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NEW HERMITE–HADAMARD TYPE INEQUALITIES FOR (α,m)–CONVEX FUNCTIONS AND APPLICATIONS TO SPECIAL MEANS

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Abstract. In this paper, some new results involving the left-hand side of the Hermite-Hadamard type inequalities for the class of functions whose second derivatives in absolute value at certain powers are (α,m)-convex functions are obtained. Some applications to special means of real numbers are also given.

1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $a,b \in I$ with $a < b$, then the following inequality holds:

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2},$$

which is well known as Hermite-Hadamard's inequality.

For several recent results concerning Hermite-Hadamard's inequality, we refer the interested reader to [1–5].

In [6], Miheşan gave the definition of (α,m)-convexity as following.

DEFINITION 1. The function $f : [0,d] \to \mathbb{R}$, $d > 0$, is said to be (α,m)-convex function, where $(\alpha,m) \in [0,1]^2$, if we have

$$f(tx+m(1-t)y) \leqslant t^{\alpha} f(x) + m(1-t^{\alpha})f(y),$$

for all $x,y \in [0,d]$ and $t \in [0,1]$.

Denote by $K_m^{\alpha}(d)$ the class of all (α,m)-convex functions on $[0,d]$ for which $f(0) \leqslant 0$.

REMARK 1. If we choose $(\alpha,m) = (1,m)$, we can obtain m-convex function and for $(\alpha,m) = (1,1)$, we have ordinary convex functions on $[0,d]$.

In recent years, with the generalization of the concept of convex function, on some new results of Hadamard-type inequalities have been obtained, see [7–13].

In [14], Özdemir et al. proved the following inequalities for (α,m)-convex functions.

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THEOREM 1. *Let $f : I \subset [0,b^*] \to \mathbb{R}$, be a differentiable mapping on I^o such that $f'' \in L[a,b]$ where $a,b \in I$ with $a < b$, $b^* > 0$. If $|f''|^q$ is (α,m)-convex on $[a,b]$ for $(\alpha,m) \in [0,1]^2$, $q > 1$, then the following inequality holds*

$$\left|\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb} f(x)dx\right|$$
$$\leqslant \frac{(mb-a)^2}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\cdot\left[\left|f''(a)\right|^q\frac{1}{(\alpha+2)(\alpha+3)}+m\left|f''(b)\right|^q\left(\frac{1}{6}-\frac{1}{(\alpha+2)(\alpha+3)}\right)\right]^{\frac{1}{q}},$$

where I^o is the interior of I and $L[a,b]$ is the set of integrable functions on $[a,b]$.

THEOREM 2. *Let $f : I \subset [0,b^*] \to \mathbb{R}$, be a differentiable mapping on I^o such that $f'' \in L[a,b]$ where $a,b \in I$ with $a < b$, $b^* > 0$. If $|f''|^q$ is (α,m)-convex on $[a,b]$ for $(\alpha,m) \in [0,1]^2$, $q > 1$ with $\frac{1}{p}+\frac{1}{q} = 1$, then the following inequality holds*

$$\left|\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb} f(x)dx\right|$$
$$\leqslant \frac{(mb-a)^2}{2}\left\{|f''(a)|^q\left(\frac{q}{\alpha+q+1}\right)\frac{\Gamma(\alpha+1)\Gamma(q)}{\Gamma(\alpha+q+1)}\right.$$
$$\left.+m|f''(b)|^q\left[\left(\frac{1}{q+1}\right)-\left(\frac{q}{\alpha+q+1}\right)\frac{\Gamma(\alpha+1)\Gamma(q)}{\Gamma(\alpha+q+1)}\right]\right\}^{\frac{1}{q}}.$$

In [15], Özdemir et al. proved the following lemma in order to establish some inequalities for s-convex functions in the second sense.

LEMMA 1. *Let $f : \mathrm{I} \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^o where $a,b \in I$ with $a < b$. If $f'' \in L[a,b]$, then the following equality holds*

$$\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) = \frac{(b-a)^2}{16}\left[\int_0^1 t^2 f''\left(t\frac{a+b}{2}+(1-t)a\right)dt\right.$$
$$\left.+\int_0^1 (t-1)^2 f''\left(tb+(1-t)\frac{a+b}{2}\right)dt\right].$$

In [15], Özdemir et al. obtained the following inequalities concerning Hermite-Hadamard type for s-convex functions by using Lemma 1.

THEOREM 3. *Let $f : I \subset [0,\infty) \to \mathbb{R}$, be a differentiable mapping on I^o such that $f'' \in L[a,b]$ where $a,b \in I$ with $a < b$. If $|f''|^q$, $q > 1$ is s-convex on $[a,b]$, for some fixed $s \in (0,1]$, then the following inequality holds*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leqslant \frac{(b-a)^2}{16}\left(\frac{1}{3}\right)^{\frac{1}{p}}\left\{\left(\frac{2}{(s+1)(s+2)(s+3)}|f''(a)|^q+\frac{1}{s+3}\left|f''\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{1}{s+3}\left|f''\left(\frac{a+b}{2}\right)\right|^q+\frac{2}{(s+1)(s+2)(s+3)}|f''(b)|^q\right)^{\frac{1}{q}}\right\}.$$

COROLLARY 1. *In Theorem 3, if we choose* $s=1$, *we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leqslant \frac{(b-a)^2}{48}\left(\frac{3}{4}\right)^{\frac{1}{q}}\left\{\left(\frac{1}{3}|f''(a)|^q+\left|f''\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(\left|f''\left(\frac{a+b}{2}\right)\right|^q+\frac{1}{3}|f''(b)|^q\right)^{\frac{1}{q}}\right\}.$$

In this paper, we denote that I^o is the interior of I.

The main aim of this paper is to establish new Hermite-Hadamard type inequalities for the class of functions whose second derivatives in absolute value at certain powers are (α,m)-convex functions.

