Journal of Mathematical Inequalities
Volume 11, Number 2 (2017), 399–411 doi:10.7153/jmi-11-34

PROOF OF AN INEQUALITY CONJECTURE FOR A POINT IN THE PLANE OF A TRIANGLE

YONG YANG, SHENGLI CHEN, DONG HUANG, XIANG WANG AND XIAOGUANG LIN

(*Communicated by L. Yang*)

Abstract. In [1] Jian Liu established a novel inequality about an arbitrary point in the plane of a triangle. He also put forward a conjecture about a parameterized version of this inequality. In this paper, we proceed to give a proof of this inequality facilitated by a combination of computer-aided calculations and traditional planar geometry. This proof demonstrates again the strengths of the real algebra methodology develped over time by Ritt, Wu, Yang, Yang, Xia, et. al.

1. Introduction

For a given triangle ABC, let a, b, c denote the side lengths BC, CA, AB respectively. Let P be a point in the plane of the triangle. Denote the distances from P to the vertices A, B, C by R_1, R_2, R_3 and the distances from P to the sides BC, CA, AB by r_1, r_2, r_3 respectively.

In a recent paper [1], Jian Liu gave the following two inequalities:

PROPOSITION 1. *For any point P in a plane of the triangle ABC, the following inequality holds:*

$$\frac{R_1^2-r_1^2}{b^2+c^2}+\frac{R_2^2-r_2^2}{c^2+a^2}+\frac{R_3^2-r_3^2}{a^2+b^2}\geqslant\frac{3}{8}. \tag{1.1}$$

And the conjecture was proposed at the end of the paper (A form of generalization for Proposition 1):

PROPOSITION 2. *Let λ be a real number, and $0<\lambda<1$, for any point P in a plane of the triangle ABC, the following inequality holds:*

$$\frac{R_1^2-r_1^2}{b^2+c^2+\lambda a^2}+\frac{R_2^2-r_2^2}{c^2+a^2+\lambda b^2}+\frac{R_3^2-r_3^2}{a^2+b^2+\lambda c^2}\geqslant\frac{3}{4(\lambda+2)}. \tag{1.2}$$

In this paper, we present a proof of the above-mentioned proposition.

Mathematics subject classification (2010): 51M16, 51N20.
Keywords and phrases: Triangle, point, difference substitution, conjecture.

© ELEMENT, Zagreb
Paper JMI-11-34

2. Several lemmas

In order to prove proposition 2, we first give several lemmas:

LEMMA 1. [2, p. 329] *Given a cubic polynomial of real coefficients:*

$$Q(\lambda) = \lambda^3 + p\lambda^2 + q\lambda + r,$$

where $r \geqslant 0$, *then*

$$(\forall \lambda \geqslant 0) Q(\lambda) \geqslant 0$$

holds only if $Q(\lambda)$*'s discriminant:*

$$-27r^2 + 18pqr - 4q^3 + p^2q^2 - 4p^3r \leqslant 0$$

holds.

LEMMA 2. [3, pp. 55–56] *Given a quartic polynomial of real coefficients:*

$$Q(\lambda) = \lambda^4 + p\lambda^3 + q\lambda^2 + r\lambda + s,$$

where $s \neq 0$, *then*

$$(\forall \lambda \geqslant 0) Q(\lambda) \geqslant 0$$

is equivalent to

$$\begin{aligned}
&s \geqslant 0 \wedge ((p \geqslant 0 \wedge q \geqslant 0 \wedge r \geqslant 0) \vee (d_8 > 0 \wedge (d_6 \leqslant 0 \vee d_4 \leqslant 0)) \vee (d_8 = 0 \wedge d_6 < 0) \\
&\vee (d_8 < 0 \wedge d_7 \geqslant 0 \wedge (p \geqslant 0 \vee d_5 < 0)) \vee (d_8 < 0 \wedge d_7 < 0 \wedge p > 0 \wedge d_5 > 0) \\
&\vee (d_8 = 0 \wedge d_6 > 0 \wedge d_7 > 0 \wedge (p \geqslant 0 \vee d_5 < 0)) \\
&\vee (d_8 = 0 \wedge d_6 = 0 \wedge (d_4 \leqslant 0 \vee E_1 = 0))),
\end{aligned}$$

