, 1]$ , also the function  $L: [0, 1]^2 \rightarrow [0, 1]$  given by

$$L(x, y) = R(Q_1(f_1(x), g_1(y)), Q_2(f_2(x), g_2(y))) \tag{5.2}$$

is a quasi-copula.

*Proof.* As a consequence of Corollary 5.1, the function  $L$  in (5.2) is a semicopula, so only the 1-Lipschitz continuity of  $L$  remains to be shown. Because of [49, 51] there are two families of copulas  $(C_\alpha)_{\alpha \in A}$  and  $(C_\beta)_{\beta \in B}$  such that  $Q_1 = \sup\{C_\alpha \mid \alpha \in A\}$  and  $Q_2 = \sup\{C_\beta \mid \beta \in B\}$ . Since each ultramodular quasi-copula is necessarily an ultramodular copula, Theorem 2.3 implies that, for each  $(\alpha, \beta) \in A \times B$ , the function  $D_{\alpha, \beta}: [0, 1]^2 \rightarrow [0, 1]$  given by

$$D_{\alpha, \beta}(x, y) = R(C_\alpha(f_1(x), g_1(y)), C_\beta(f_2(x), g_2(y)))$$

is a copula. The uniform continuity of  $R$  yields  $L = \sup\{D_{\alpha, \beta} \mid (\alpha, \beta) \in A \times B\}$ , which is a quasi-copula because of [49, 51].  $\square$

Example 2.5 shows that Corollary 5.2 does no longer hold if the quasi-copula  $R$  fails to be ultramodular.

When looking for a version of Theorem 3.1 in the context of quasi-copulas, it turns out that the situation is much simpler:

**PROPOSITION 5.3.** *Let  $Q$  and  $R$  be two binary quasi-copulas. Then  $R(Q, Q^*)$  is a quasi-copula.*

*Proof.* Obviously,  $R(Q, Q^*)$  satisfies the boundary conditions. Since  $Q$  and  $R$  are 1-Lipschitz, for each  $(x, y) \in [0, 1]^2$  and for all  $\varepsilon \in ]0, 1 - x]$  there is a  $\delta > 0$  such that

$$\begin{aligned} R(Q, Q^*)(x + \varepsilon, y) &= R(Q(x + \varepsilon, y), x + \varepsilon + y - Q(x + \varepsilon, y)) \\ &= R(Q(x, y) + \delta, x + y - Q(x, y) + \varepsilon - \delta) \\ &\leq R(Q, Q^*)(x, y) + \delta + \varepsilon - \delta \\ &= R(Q, Q^*)(x, y) + \varepsilon. \end{aligned}$$

Therefore,  $R(Q, Q^*)$  is 1-Lipschitz in the first component. In complete analogy it is shown that  $R(Q, Q^*)$  is 1-Lipschitz in the second component, i.e.,  $R(Q, Q^*)$  is a quasi-copula.  $\square$

In particular, the three functions  $D(C, C^*)$ ,  $D(\Pi, \Pi^*)$ , and  $D(M, M^*)$  considered in Example 4.1(i), (ii) and (iv), respectively, which were shown not to be copulas, are indeed quasi-copulas.

If we try to apply a construction à la Theorem 3.1 and Proposition 5.3 to semicopulas  $S$  and  $T$ , we see that  $T(S, S^*)$  is not well-defined if  $S \not\leq W$ , in which case there are arguments  $(x, y) \in [0, 1]^2$  such that  $x + y - S(x, y) > 1$ . However, taking into account that the product  $\Pi$  is defined on  $\mathbb{R}^2$ , then, for each semicopula  $S$ , the range of the function  $\Pi_S: [0, 1]^2 \rightarrow \mathbb{R}$  defined by

$$\Pi_S(x, y) = \Pi(S(x, y), x + y - S(x, y))$$

is a subset of  $[0, 1]$ .

**PROPOSITION 5.4.** *If  $S$  is a binary semicopula then also  $\Pi_S$  is a semicopula.*

*Proof.* Obviously,  $\Pi_S$  satisfies the boundary conditions of semicopulas. In order to show the monotonicity of  $\Pi_S$  in its first component, fix  $\varepsilon > 0$ ,  $(x_0, y_0) \in [0, 1]^2$  such that  $x_0 + \varepsilon \leq 1$ , and put  $\delta = S(x_0 + \varepsilon, y_0) - S(x_0, y_0) \geq 0$ . Then we get

$$\begin{aligned} \Pi_S(x_0 + \varepsilon, y_0) &= S(x_0 + \varepsilon, y_0) \cdot (x_0 + \varepsilon + y_0 - S(x_0 + \varepsilon, y_0)) \\ &= (S(x_0, y_0) + \delta) \cdot (x_0 + \varepsilon + y_0 - (S(x_0, y_0) + \delta)) \\ &= \Pi_S(x_0, y_0) + \delta \cdot (x_0 + \varepsilon + y_0 - 2S(x_0, y_0) - \delta) + \varepsilon \cdot S(x_0, y_0) \\ &\geq \Pi_S(x_0, y_0), \end{aligned}$$

where the inequality is a consequence of  $S \leq M$ , implying  $S(x_0, y_0) \leq y_0$  and  $S(x_0, y_0) + \delta = S(x_0 + \varepsilon, y_0) \leq x_0 + \varepsilon$ , i.e.,  $x_0 + \varepsilon + y_0 - 2S(x_0, y_0) - \delta \geq 0$ . In complete analogy, the monotonicity of  $\Pi_S$  in its second component is shown, thus  $\Pi_S$  is a semicopula.  $\square$