2. Inequalities for (α,m)-convex functions

The following theorems give some new results of Hermite-Hadamard type inequalities for (α,m)-convex functions.

THEOREM 4. *Let* $f: I\subset[0,\infty]\to\mathbb{R}$, *be a differentiable mapping on* I^o *such that* $f''\in L[a,b]$ *where* $a,b\in I$ *with* $a<b$. *If* $|f''|$ *is* (α,m)*-convex on* $[a,b]$ *for* $\alpha\in[0,1]$ *and* $m\in(0,1]$, *then the following inequality holds*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\\&\leqslant \frac{(b-a)^2}{48(\alpha+3)}\left[3\left|f''\left(\frac{a+b}{2}\right)\right|+\alpha m\left|f''\left(\frac{a}{m}\right)\right|\right]\\&\quad+\frac{(b-a)^2}{48(\alpha+1)(\alpha+2)(\alpha+3)}\left\{6\left|f''(b)\right|-m[(\alpha+1)(\alpha+2)(\alpha+3)+6]\left|f''\left(\frac{a+b}{2m}\right)\right|\right\}\end{aligned} \tag{2.1}$$

Proof. From Lemma 1 and using the (α,m)-convexity of $|f''|$, we have

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\\&\leqslant \frac{(b-a)^2}{16}\left[\int_0^1 t^2\left|f''\left(t\frac{a+b}{2}+(1-t)a\right)\right|dt+\int_0^1(t-1)^2\left|f''\left(tb+(1-t)\frac{a+b}{2}\right)\right|dt\right]\\&\leqslant \frac{(b-a)^2}{16}\int_0^1 t^2\left[t^\alpha\left|f''\left(\frac{a+b}{2}\right)\right|+m(1-t^\alpha)\left|f''\left(\frac{a}{m}\right)\right|\right]dt\\&\quad+\frac{(b-a)^2}{16}\int_0^1(t-1)^2\left[t^\alpha\left|f''(b)\right|+m(1-t^\alpha)\left|f''\left(\frac{a+b}{2m}\right)\right|\right]dt\end{aligned}$$

$$= \frac{(b-a)^2}{16}\Big[\frac{1}{\alpha+3}\Big|f''\Big(\frac{a+b}{2}\Big)\Big| + \frac{\alpha m}{3(\alpha+3)}\,\Big|f''\Big(\frac{a}{m}\Big)\Big|\Big]$$
$$+\frac{(b-a)^2}{16}\Big[\frac{2}{(\alpha+1)(\alpha+2)(\alpha+3)}\,|f''(b)|$$
$$-m\Big(\frac{1}{3}+\frac{2}{(\alpha+1)(\alpha+2)(\alpha+3)}\Big)\Big|f''\Big(\frac{a+b}{2m}\Big)\Big|\Big]$$
$$= \frac{(b-a)^2}{48(\alpha+3)}\Big[3\Big|f''\Big(\frac{a+b}{2}\Big)\Big| + \alpha m\Big|f''\Big(\frac{a}{m}\Big)\Big|\Big]$$
$$+\frac{(b-a)^2}{48(\alpha+1)(\alpha+2)(\alpha+3)}\Big\{6\,|f''(b)| - m\Big[(\alpha+1)(\alpha+2)(\alpha+3)+6\Big]\Big|f''\Big(\frac{a+b}{2m}\Big)\Big|\Big\}$$

where we used the fact that

$$\int_0^1 (t-1)^2 t^\alpha dt = \int_0^1 t^2 (t-1)^\alpha dt = \frac{2}{(\alpha+1)(\alpha+2)(\alpha+3)}.$$

Thus the proof is completed. □

COROLLARY 2. *If we choose $\alpha = 1$ in (2.1), we obtain the following inequality*

$$\Big|f\Big(\frac{a+b}{2}\Big) - \frac{1}{b-a}\int_a^b f(x)dx\Big|$$
$$\leqslant \frac{(b-a)^2}{192}\Big[3\Big|f''\Big(\frac{a+b}{2}\Big)\Big| + m\Big|f''\Big(\frac{a}{m}\Big)\Big| + |f''(b)| - 5m\Big|f''\Big(\frac{a+b}{2m}\Big)\Big|\Big] \quad (2.2)$$

COROLLARY 3. *If we choose $\alpha = m = 1$ in (2.1), we obtain the following inequality*

$$\Big|f\Big(\frac{a+b}{2}\Big) - \frac{1}{b-a}\int_a^b f(x)dx\Big| \leqslant \frac{(b-a)^2}{192}\Big[|f''(a)| + |f''(b)| - 2\Big|f''\Big(\frac{a+b}{2}\Big)\Big|\Big] \quad (2.3)$$

THEOREM 5. *Let $f : I \subset [0,\infty) \to \mathbb{R}$, be a differentiable mapping on I^o such that $f'' \in L[a,b]$ where $a,b \in I$ with $a < b$. If $|f''|^q$ is (α,m)-convex on $[a,b]$ for $\alpha \in [0,1]$ and $m \in (0,1]$, and $q > 1$ with $\frac{1}{p}+\frac{1}{q} = 1$, then the following inequality holds*

$$\Big|f\Big(\frac{a+b}{2}\Big) - \frac{1}{b-a}\int_a^b f(x)dx\Big|$$
$$\leqslant \frac{(b-a)^2}{16}\Big(\frac{1}{\alpha+1}\Big)^{\frac{1}{q}} \cdot \Big[\Big(\Big|f''\Big(\frac{a+b}{2}\Big)\Big|^q + m\alpha\Big|f''\Big(\frac{a}{m}\Big)\Big|^q\Big)^{\frac{1}{q}}$$
$$+\Big(|f''(b)|^q + m\alpha\Big|f''\Big(\frac{a+b}{2m}\Big)\Big|^q\Big)^{\frac{1}{q}}\Big]. \quad (2.4)$$