where

$$\begin{aligned}
&d_4 = 3p^2 - 8q, \\
&d_5 = 3pr + p^2q - 4q^2, \\
&d_6 = 14pqr - 4q^3 + 16sq - 3p^3r + p^2q^2 - 6p^2s - 18r^2, \\
&d_7 = 7p^2sr - 18pqr^2 - 3qp^3s - p^2q^2r + 16ps^2 + 4p^3r^2 + 12pq^2s + 4rq^3 - 48rqs + 27r^3, \\
&d_8 = 144sr^2q + q^2p^2r^2 + 144p^2qs^2 + 18pr^3q - 192prs^2 - 6p^2r^2s - 27s^2p^4 - 27r^4 \\
&\qquad -4p^3r^3 - 128q^2s^2 + 16q^4s - 80prq^2s + 18p^3rqs + 256s^3 - 4q^3r^2 - 4q^3p^2s, \\
&E_1 = p^3 - 4pq + 8r.
\end{aligned}$$

LEMMA 3. *Let* x, k *be non-negative real numbers, then the following inequality strictly holds:*

$$\begin{aligned}
f_1(x,k) = &(2k^6 - k^5 - 9k^4 + 15k^3 - 9k^2 + 2k)x^4 \\
&+ (24k^6 - 12k^5 - 108k^4 + 167k^3 - 88k^2 + 15k + 2)x^3 \\
&+ (96k^6 - 48k^5 - 416k^4 + 636k^3 - 278k^2 + 8k + 10)x^2 \\
&+ (128k^6 - 64k^5 - 448k^4 + 912k^3 - 248k^2 - 76k)x \\
&+ 256k^4 + 320k^3 + 128k^2 + 16k \geqslant 0. \qquad (2.1)
\end{aligned}$$

Proof.

$$f_1(x,0) = 2x^3 + 10x^2 \geqslant 0, \quad f_1(x,1) = 8x^2 + 204x + 720 \geqslant 0,$$

When $k = 0,1$, formula (2.1) holds.

When $k \neq 0,1$, deducing from $k \geqslant 0$ and lemma 1, the following inequality holds:

$$2k^6 - k^5 - 9k^4 + 15k^3 - 9k^2 + 2k = k(k-1)^2(2k^3 + 3k^2 - 5k + 2) > 0,$$

and formula (2.1) can be transformed into the form:

$$F(x,k) = x^4 + px^3 + qx^2 + rx + s \geqslant 0, \tag{2.2}$$

where

$$p = \frac{24k^6 - 12k^5 - 108k^4 + 167k^3 - 88k^2 + 15k + 2}{2k^6 - k^5 - 9k^4 + 15k^3 - 9k^2 + 2k},$$
$$q = \frac{96k^6 - 48k^5 - 416k^4 + 636k^3 - 278k^2 + 8k + 10}{2k^6 - k^5 - 9k^4 + 15k^3 - 9k^2 + 2k},$$
$$r = \frac{128k^6 - 64k^5 - 448k^4 + 912k^3 - 248k^2 - 76k}{2k^6 - k^5 - 9k^4 + 15k^3 - 9k^2 + 2k},$$
$$s = \frac{256k^4 + 320k^3 + 128k^2 + 16k}{2k^6 - k^5 - 9k^4 + 15k^3 - 9k^2 + 2k}.$$

When $0 < k \leqslant \frac{3}{4}$, let $k = \frac{3}{4(1+t)}$, $t \geqslant 0$, then formula (2.1) holds. As a matter of fact, if $t = 2$, namely $k = \frac{1}{4}$, then the following inequality holds:

$$F(x,k) = \frac{(31x^2 + 636x + 2304)(3x-4)^2}{279} \geqslant 0.$$

If $t \neq 2$, then according to lemma 2, the following inequalities hold:

$$p > 0 \iff P = 256t^5 + 2912t^4 + 3976t^3 + 2200t^2 + 2354t + 745 \geqslant 0,$$
$$\begin{aligned} d_8 < 0 \iff D_8 = {} & 16777216000t^{16} + 452737368064t^{15} + 5492793409536t^{14} \\ & + 40388043800576t^{13} + 204709123801088t^{12} + 764617681330176t^{11} \\ & + 2164135136708608t^{10} + 4659323797288192t^9 + 7600945861071936t^8 \\ & + 9389474123185792t^7 + 8848816607680768t^6 + 6477274429869648t^5 \\ & + 3784566061553600t^4 + 1794395191836500t^3 + 656082934163589t^2 \\ & + 153990040466752t + 15259442511100 > 0, \end{aligned}$$
$$\begin{aligned} d_7 \geqslant 0 \iff D_7 = {} & 55700357120t^{20} + 1455255715840t^{19} + 17596839624704t^{18} \\ & + 132689419567104t^{17} + 705781783855104t^{16} + 2815906269167616t^{15} \\ & + 8672463622299648t^{14} + 20823326754035712t^{13} + 38970441922437120t^{12} \\ & + 56538553659942400t^{11} + 63089415207186944t^{10} + 53708501972210560t^9 \\ & + 34721931897594048t^8 + 17110250565185760t^7 + 6401560382411088t^6 \\ & + t^2 \cdot T(t) + 43882903763107t + 7244967625100 \geqslant 0. \end{aligned}$$

where

$$T(t) = 1558050794960736t^3 - 20222743938600t^2 - 129987454776360t + 36447729930224.$$

Note that $T(t)$'s discriminant < 0, by applying lemma 1, $T(t) \geqslant 0$.

When $1 < k \leqslant \frac{5}{4}$, let $k = 1 + \frac{1}{4(1+t)}$, $t \geqslant 0$, by applying lemma 2, then the following inequalities hold:

$$\begin{aligned}
d_8 < 0 \iff D_8 = {} & 12466142576640t^{16} + 500434087182336t^{15} \\
& + 6742630075465728t^{14} + 49687811913678848t^{13} \\
& + 237983768869404672t^{12} + 806127835204251648t^{11} \\
& + 2024892983757444096t^{10} + 3878251354845044992t^{9} \\
& + 5753158640324326336t^{8} + 6655078477210731392t^{7} \\
& + 5997762895076430016t^{6} + 4174439498511965968t^{5} \\
& + 2202577055307284608t^{4} + 852575550107577100t^{3} \\
& + 228508300424146091t^{2} + 37912728080055388t \\
& + 2934949983406960 > 0. \\
d_7 \geqslant 0 \iff D_7 = {} & 39427799777280t^{20} + 1767578221412352t^{19} \\
& + 29609486217904128t^{18} + 281847590008389632t^{17} \\
& + 1789685013845901312t^{16} + 8230295633013604352t^{15} \\
& + 28771548429660569600t^{14} + 78840718231499520000t^{13} \\
& + 172776969639545382912t^{12} + 306753637240926299648t^{11} \\
& + 444636509130302952448t^{10} + 527976960053189902720t^{9} \\
& + 513343995080084259008t^{8} + 406860115177863086048t^{7} \\
& + 260496715385419954928t^{6} + 132745508179694007456t^{5} \\
& + 52604266190856133576t^{4} + 15630063913339880328t^{3} \\
& + 3276945401920388804t^{2} + 432379509256460815t \\
& + 27009921359339330 \geqslant 0, \\
d_5 < 0 \iff D_5 = {} & 968884224t^{13} + 14701821952t^{12} + 102537674752t^{11} \\
& + 435079538688t^{10} + 1252537227264t^{9} + 2582192426112t^{8} \\
& + 3919402912256t^{7} + 4431343830912t^{6} + 3728206501088t^{5} \\
& + 2302276676656t^{4} + 1012593315248t^{3} + 299686426256t^{2} \\
& + 53321404077t + 4287198330 > 0.
\end{aligned}$$

When $(0 < k < \frac{1}{4}) \vee (\frac{1}{4} < k \leqslant \frac{3}{4}) \vee (1 < k \leqslant \frac{5}{4})$ and

$$d_8 < 0 \wedge d_7 \geqslant 0 \wedge (p \geqslant 0 \vee d_5 < 0), \tag{2.3}$$

then $F(x,k) \geqslant 0$ holds.