**PROPOSITION 5.5.** *Let  $T$  be a binary semicopula which is Schur concave on the upper left triangle  $\Delta$ . Then for each binary semicopula  $S$  the function  $T_S: [0, 1]^2 \rightarrow [0, 1]$  given by*

$$T_S(x, y) = T(S(x, y), \min(x + y - S(x, y), 1)) \tag{5.3}$$

*is a semicopula.*



*Proof.* First of all,  $S \leq M$  implies  $S(x, y) \leq x + y$  for all  $(x, y) \in [0, 1]^2$ , so  $T_S$  is well-defined. Also,  $T_S(x, 0) = T_S(0, x) = 0$  and  $T_S(x, 1) = T_S(1, x) = x$  for each  $x \in [0, 1]$ . Therefore, it suffices to show the monotonicity of  $T_S$  in the first coordinate (the proof of the monotonicity in the second coordinate being completely analogous). We fix an arbitrary point  $(x, y) \in ]0, 1[^2$  and  $\varepsilon > 0$  with  $x + \varepsilon \leq 1$  and consider the following four cases:

(i)  $x + y - S(x, y) \leq 1$  and  $x + \varepsilon + y - S(x + \varepsilon, y) \leq 1$ : then

$$\begin{aligned} T_S(x, y) &= T(S(x, y), x + y - S(x, y)) \\ &\leq T(S(x + \varepsilon, y), x + y - S(x + \varepsilon, y)) \\ &\leq T(S(x + \varepsilon, y), x + \varepsilon + y - S(x + \varepsilon, y)) \\ &= T_S(x + \varepsilon, y), \end{aligned}$$

where the first inequality holds because of the Schur concavity of  $T$  on  $\Delta$ , and the second inequality because of the monotonicity of  $T$  in the second coordinate.

(ii)  $x + y - S(x, y) \leq 1$  and  $x + \varepsilon + y - S(x + \varepsilon, y) > 1$ : then

$$\begin{aligned} T_S(x, y) &= T(S(x, y), x + y - S(x, y)) \\ &\leq S(x, y) \\ &\leq S(x + \varepsilon, y) \\ &= T(S(x + \varepsilon, y), \min(x + \varepsilon + y - S(x + \varepsilon, y), 1)) \\ &= T_S(x + \varepsilon, y), \end{aligned}$$

where the first inequality follows from  $T \leq M$ , and the second inequality because of the monotonicity of  $S$  in the first coordinate.

(iii)  $x + y - S(x, y) > 1$  and  $x + \varepsilon + y - S(x + \varepsilon, y) \leq 1$ : then

$$\begin{aligned} T_S(x, y) &= S(x, y) \\ &< x + y - 1 \\ &< x + \varepsilon + y - 1 \\ &= T(1, x + \varepsilon + y - 1) \\ &\leq T(S(x + \varepsilon, y), x + \varepsilon + y - S(x + \varepsilon, y)) \\ &= T_S(x + \varepsilon, y), \end{aligned}$$

where the last inequality holds because of the Schur concavity of  $T$  on  $\Delta$ .

(iv)  $x + y - S(x, y) > 1$  and  $x + \varepsilon + y - S(x + \varepsilon, y) > 1$ : then we have

$$T_S(x, y) = S(x, y) \leq S(x + \varepsilon, y) = T_S(x + \varepsilon, y),$$

because of the monotonicity of  $S$  in the first coordinate.

Thus  $T_S(x, y) \leq T_S(x + \varepsilon, y)$  for all  $(x, y) \in [0, 1]^2$  and all  $\varepsilon > 0$  with  $x + \varepsilon \leq 1$ .  $\square$

Note that the dual  $S^*$  of a semicopula  $S$  is monotone if and only if  $S$  is 1-Lipschitz, i.e., a quasi-copula. Observe that the range of the dual  $S^*$  of a semicopula  $S$  defined by  $S^*(x, y) = x + y - S(x, y)$  is a subset of  $[0, 1]$  (in fact, equal to  $[0, 1]$ ) only if  $S \geq W$ . Therefore we have as an immediate consequence of Proposition 5.5:

**COROLLARY 5.6.** *Let  $T$  be a binary semicopula which is Schur concave on the upper left triangle  $\Delta$  and  $S$  be a semicopula with  $S \geq W$ . Then the function  $T(S, S^*)$  is a semicopula.*

## 6. Concluding remarks

The main result of this paper is a new construction method for copulas: for each copula  $D$  which is ultramodular and Schur concave on the left upper triangle  $\Delta$ , the  $D$ -product  $D(C, C^*)$  yields a copula for an arbitrary copula  $C$ . We have shown that the ultramodularity of  $D$  on  $\Delta$  cannot be relaxed. However, it is an open question whether the Schur concavity on  $\Delta$  can be replaced by some weaker condition.

In all our considerations we restricted ourselves to two-dimensional copulas. In a next step we will consider similar problems for copulas of higher dimensions. Observe that if an  $n$ -dimensional copula  $C$  has a probabilistic representation in the form  $C(x_1, \dots, x_n) = P(\{X_1 \leq x_1, \dots, X_n \leq x_n\})$ , then its  $n$ -dimensional dual  $C^*$  can be represented in the form  $C^*(x_1, \dots, x_n) = 1 - P(\{X_1 > x_1, \dots, X_n > x_n\})$ .

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