Proof. Suppose that $p>1$. Since Lemma 1 and using the Hölder inequality, we have

$$\begin{aligned}
&\left|f\Big(\frac{a+b}{2}\Big)-\frac{1}{b-a}\int_a^b f(x)dx\right|\\
&\leqslant \frac{(b-a)^2}{16}\Big[\int_0^1 t^2\left|f''\Big(t\frac{a+b}{2}+(1-t)a\Big)\right|dt+\int_0^1 (t-1)^2\left|f''\Big(tb+(1-t)\frac{a+b}{2}\Big)\right|dt\Big]\\
&\leqslant \frac{(b-a)^2}{16}\Big(\int_0^1 t^{2p}dt\Big)^{\frac{1}{p}}\Big(\int_0^1\left|f''\Big(t\frac{a+b}{2}+(1-t)a\Big)\right|^q dt\Big)^{\frac{1}{q}}\\
&\quad+\frac{(b-a)^2}{16}\Big(\int_0^1 (t-1)^{2p}dt\Big)^{\frac{1}{p}}\Big(\int_0^1\left|f''\Big(tb+(1-t)\frac{a+b}{2}\Big)\right|^q dt\Big)^{\frac{1}{q}}.
\end{aligned}$$

From $|f''|^q$ is (α,m)-convex on $[a,b]$, we known that for every $t\in[0,1]$

$$\int_0^1\left|f''\Big(t\frac{a+b}{2}+(1-t)a\Big)\right|^q dt\leqslant\frac{1}{\alpha+1}\Big[\left|f''\Big(\frac{a+b}{2}\Big)\right|^q+m\alpha\left|f''\Big(\frac{a}{m}\Big)\right|^q\Big]$$

and

$$\left|f''\Big(tb+(1-t)\frac{a+b}{2}\Big)\right|^q\leqslant\frac{1}{\alpha+1}\Big[|f''(b)|^q+m\alpha\left|f''\Big(\frac{a+b}{2m}\Big)\right|^q\Big]$$

Therefore, by a simple computation, we have

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)dx-f\Big(\frac{a+b}{2}\Big)\right|\\
&\leqslant\frac{(b-a)^2}{16}\Big(\int_0^1 t^{2p}dt\Big)^{\frac{1}{p}}\Big(\frac{1}{\alpha+1}\Big)^{\frac{1}{q}}\Big[\left|f''\Big(\frac{a+b}{2}\Big)\right|^q+m\alpha\left|f''\Big(\frac{a}{m}\Big)\right|^q\Big]^{\frac{1}{q}}\\
&\quad+\frac{(b-a)^2}{16}\Big(\int_0^1 (t-1)^{2p}dt\Big)^{\frac{1}{p}}\Big(\frac{1}{\alpha+1}\Big)^{\frac{1}{q}}\Big[|f''(b)|^q+m\alpha\left|f''\Big(\frac{a+b}{2m}\Big)\right|^q\Big]^{\frac{1}{q}}\\
&=\frac{(b-a)^2}{16}\Big(\frac{1}{2p+1}\Big)^{\frac{1}{p}}\Big(\frac{1}{\alpha+1}\Big)^{\frac{1}{q}}\cdot\Big[\Big(\left|f''\Big(\frac{a+b}{2}\Big)\right|^q+m\alpha\left|f''\Big(\frac{a}{m}\Big)\right|^q\Big)^{\frac{1}{q}}\\
&\quad+\Big(|f''(b)|^q+m\alpha\left|f''\Big(\frac{a+b}{2m}\Big)\right|^q\Big)^{\frac{1}{q}}\Big].
\end{aligned}$$

Since $\frac{1}{3}<(\frac{1}{2p+1})^{\frac{1}{p}}<1$, we obtain

$$\begin{aligned}
&\left|f\Big(\frac{a+b}{2}\Big)-\frac{1}{b-a}\int_a^b f(x)dx\right|\\
&\leqslant\frac{(b-a)^2}{16}\Big(\frac{1}{\alpha+1}\Big)^{\frac{1}{q}}\cdot\Big[\Big(\left|f''\Big(\frac{a+b}{2}\Big)\right|^q+m\alpha\left|f''\Big(\frac{a}{m}\Big)\right|^q\Big)^{\frac{1}{q}}\\
&\quad+\Big(|f''(b)|^q+m\alpha\left|f''\Big(\frac{a+b}{2m}\Big)\right|^q\Big)^{\frac{1}{q}}\Big].
\end{aligned}$$

This completes the proof. □

COROLLARY 4. *Let $f: I \subset [0,\infty) \to \mathbb{R}$, be a differentiable mapping on I^o such that $f'' \in L[a,b]$ where $a,b \in I$ with $a < b$. If $|f''|^q$ is (α,m)-convex on $[a,b]$ for $\alpha \in [0,1]$ and $m \in (0,1]$, $q > 1$, then the following inequality holds*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \tag{2.5}$$
$$\leqslant \frac{(b-a)^2}{16}\left(\frac{1}{\alpha+1}\right)^{\frac{1}{q}} \cdot \left[\left|f''\left(\frac{a+b}{2}\right)\right| + |f''(b)| + (m\alpha)^{\frac{1}{q}}\left(\left|f''\left(\frac{a}{m}\right)\right| + \left|f''\left(\frac{a+b}{2m}\right)\right|\right)\right].$$

Proof. Let $a_1 = \left|f''\left(\frac{a+b}{2}\right)\right|^q$, $b_1 = m\alpha\left|f''\left(\frac{a}{m}\right)\right|^q$, $a_2 = |f''(b)|^q$, $b_2 = m\alpha\left|f''\left(\frac{a+b}{2m}\right)\right|^q$. Since,

$$\sum_{i=1}^{n}(a_i+b_i)^r \leqslant \sum_{i=1}^{n} a_i{}^r + \sum_{i=1}^{n} b_i{}^r$$

for $0 < r < 1$, $a_1, a_2, \ldots, a_n \geqslant 0$ and $b_1, b_2, \ldots, b_n \geqslant 0$, from Theorem 5, we obtain the inequalities