When $\frac{3}{4} < k < 1$, let $k = \frac{3}{4} + \frac{1}{4(1+t)}$, $t > 0$, then

$$\begin{aligned}
&F(x,k) \geqslant 0 \iff \\
&(75t^6 + 394t^5 + 800t^4 + 736t^3 + 256t^2)x^4 \\
&+(2980t^6 + 16728t^5 + 37216t^4 + 40480t^3 + 21120t^2 + 4096t)x^3 \\
&+(4112t^6 + 36448t^5 + 120832t^4 + 198784t^3 + 175872t^2 + 81920t + 16384)x^2 \\
&+(110784t^6 + 877696t^5 + 2807296t^4 + 4677120t^3 + 4305920t^2 + 2086912t + 417792)x \\
&+2048(3t+4)(4t+5)(5t+6)^2(t+1)^2 \geqslant 0
\end{aligned}$$

holds.

When $k > \frac{5}{4}$, let $k = \frac{5}{4} + t$, $t > 0$, then

$$\begin{aligned}
&F(x,k) \geqslant 0 \iff \\
&(4096t^6 + 28672t^5 + 64768t^4 + 66560t^3 + 33968t^2 + 8016t + 695)x^4 \\
&+(49152t^6 + 344064t^5 + 777216t^4 + 772096t^3 + 348736t^2 + 55360t + 1396)x^3 \\
&+(196608t^6 + 1376256t^5 + 3141632t^4 + 3186688t^3 + 1607936t^2 + 442624t + 65360)x^2 \\
&+(262144t^6 + 1835008t^5 + 4407296t^4 + 5472256t^3 + 4934656t^2 + 3361792t + 1019840)x \\
&+524288t^4 + 3276800t^3 + 7634944t^2 + 7856128t + 3010560 \geqslant 0
\end{aligned}$$

holds.

So when $(\frac{3}{4} < k < 1) \vee (k > \frac{5}{4})$, $F(x,k) \geqslant 0$ holds.

So the proof of lemma 3 is completed. □

LEMMA 4. *Let x,k be non-negative real number, then*

$$\begin{aligned}
f_2(x,k) = &(1024k^7 - 256k^6 - 1792k^5 + 832k^4 + 2560k^3 + 1248k^2 + 176k)x^4 \\
&+ (1024k^7 + 384k^6 - 2624k^5 - 912k^4 + 3920k^3 + 120k^2 - 372k + 4)x^3 \\
&+ (384k^7 + 384k^6 - 1296k^5 - 884k^4 + 2136k^3 - 412k^2 - 156k + 60)x^2 \\
&+ (64k^7 + 104k^6 - 268k^5 - 219k^4 + 493k^3 - 163k^2 - 17k + 22)x \\
&+ 4k^7 + 9k^6 - 20k^5 - 17k^4 + 41k^3 - 18k^2 - k + 2 \geqslant 0. \qquad (2.4)
\end{aligned}$$

Proof.

(1) When $k \leqslant \frac{1}{15}$, let $k = \frac{1}{15(1+y)}$, $y \geqslant 0$, then

$$\begin{aligned}
&f_2(x,k) \geqslant 0 \iff \\
g(y,x) = &(683437500x^3 + 10251562500x^2 + 3758906250x + 341718750)y^7 \\
&+(2004750000x^4 + 546750000x^3 + 69984000000x^2 + 26118703125x \\
&+2380640625)y^6 \\
&+(x(12976200000x^3 - 10980562500x^2 + 204308325000x + 77651409375) \\
&+7094081250)y^5 \\
&+(x(34939350000x^3 - 38985300000x^2 + 330694447500x + 128063176875) \\
&+11723028750)y^4
\end{aligned}$$

$$\begin{aligned}&+(x(50093208000x^3-59123965500x^2+320566869000x+126550218375)\\&+11603901375)y^3\\&+(x(40333870800x^3-47115374400x^2+186139692900x+74942111700)\\&+6880824000)y^2\\&+(x(17293013760x^3-19400796540x^2+59958012060x+24629105460)\\&+2263465260)y\\&+x(3084451984x^3-3267961616x^2+8266628544x+3465647824)\\&+318673264\geqslant 0.\end{aligned}$$

By lemma 1, it is easy to obtain the following inequalities:

$$\begin{aligned}&12976200000x^3-10980562500x^2+204308325000x+77651409375\geqslant 0\\&34939350000x^3-38985300000x^2+330694447500x+128063176875\geqslant 0\\&50093208000x^3-59123965500x^2+320566869000x+126550218375\geqslant 0\\&40333870800x^3-47115374400x^2+186139692900x+74942111700\geqslant 0\\&17293013760x^3-19400796540x^2+59958012060x+24629105460\geqslant 0\\&3084451984x^3-3267961616x^2+8266628544x+3465647824\geqslant 0.\end{aligned}$$

So $g(y,x)\geqslant 0$ holds.

(2) When $k\geqslant \frac{2}{3}$, let $k=\frac{2}{3}+\frac{m}{n}$, $m\geqslant 0$, $n>0$, then

$$\begin{aligned}&f_2(x,k)\geqslant 0 \iff\\&h(x,m,n)=c_0(m,n)x^4+c_1(m,n)x^3+c_2(m,n)x^2+c_3(m,n)x+c_4(m,n)\geqslant 0\end{aligned}$$

holds.

Where

$$\begin{aligned}c_0(m,n)=&\ 2239488m^7+9891072m^6n+14743296m^5n^2+8247744m^4n^3\\&+5197824m^3n^4+11701152m^2n^5+10708752mn^6+3053792n^7\geqslant 0,\\c_1(m,n)=&\ 2239488m^7+11290752m^6n+18522432m^5n^2+7699536m^4n^3\\&-1791504m^3n^4+3767688m^2n^5+4975140mn^6+1178276n^7,\\c_2(m,n)=&\ 839808m^7+4758912m^6n+8363088m^5n^2+2926692m^4n^3-2298456m^3n^4\\&-300996m^2n^5+774972mn^6+255180n^7,\\c_3(m,n)=&\ 139968m^7+880632m^6n+1630044m^5n^2+535167m^4n^3-488457m^3n^4\\&-152955m^2n^5+44301mn^6+40724n^7,\\c_4(m,n)=&\ (972m^5+7371m^4n+17766m^3n^2+15354m^2n^3+5289mn^4+1124n^5)\\&\times(3m-n)^2\geqslant 0,\end{aligned}$$

so $c_0(m,n)\geqslant 0$, $c_4(m,n)\geqslant 0$ hold.

In order to prove $c_i(m,n)\geqslant 0$ $(i=1,2,3)$, presuming:

$$u=(m-n)^2\geqslant 0,\quad v=mn\geqslant 0,\quad p=(u-v)^2,\quad q=uv \tag{2.5}$$

(by ref [4] example 1, lemma 3 and lemma 4), then

$$(c_1(m,n)+c_1(n,m))(m+n)=3417764u^4+47025768u^3v+\cdots+191527232v^4\geqslant 0,$$

$$c_1(m,n)c_1(n,m) = 2638734962688u^7 + 61387677909504u^6v + \cdots$$
$$+2292667537348864v^7 \geqslant 0;$$
$$(c_2(m,n)+c_2(n,m))(m+n) = 1094988u^4 + 15388776u^3v + \cdots + 61276800v^4 \geqslant 0,$$
$$c_2(m,n)c_2(n,m) = 214302205440u^7 + 4865437725696u^6v + \cdots$$
$$+234677888640000v^7 \geqslant 0;$$
$$c(u,v) = (c_3(m,n)+c_3(n,m))(m+n)$$
$$= 180692u^4 + 2551161u^3v + \cdots + 10517696v^4 \geqslant 0,$$
$$d(u,v) = c_3(m,n)c_3(n,m)$$
$$= 5700056832u^7 + 121864375584u^6v + \cdots + 6913870571776v^7;$$
$$(d(u,v)+d(v,u))(u+v) = 6919570628608p^4 + \cdots + \cdots + 88070212611436q^4 \geqslant 0,$$
$$d(u,v)d(v,u) = 39409455188215535173632p^7 + \cdots$$
$$+48477264683897129141 9873881q^7 \geqslant 0.$$

and $c_i(m,n) \geqslant 0 (i=0,1,2,3,4)$ hold.