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leqslant \frac{(b-a)^2}{16}\left(\frac{1}{\alpha+1}\right)^{\frac{1}{q}} \cdot \left[\left(\left|f''\left(\frac{a+b}{2}\right)\right|^q + m\alpha\left|f''\left(\frac{a}{m}\right)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(|f''(b)|^q + m\alpha\left|f''\left(\frac{a+b}{2m}\right)\right|^q\right)^{\frac{1}{q}}\right]$$
$$\leqslant \frac{(b-a)^2}{16}\left(\frac{1}{\alpha+1}\right)^{\frac{1}{q}} \cdot \left[\left|f''\left(\frac{a+b}{2}\right)\right| + |f''(b)| + (m\alpha)^{\frac{1}{q}}\left(\left|f''\left(\frac{a}{m}\right)\right| + \left|f''\left(\frac{a+b}{2m}\right)\right|\right)\right],$$

where $0 < \frac{1}{q} < 1$. □

COROLLARY 5. *If we choose $\alpha = m = 1$ in (2.5), we obtain the following inequality*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leqslant \frac{(b-a)^2}{16}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left[2\left|f''\left(\frac{a+b}{2}\right)\right| + |f''(b)| + |f''(a)|\right] \tag{2.6}$$

THEOREM 6. *Let $f: I \subset [0,\infty) \to \mathbb{R}$, be a differentiable mapping on I^o such that $f'' \in L[a,b]$ where $a,b \in I$ with $a < b$. If $|f''|^q$ is (α,m)-convex on $[a,b]$ for $\alpha \in [0,1]$ and $m \in (0,1]$, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \tag{2.7}$$
$$\leqslant \frac{(b-a)^2}{16}\left(\frac{1}{q+\alpha+1}\right)^{\frac{1}{q}}\left[\left|f''\left(\frac{a+b}{2}\right)\right|^q + \frac{m\alpha}{q+1}\left|f''\left(\frac{a}{m}\right)\right|^q\right]^{\frac{1}{q}}$$
$$+ \frac{(b-a)^2}{16}\left[|f''(b)|^q \beta(\alpha+1, q+1) + m\left|f''\left(\frac{a+b}{2m}\right)\right|^q\left(\frac{1}{q+1} - \beta(\alpha+1, q+1)\right)\right]^{\frac{1}{q}},$$

where $\beta(\cdot,\cdot)$ *denotes the Beta function, which is defined as*

$$\beta(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt, \quad x>0,\; y>0.$$

Proof. Suppose that $p>1$. From Lemma 1 and using the Hölder inequality, we have

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\\
&\leqslant \frac{(b-a)^2}{16}\Big[\int_0^1 t\Big(t\left|f''\Big(t\frac{a+b}{2}+(1-t)a\Big)\right|\Big)dt\\
&\quad+\int_0^1 (1-t)(1-t)\left|f''\Big(tb+(1-t)\frac{a+b}{2}\Big)\right|dt\Big]\\
&\leqslant \frac{(b-a)^2}{16}\Big(\int_0^1 t^p dt\Big)^{\frac{1}{p}}\Big(\int_0^1 t^q\left|f''\Big(t\frac{a+b}{2}+(1-t)a\Big)\right|^q dt\Big)^{\frac{1}{q}}\\
&\quad+\frac{(b-a)^2}{16}\Big(\int_0^1 (1-t)^p dt\Big)^{\frac{1}{p}}\Big(\int_0^1 (1-t)^q\left|f''\Big(tb+(1-t)\frac{a+b}{2}\Big)\right|^q dt\Big)^{\frac{1}{q}}
\end{aligned}$$

Since, $|f''|^q$ is (α,m)-convex on $[a,b]$, and using the facts that

$$\begin{aligned}
&\int_0^1 t^p dt=\int_0^1 (1-t)^p dt=\frac{1}{p+1}\\
&\int_0^1 t^p(1-t^\alpha)dt=\frac{\alpha}{(p+1)(p+\alpha+1)}\\
&\int_0^1 t^\alpha(1-t)^q dt=\beta(\alpha+1,q+1)\\
&\int_0^1 (1-t)^q(1-t^\alpha)dt=\frac{1}{q+1}-\beta(\alpha+1,q+1),
\end{aligned}$$

and

$$\beta(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt, \quad x>0,\; y>0,$$

we have

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\\
&\leqslant \frac{(b-a)^2}{16}\Big(\int_0^1 t^p dt\Big)^{\frac{1}{p}}\Big(\int_0^1 t^q\Big(t^\alpha\left|f''\Big(\frac{a+b}{2}\Big)\right|^q+m(1-t^\alpha)\left|f''\Big(\frac{a}{m}\Big)\right|^q\Big)dt\Big)^{\frac{1}{q}}\\
&\quad+\frac{(b-a)^2}{16}\Big(\int_0^1 (1-t)^p dt\Big)^{\frac{1}{p}}\Big(\int_0^1 (1-t)^q\Big(t^\alpha|f''(b)|^q+m(1-t^\alpha)\left|f''\Big(\frac{a+b}{2m}\Big)\right|^q\Big)dt\Big)^{\frac{1}{q}}
\end{aligned}$$

$$= \frac{(b-a)^2}{16}\Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\Big(\Big|f''\Big(\frac{a+b}{2}\Big)\Big|^q\int_0^1 t^{q+\alpha}dt + m\Big|f''\Big(\frac{a}{m}\Big)\Big|^q\int_0^1 t^q(1-t^\alpha)dt\Big)^{\frac{1}{q}}$$
$$+\frac{(b-a)^2}{16}\Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\Big(|f''(b)|^q\int_0^1 t^\alpha(1-t)^q dt + m\Big|f''\Big(\frac{a+b}{2m}\Big)\Big|^q\int_0^1 (1-t)^q(1-t^\alpha)dt\Big)^{\frac{1}{q}}$$
$$= \frac{(b-a)^2}{16}\Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\Big(\frac{1}{q+\alpha+1}\Big|f''\Big(\frac{a+b}{2}\Big)\Big|^q + m\Big|f''\Big(\frac{a}{m}\Big)\Big|^q\frac{\alpha}{(q+1)(q+\alpha+1)}\Big)^{\frac{1}{q}}$$
$$+\frac{(b-a)^2}{16}\Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\Big(|f''(b)|^q\beta(\alpha+1,q+1)$$
$$+m\Big|f''\Big(\frac{a+b}{2m}\Big)\Big|^q\Big(\frac{1}{q+1}-\beta(\alpha+1,q+1)\Big)\Big)^{\frac{1}{q}}.$$