(3) When $\frac{1}{15} < k \leqslant \frac{2}{3}$, let $k = \frac{1}{15} + \frac{3}{5(1+t)}$, $t \geqslant 0$. By lemma 1, $32k^3 - 24k^2 - 44k + 48 \geqslant 0$ holds, then the following inequality holds:

$$1024k^7 - 256k^6 - 1792k^5 + 832k^4 + 2560k^3 + 1248k^2 + 176k$$
$$= 16k(2*k+1)(k^2(32k^3 - 24k^2 - 44k + 48) + 56k + 11) > 0.$$

Meanwhile $f_2(x,k)$ can be transformed into the form of formula (2.2).

By lemma 2, the following inequalities hold:

$$d_5 < 0 \iff$$
$$D_5 = 132661478750923269090092 8t^{19} + 3672128045434955455446408t^{18} + \cdots$$
$$+164132818639559936523437500t + 113131239345169067382812 5 > 0,$$
$$d_8 < 0 \iff$$
$$D_8 = 133795739331005652719583931321548 8t^{24} + \cdots$$
$$+2555556344459055689239501953125000000 > 0,$$
$$d_7 \geqslant 0 \iff$$
$$D_7 = 1290431251099964758216185881782299852 8t^{27} + \cdots$$
$$+278274927052716728833913803100585937 5000 \geqslant 0.$$

So the conditions of formula (2.3) in lemma 2 holds.

Hence $f_2(x,k) \geqslant 0$ holds.

So the proof of lemma 4 is completed. □

LEMMA 5. *Let x,k be the non-negative real number, then the following inequalities hold:*

$$f_3(x,k) = (2k^5 + 19k^4 + 34k^3 - 7k^2 - 6k + 6)(k-1)^2x^4$$
$$+(32k^7 + 240k^6 - 84k^5 - 751k^4 + 519k^3 + 215k^2 - 179k + 56)x^3$$

$$
\begin{aligned}
&+(192k^7+1440k^6-816k^5-3636k^4+2358k^3+1438k^2-582k+134)x^2\\
&+8(64k^7+480k^6-376k^5-922k^4+588k^3+500k^2-53k+3)x\\
&+768k+512k^7+3936k^2+3840k^6-3840k^5-5056k^4+3552k^3\geqslant 0,\\
f_4(x,k)=\ &(7k^6+16k^5-34k^4-18k^3+45k^2-22k+6)x^4\\
&+(112k^6+256k^5-545k^4-253k^3+619k^2-195k+54)x^3\\
&+(672k^6+1536k^5-3276k^4-1308k^3+3116k^2-400k+132)x^2\\
&+(1792k^6+4096k^5-8752k^4-2928k^3+6736k^2+528k+48)x\\
&+1664k+5184k^2+1792k^6+4096k^5-8768k^4-2368k^3\geqslant 0,\\
f_5(x,k)=\ &(4k^5+k^4-14k^3+10k^2-2k+1)x^4\\
&+(64k^5+16k^4-215k^3+135k^2-k+9)x^3\\
&+(384k^5+96k^4-1236k^3+660k^2+144k+24)x^2\\
&+(1024k^5+256k^4-3152k^3+1360k^2+688k+16)x\\
&+1024k^5+256k^4-3008k^3+960k^2+896k\geqslant 0.
\end{aligned}
$$

Proof. The coefficients of x^j $(j=0,1,2,3,4)$ in $f_i(x,k)$ $(i=3,4,5)$ are non-negative.

For example, for the proof:

$$d(k)=64k^7+480k^6-376k^5-922k^4+588k^3+500k^2-53k+3\geqslant 0$$

holds, only if

$$c(a,b)=64a^7+480a^6b-376a^5b^2-922a^4b^3+588a^3b^4+500a^2b^5-53ab^6+3b^7\geqslant 0.$$

(where $a\geqslant 0,\ b>0$).

Apply the substitution type formula (2.5), then:

$$
\begin{aligned}
&c(a,b)+c(b,a)=(67u^3+762u^2v+1807uv^2+284v^3)(a+b)\geqslant 0,\\
&d(u,v)=c(a,b)c(b,a)=192u^7+736u^6v-3208u^5v^2+284026u^4v^3+2472158u^3v^4\\
&\qquad+5336948u^2v^5+773011uv^6+80656v^7;\\
&d(u,v)+d(v,u)=(80848p^3+1177987p^2q+8140069pq^2+8944519q^3)(u+v)\geqslant 0,\\
&d(u,v)d(v,u)=15485952p^7+424584256p^6q+5020675104p^5q^2+52652306504p^4q^3\\
&\qquad+676897880750p^3q^4+6860009326662p^2q^5+37553412857030pq^6\\
&\qquad+80004420141361q^7\geqslant 0.
\end{aligned}
$$

So $c(a,b)\geqslant 0$ holds. □

LEMMA 6. *Let* $p_1\geqslant 0,\ p_2\geqslant 0,\ p_3\geqslant 0$ *and*

$$4p_1p_2-q_3^2\geqslant 0,\quad 4p_2p_3-q_1^2\geqslant 0,\quad 4p_3p_1-q_2^2\geqslant 0, \tag{2.6}$$

if

$$D(p_1,p_2,p_3,q_1,q_2,q_3)=4p_1p_2p_3-q_1q_2q_3-p_1q_1^2-p_2q_2^2-p_3q_3^2\geqslant 0 \tag{2.7}$$

holds, in which $p_1,\ p_2,\ p_3,\ q_1,\ q_2,\ q_3$ *are real number, then for arbitrary real number* x,y,z*, the inequality*

$$p_1x^2+p_2y^2+p_3z^2-(q_1yz+q_2zx+q_3xy)\geqslant 0 \tag{2.8}$$

holds.

Proof. Note from formula (2.6), when $p_1=0$, the following inequalities hold: $q_2=0,\ q_3=0,\ 4p_2p_3-q_1^2\geqslant 0$, and $p_2\geqslant 0,\ p_3\geqslant 0$.

Hence

$$p_2y^2+p_3z^2-q_1yz\geqslant 2\sqrt{p_2p_3y^2z^2}-q_1yz\geqslant 0.$$

So formula (2.8) holds.

Similarly, when $p_2=0\vee p_3=0$, formula (2.8) holds too.

When $p_1\neq 0\wedge p_2\neq 0\wedge p_3\neq 0$, the inequality (2.8) is equivalent to

$$p_1x^2-(q_2z+q_3y)x+p_2y^2+p_3z^2-q_1yz\geqslant 0.$$

For arbitrary real numbers $x,\ y,\ z$, the inequality holds only if the inequality

$$\begin{aligned}&4p_1(p_2y^2+p_3z^2-q_1yz)-(q_2z+q_3y)^2\\&\quad=(4p_1p_2-q_3^2)y^2+(4p_3p_1-q_2^2)z^2-(4p_1q_1+2q_2q_3)yz\geqslant 0\end{aligned}$$

and

$$4p_1p_2-q_3^2\geqslant 0,\quad 4p_3p_1-q_2^2\geqslant 0$$

hold.

So

$$\begin{aligned}&4(4p_1p_2-q_3^2)(4p_3p_1-q_2^2)-(4p_1q_1+2q_2q_3)^2\\&\quad=16p_1(4p_1p_2p_3-p_1q_1^2-p_2q_2^2-p_3q_3^2-q_1q_2q_3)\geqslant 0\end{aligned}$$

holds.

Based on the argument presented above, the proof of the proposition is completed. □

3. Proof of Proposition 2

Proof. Let (x,y,z) be the coordinates of point P with respect to triangle ABC (where $x,\ y,\ z$ are real numbers such that $x+y+z\geqslant 0$), and

$$p=a^2,\quad q=b^2,\quad r=c^2, \tag{3.1}$$

then (for ref [1])

$$R_1^2=\frac{(x+y+z)(zq+yr)-(yzp+zxq+xyr)}{(x+y+z)^2},$$

$$R_2^2=\frac{(x+y+z)(xr+zp)-(yzp+zxq+xyr)}{(x+y+z)^2},$$

$$R_3^2 = \frac{(x+y+z)(yp+xq)-(yzp+zxq+xyr)}{(x+y+z)^2},$$
$$r_1^2 = \frac{x^2(2pq+2qr+2rp-p^2-q^2-r^2)}{4(x+y+z)^2p},$$
$$r_2^2 = \frac{y^2(2pq+2qr+2rp-p^2-q^2-r^2)}{4(x+y+z)^2q},$$
$$r_3^2 = \frac{z^2(2pq+2qr+2rp-p^2-q^2-r^2)}{4(x+y+z)^2r}.$$