Since $\frac{1}{2} < (\frac{1}{p+1})^{\frac{1}{p}} < 1$, we obtain

$$\Big|f\Big(\frac{a+b}{2}\Big) - \frac{1}{b-a}\int_a^b f(x)dx\Big|$$
$$\leqslant \frac{(b-a)^2}{16}\Big(\frac{1}{q+\alpha+1}\Big)^{\frac{1}{q}}\Big[\Big|f''\Big(\frac{a+b}{2}\Big)\Big|^q + m\Big|f''\Big(\frac{a}{m}\Big)\Big|^q\frac{\alpha}{q+1}\Big]^{\frac{1}{q}}$$
$$+\frac{(b-a)^2}{16}\Big[|f''(b)|^q\beta(\alpha+1,q+1) + m\Big|f''\Big(\frac{a+b}{2m}\Big)\Big|^q\Big(\frac{1}{q+1}-\beta(\alpha+1,q+1)\Big)\Big]^{\frac{1}{q}}.$$

This completes the proof. □

COROLLARY 6. *Let $f : I \subset [0,\infty) \to \mathbb{R}$, be a differentiable mapping on I^o such that $f'' \in L[a,b]$ where $a,b \in I$ with $a < b$. If $|f''|^q$ is convex on $[a,b]$ for $q > 1$ with $\frac{1}{p}+\frac{1}{q} = 1$, then the following inequality holds*

$$\Big|f\Big(\frac{a+b}{2}\Big) - \frac{1}{b-a}\int_a^b f(x)dx\Big| \tag{2.8}$$
$$\leqslant \frac{(b-a)^2}{16}\Big[\Big(|f''(a)|^q+(q+1)\Big|f''\Big(\frac{a+b}{2}\Big)\Big|^q\Big)^{\frac{1}{q}} + \Big(|f''(b)|^q+(q+1)\Big|f''\Big(\frac{a+b}{2}\Big)\Big|^q\Big)^{\frac{1}{q}}\Big].$$

Proof. From Theorem 5, using the property of Beta and Gamma functions which is for $x,y > 0$ defined as

$$\beta(x,y+1) = \frac{y}{x}\beta(x+1,y) = \frac{y}{x+y}\beta(x,y)$$

and

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad \Gamma(x+1) = x\Gamma(x), \quad \Gamma(n) = (n-1)!,$$

we obtain

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leqslant \frac{(b-a)^2}{16}\left(\frac{1}{q+\alpha+1}\left|f''\left(\frac{a+b}{2}\right)\right|^q+m\left|f''\left(\frac{a}{m}\right)\right|^q\frac{\alpha}{(q+1)(q+\alpha+1)}\right)^{\frac{1}{q}} \\
&\quad+\frac{(b-a)^2}{16}\left(\left|f''(b)\right|^q\left(\frac{q}{\alpha+q+1}\right)\frac{\Gamma(\alpha+1)\Gamma(q)}{\Gamma(\alpha+q+1)}\right. \\
&\quad\left.+m\left|f''\left(\frac{a+b}{2m}\right)\right|^q\left(\frac{1}{q+1}-\left(\frac{q}{\alpha+q+1}\right)\frac{\Gamma(\alpha+1)\Gamma(q)}{\Gamma(\alpha+q+1)}\right)\right)^{\frac{1}{q}} \\
&=\frac{(b-a)^2}{16}\left[\frac{1}{q+\alpha+1}\left|f''\left(\frac{a+b}{2}\right)\right|^q+m\left|f''\left(\frac{a}{m}\right)\right|^q\frac{\alpha}{(q+1)(q+\alpha+1)}\right]^{\frac{1}{q}} \\
&\quad+\frac{(b-a)^2}{16}\left[\left|f''(b)\right|^q\left(\frac{q}{\alpha+q+1}\right)\frac{\Gamma(\alpha+1)\Gamma(q)}{\Gamma(\alpha+q+1)}\right. \\
&\quad\left.+m\left|f''\left(\frac{a+b}{2m}\right)\right|^q\left(\frac{1}{q+1}-\left(\frac{q}{\alpha+q+1}\right)\frac{\Gamma(\alpha+1)\Gamma(q)}{\Gamma(\alpha+q+1)}\right)\right]^{\frac{1}{q}}. \qquad (2.9)
\end{aligned}$$

If we substitute $\alpha=m=1$ into the inequality (2.9), we have

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leqslant \frac{(b-a)^2}{16}\left(\frac{1}{q+2}\left|f''\left(\frac{a+b}{2}\right)\right|^q+\left|f''(a)\right|^q\frac{1}{(q+1)(q+2)}\right)^{\frac{1}{q}} \\
&\quad+\frac{(b-a)^2}{16}\left(\left|f''(b)\right|^q\left(\frac{q}{q+2}\right)\frac{\Gamma(2)\Gamma(q)}{\Gamma(q+2)}+\left|f''\left(\frac{a+b}{2}\right)\right|^q\left(\frac{1}{q+1}-\left(\frac{q}{q+2}\right)\frac{\Gamma(2)\Gamma(q)}{\Gamma(q+2)}\right)\right)^{\frac{1}{q}}.
\end{aligned}$$

By the facts that

$$\frac{\Gamma(2)\Gamma(q)}{\Gamma(q+2)}=\frac{1}{q(q+1)}, \qquad \left(\frac{1}{(q+1)(q+2)}\right)^{\frac{1}{q}}\leqslant 1,$$

for $q\in[1,\infty)$, we obtain

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leqslant \frac{(b-a)^2}{16}\left(\frac{1}{q+2}\left|f''\left(\frac{a+b}{2}\right)\right|^q+\left|f''(a)\right|^q\frac{1}{(q+1)(q+2)}\right)^{\frac{1}{q}} \\
&\quad+\frac{(b-a)^2}{16}\left(\left|f''(b)\right|^q\frac{1}{(q+1)(q+2)}+\left|f''\left(\frac{a+b}{2}\right)\right|^q\left(\frac{1}{q+1}-\frac{1}{(q+1)(q+2)}\right)\right)^{\frac{1}{q}} \\
&=\frac{(b-a)^2}{16}\left(\frac{1}{(q+1)(q+2)}\right)^{\frac{1}{q}}\left[\left(\left|f''(a)\right|^q+(q+1)\left|f''\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\left|f''(b)\right|^q+(q+1)\left|f''\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right]
\end{aligned}$$

$$\leqslant \frac{(b-a)^2}{16}\Big[\Big(|f''(a)|^q+(q+1)\Big|f''\Big(\frac{a+b}{2}\Big)\Big|^q\Big)^{\frac{1}{q}}+\Big(|f''(b)|^q+(q+1)\Big|f''\Big(\frac{a+b}{2}\Big)\Big|^q\Big)^{\frac{1}{q}}\Big].$$

This completes the proof. □

THEOREM 7. *Let $f: I\subset[0,\infty)\to\mathbb{R}$, be a differentiable mapping on I^o such that $f''\in L[a,b]$ where $a,b\in I$ with $a<b$. If $|f''|^q$ is (α,m)-convex on $[a,b]$ for $\alpha\in[0,1]$ and $m\in(0,1]$, $q>1$, then the following inequality holds*

$$\begin{aligned}
&\Big|f\Big(\frac{a+b}{2}\Big)-\frac{1}{b-a}\int_a^b f(x)dx\Big|\\
&\leqslant \frac{(b-a)^2}{16}\Big(\frac{1}{3}\Big)^{1-\frac{1}{q}}\Big[\Big(\frac{1}{\alpha+3}\Big|f''\Big(\frac{a+b}{2}\Big)\Big|^q+\frac{m\alpha}{3(\alpha+3)}\Big|f''\Big(\frac{a}{m}\Big)\Big|^q\Big)^{\frac{1}{q}}\\
&+\Big(|f''(b)|^q\frac{2}{(\alpha+1)(\alpha+2)(\alpha+3)}\\
&+m\Big|f''\Big(\frac{a+b}{2m}\Big)\Big|^q\Big(\frac{1}{3}-\frac{2}{(\alpha+1)(\alpha+2)(\alpha+3)}\Big)\Big)^{\frac{1}{q}}\Big]. \qquad (2.10)
\end{aligned}$$

Proof. From Lemma 1 and using the power mean inequality, we have

$$\begin{aligned}
&\Big|f\Big(\frac{a+b}{2}\Big)-\frac{1}{b-a}\int_a^b f(x)dx\Big|\\
&\leqslant \frac{(b-a)^2}{16}\Big[\int_0^1 t^2\Big|f''\Big(t\frac{a+b}{2}+(1-t)a\Big)\Big|dt+\int_0^1(t-1)^2\Big|f''\Big(tb+(1-t)\frac{a+b}{2}\Big)\Big|dt\Big]\\
&\leqslant \frac{(b-a)^2}{16}\Big(\int_0^1 t^2dt\Big)^{1-\frac{1}{q}}\Big(\int_0^1 t^2\Big|f''\Big(t\frac{a+b}{2}+(1-t)a\Big)\Big|^q dt\Big)^{\frac{1}{q}}\\
&+\frac{(b-a)^2}{16}\Big(\int_0^1(t-1)^2dt\Big)^{1-\frac{1}{q}}\Big(\int_0^1(t-1)^2\Big|f''\Big(tb+(1-t)\frac{a+b}{2}\Big)\Big|^q dt\Big)^{\frac{1}{q}}
\end{aligned}$$

Since $|f''|^q$ is (α,m)-convex on $[a,b]$, we have

$$\int_0^1 t^2\Big|f''\Big(t\frac{a+b}{2}+(1-t)a\Big)\Big|^q dt\leqslant\frac{1}{\alpha+3}\Big|f''\Big(\frac{a+b}{2}\Big)\Big|^q+\frac{m\alpha}{3(\alpha+3)}\Big|f''\Big(\frac{a}{m}\Big)\Big|^q$$

and

$$\begin{aligned}
&\int_0^1(t-1)^2\Big|f''\Big(tb+(1-t)\frac{a+b}{2}\Big)\Big|^q dt\\
&\leqslant|f''(b)|^q\beta(\alpha+1,3)+m\Big|f''\Big(\frac{a+b}{2m}\Big)\Big|^q\Big(\frac{1}{3}-\beta(\alpha+1,3)\Big).
\end{aligned}$$

Thus, combining the above inequalities, we obtain

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leqslant\ &\frac{(b-a)^2}{16}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\Bigg[\left(\frac{1}{\alpha+3}\left|f''\left(\frac{a+b}{2}\right)\right|^q+\frac{m\alpha}{3(\alpha+3)}\left|f''\left(\frac{a}{m}\right)\right|^q\right)^{\frac{1}{q}} \\
&+\left(|f''(b)|^q\beta(\alpha+1,3)+m\left|f''\left(\frac{a+b}{2m}\right)\right|^q\left(\frac{1}{3}-\beta(\alpha+1,3)\right)\right)^{\frac{1}{q}}\Bigg] \\
=\ &\frac{(b-a)^2}{16}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\Bigg[\left(\frac{1}{\alpha+3}\left|f''\left(\frac{a+b}{2}\right)\right|^q+\frac{m\alpha}{3(\alpha+3)}\left|f''\left(\frac{a}{m}\right)\right|^q\right)^{\frac{1}{q}} \\
&+\Bigg(|f''(b)|^q\frac{2}{(\alpha+1)(\alpha+2)(\alpha+3)} \\
&+m\left|f''\left(\frac{a+b}{2m}\right)\right|^q\left(\frac{1}{3}-\frac{2}{(\alpha+1)(\alpha+2)(\alpha+3)}\right)\Bigg)^{\frac{1}{q}}\Bigg].
\end{aligned}$$

This completes the proof. □

COROLLARY 7. *If we choose* $\alpha=m=1$ *in (2.10), we obtain the following inequality*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \qquad (2.11)\\
\leqslant\ &\frac{(b-a)^2}{48}\left(\frac{3}{4}\right)^{\frac{1}{q}}\Bigg[\left(\left|f''\left(\frac{a+b}{2}\right)\right|^q+\frac{1}{3}|f''(a)|^q\right)^{\frac{1}{q}}+\left(\left|f''\left(\frac{a+b}{2}\right)\right|^q+\frac{1}{3}|f''(b)|^q\right)^{\frac{1}{q}}\Bigg].
\end{aligned}$$

REMARK 2. The inequality (2.11) is the result in Corollary 1.

COROLLARY 8. *From Theorem 4–7, we obtain the following inequality*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\leqslant\min\{I_1,I_2,I_3,I_4\}, \qquad (2.12)$$

where

$$\begin{aligned}
I_1=\ &\frac{(b-a)^2}{48(\alpha+3)}\left[3\left|f''\left(\frac{a+b}{2}\right)\right|+\alpha m\left|f''\left(\frac{a}{m}\right)\right|\right] \\
&+\frac{(b-a)^2}{48(\alpha+1)(\alpha+2)(\alpha+3)}\left\{6\,|f''(b)|-m[(\alpha+1)(\alpha+2)(\alpha+3)+6]\left|f''\left(\frac{a+b}{2m}\right)\right|\right\}, \\
I_2=\ &\frac{(b-a)^2}{16}\left(\frac{1}{\alpha+1}\right)^{\frac{1}{q}}\Bigg[\left(\left|f''\left(\frac{a+b}{2}\right)\right|^q+m\alpha\left|f''\left(\frac{a}{m}\right)\right|^q\right)^{\frac{1}{q}} \\
&+\left(|f''(b)|^q+m\alpha\left|f''\left(\frac{a+b}{2m}\right)\right|^q\right)^{\frac{1}{q}}\Bigg],
\end{aligned}$$

$$I_3=\frac{(b-a)^2}{16}\Big(\frac{1}{q+\alpha+1}\Big)^{\frac{1}{q}}\Big(\Big|f''\Big(\frac{a+b}{2}\Big)\Big|^q+m\Big|f''\Big(\frac{a}{m}\Big)\Big|^q\frac{\alpha}{q+1}\Big)^{\frac{1}{q}}$$
$$+\frac{(b-a)^2}{16}\Big(|f''(b)|^q\beta(\alpha+1,q+1)+m\Big|f''\Big(\frac{a+b}{2m}\Big)\Big|^q\Big(\frac{1}{q+1}-\beta(\alpha+1,q+1)\Big)\Big)^{\frac{1}{q}},$$
$$I_4=\frac{(b-a)^2}{16}\Big(\frac{1}{3}\Big)^{1-\frac{1}{q}}\Big[\Big(\frac{1}{\alpha+3}\Big|f''\Big(\frac{a+b}{2}\Big)\Big|^q+\frac{m\alpha}{3(\alpha+3)}\Big|f''\Big(\frac{a}{m}\Big)\Big|^q\Big)^{\frac{1}{q}}$$
$$+\Big(|f''(b)|^q\frac{2}{(\alpha+1)(\alpha+2)(\alpha+3)}$$
$$+m\Big|f''\Big(\frac{a+b}{2m}\Big)\Big|^q\Big(\frac{1}{3}-\frac{2}{(\alpha+1)(\alpha+2)(\alpha+3)}\Big)\Big)^{\frac{1}{q}}\Big].$$

3. Applications to special means

Consider the following special means for arbitrary real numbers $\alpha,\beta,\alpha\neq\beta$ as follows:

(1) The arithmetic mean:

$$A=A(\alpha,\beta)=\frac{\alpha+\beta}{2},\quad \alpha,\beta\in\mathbb{R}\text{ with }\alpha,\beta>0$$

(2) The logarithmic mean:

$$L=L(\alpha,\beta)=\frac{\beta-\alpha}{\ln\beta-\ln\alpha},\quad \alpha\neq\beta,\ \alpha,\beta\in\mathbb{R}\text{ with }\alpha,\beta>0$$

(3) The generalized logarithmic mean:

$$L_n(\alpha,\beta)=\left[\frac{\beta^{n+1}-\alpha^{n+1}}{(n+1)(\beta-\alpha)}\right]^{\frac{1}{n}},\quad n\in Z\setminus\{-1,0\},\quad \alpha\neq\beta,\ \alpha,\beta\in\mathbb{R}\text{ with }\alpha,\beta>0$$

(4) The harmonic mean:

$$H=H(\alpha,\beta)=\frac{2\alpha\beta}{\alpha+\beta},\ \alpha,\beta\in\mathbb{R}\text{ with }\alpha,\beta>0$$

(5) The geometric mean:

$$G=G(\alpha,\beta)=\sqrt{\alpha\beta},\ \alpha,\beta\in\mathbb{R}\text{ with }\alpha,\beta>0$$

(6) The identric mean:

$$I=I(\alpha,\beta)=\begin{cases} a & \text{if}\quad a=b\\ \frac{1}{e}\Big(\frac{b^b}{a^a}\Big)^{\frac{1}{b-a}} & \text{if}\quad a\neq b.\end{cases}$$

Now using the results of Section 2, we give some applications to special means of positive real numbers.

(1) Let $f:[a,b]\subset[0,d]\to R,\ d>0,\ f(x)=x^n,\ n\in(-\infty,0)\cup[1,\infty)\setminus\{-1\}$. Then,

$$f\Big(\frac{a+b}{2}\Big)=A^n(a,b),$$
$$\frac{1}{b-a}\int_a^b f(x)dx=L_n^n(a,b).$$

(a) From Corollary 3, we obtain

$$|A^n(a,b)-L_n^n(a,b)|\leqslant\frac{(b-a)^2}{192}n(n-1)\Big[|a|^{n-2}+|b|^{n-2}-2\Big|\Big(\frac{a+b}{2}\Big)\Big|^{n-2}\Big].$$

(b) From Corollary 5, we obtain

$$|A^n(a,b)-L_n^n(a,b)|\leqslant\frac{(b-a)^2}{16}\Big(\frac{1}{2}\Big)^{\frac{1}{q}}n(n-1)\Big[2\Big|\frac{a+b}{2}\Big|^{n-2}+|b|^{n-2}+|a|^{n-2}\Big].$$

(2) Let $f:[a,b]\subset(0,d]\to R,\ d>0,\ f(x)=\frac{1}{x},\ x\in[a,b]$. Then,

$$f\Big(\frac{a+b}{2}\Big)=\frac{H(a,b)}{G^2(a,b)},$$
$$\frac{1}{b-a}\int_a^b f(x)dx=L^{-1}(a,b).$$

(a) From Corollary 6, we obtain

$$\Big|\frac{H(a,b)}{G^2(a,b)}-L^{-1}(a,b)\Big|$$
$$\leqslant\frac{(b-a)^2}{8}\Big[\Big(\Big|\frac{1}{a}\Big|^{3q}+(q+1)\Big|\frac{2}{a+b}\Big|^{3q}\Big)^{\frac{1}{q}}+\Big(\Big|\frac{1}{b}\Big|^{3q}+(q+1)\Big|\frac{2}{a+b}\Big|^{3q}\Big)^{\frac{1}{q}}\Big]$$

(b) From Corollary 7, we obtain

$$\Big|\frac{H(a,b)}{G^2(a,b)}-L^{-1}(a,b)\Big|$$
$$\leqslant\frac{(b-a)^2}{24}\Big(\frac{3}{4}\Big)^{\frac{1}{q}}\Big[\Big(\Big|\frac{2}{a+b}\Big|^{3q}+\frac{1}{3}\Big|\frac{1}{a}\Big|^{3q}\Big)^{\frac{1}{q}}+\Big(\Big|\frac{2}{a+b}\Big|^{3q}+\frac{1}{3}\Big|\frac{1}{b}\Big|^{3q}\Big)^{\frac{1}{q}}\Big].$$

(3) Let $f:[a,b]\subset(0,d]\to R,\ d>0,\ f(x)=\frac{1}{x^n},\ n\in(-\infty,-1]\cup(0,\infty)\setminus\{1\}$. Then,

$$f\Big(\frac{a+b}{2}\Big)=A^{-n}(a,b),$$

$$\frac{1}{b-a}\int_a^b f(x)dx = L_{-n}^{-n}(a,b).$$

(a) From Corollary 3, we obtain

$$\begin{aligned}&\left|A^{-n}(a,b)-L_{-n}^{-n}(a,b)\right|\\ &\leqslant \frac{(b-a)^2}{192}n(n+1)\left[|a|^{-n-2}+|b|^{-n-2}-2\left|\frac{a+b}{2}\right|^{-n-2}\right]\end{aligned}$$

(b) From Corollary 6, we obtain

$$\begin{aligned}&\left|A^{-n}(a,b)-L_{-n}^{-n}(a,b)\right|\\ &\leqslant \frac{(b-a)^2}{16}n(n+1)\left[\left(|a|^{-(n+2)q}+(q+1)\left|\frac{a+b}{2}\right|^{-(n+2)q}\right)^{\frac{1}{q}}\right.\\ &\left.+\left(|b|^{-(n+2)q}+(q+1)\left|\frac{a+b}{2}\right|^{-(n+2)q}\right)^{\frac{1}{q}}\right].\end{aligned}$$

(4) Let $f:[a,b]\subset[0,d]\to R,\ d>0,\ f(x)=-\ln x,\ x\in[a,b]$. Then,

$$\begin{aligned}f\left(\frac{a+b}{2}\right) &= -\ln A(a,b),\\ \frac{1}{b-a}\int_a^b f(x)dx &= -\ln I(a,b).\end{aligned}$$

(a) From Corollary 5, we obtain

$$\begin{aligned}&|\ln I(a,b)-\ln A(a,b)|\\ &\leqslant \frac{(b-a)^2}{16}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left[2\left|\frac{2}{a+b}\right|^2+\left|\frac{1}{b}\right|^2+\left|\frac{1}{a}\right|^2\right]\end{aligned}$$

(b) From Corollary 7, we obtain

$$\begin{aligned}&|\ln I(a,b)-\ln A(a,b)|\\ &\leqslant \frac{(b-a)^2}{48}\left(\frac{3}{4}\right)^{\frac{1}{q}}\left[\left(\left|\frac{2}{a+b}\right|^{2q}+\frac{1}{3}\left|\frac{1}{a}\right|^{2q}\right)^{\frac{1}{q}}+\left(\left|\frac{2}{a+b}\right|^{2q}+\frac{1}{3}\left|\frac{1}{b}\right|^{2q}\right)^{\frac{1}{q}}\right].\end{aligned}$$

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