Substituting the above equalities and $\lambda = 1/(1+\lambda)$ into formula (1.2), then the inequalities can be transformed into the form of formula (2.8), i.e.,

$$\begin{cases} p_1 = P(p,q,r) = qr((2p^4-5(q+r)p^3+3(q^2+r^2)p^2+(2q^3+7q^2r+7qr^2+2r^3)p \\ \quad +2q^3r-4q^2r^2+2qr^3)\lambda^3+(4p^4-9(q+r)p^3+(7q^2+2qr+7r^2)p^2 \\ \quad +(10q^3+25q^2r+25qr^2+10r^3)p+2q^4+3q^3r-10q^2r^2+3qr^3+2r^4)\lambda^2 \\ \quad +(2p^4-(q+r)p^3+(9q^2+10qr+9r^2)p^2+(17q^3+23q^2r+23qr^2+17r^3)p \\ \quad +5q^4-10q^2r^2+5r^4)\lambda+3(q+r)p^3+(9q^2+6qr+9r^2)p^2 \\ \quad +(9q^3+3q^2r+3qr^2+9r^3)p+3q^4-6q^2r^2+3r^4), \\ p_2 = P(q,r,p), \\ p_3 = P(r,p,q); \\ q_1 = Q(p,q,r) \\ \quad = 2pqr((p+q)\lambda+p+q+r)((p+r)\lambda+p+q+r)((4p-q-r)\lambda+9p-3q-3r), \\ q_2 = Q(q,r,p), \\ q_3 = Q(r,p,q). \end{cases} \tag{3.2}$$

Then we can prove the coefficients of p_1 w.r.t. λ are all positive semidefinite. That is to say $p_1 \geqslant 0$. Similarly, the inequalities $p_2 \geqslant 0$, $p_3 \geqslant 0$ hold.

For proof of formula (2.6), we put:

$$r = s(p+q), \quad u = (p-q)^2, \quad v = pq, \quad u,v,s \geqslant 0, \tag{3.3}$$

Deducing from formula (3.2), the inequality $4p_1p_2 - q_3^2 \geqslant 0$ is equivalent to

$$R = e_0\lambda^5 + e_1\lambda^4 + e_2\lambda^3 + e_3\lambda^2 + e_4\lambda + e_5 \geqslant 0,$$

where

$$e_0 = v^4 f_1\left(\frac{u}{v}, s\right),$$
$$e_1 = u^4 f_2\left(\frac{v}{u}, s\right),$$
$$e_2 = (s+1) f_3\left(\frac{u}{v}, s\right),$$
$$e_3 = (s+1)^2 v^4 f_4\left(\frac{u}{v}, s\right),$$
$$e_4 = 2(s+1)^3 v^4 f_5\left(\frac{u}{v}, s\right),$$

$$e_5 = 3s(s-1)^2(s+1)^5(u+4v)^4 \geqslant 0.$$

Due to the formulation of f_1, f_2, f_3, f_4, f_5 and their non-negatives by lemma 3, 4, 5, this completes the proof of formula (2.6).

Finally, for proof of formula (2.7), we put:

$$a = y_0 + z_0, \quad b = z_0 + x_0, \quad c = x_0 + y_0, \quad x_0, y_0, z_0 > 0,$$

so

$$\begin{cases} u = \dfrac{(x_0-y_0)^2(y_0-z_0)^2(z_0-x_0)^2(x_0+y_0+z_0)^3}{x_0y_0z_0}, \\ v = \sum\limits_{cyc} x_0(x_0-y_0)(x_0-z_0), \\ w = \sum\limits_{cyc} (y_0+z_0)(x_0-y_0)(x_0-z_0) \\ \sigma_1 = x_0+y_0+z_0, \quad \sigma_2 = x_0y_0+y_0z_0+z_0x_0, \quad \sigma_3 = x_0y_0z_0. \end{cases} \tag{3.4}$$

and $u \geqslant 0$, $v \geqslant 0$, $w \geqslant 0$, therefore (for ref [5, 6, 7])

$$\begin{cases} \sigma_1^3 = \dfrac{(v+4w)u+4(v+w)^3}{u+(2v-w)^2} \\ \sigma_2^3 = \dfrac{[wu+w(v+w)(4v+w)]^3}{[u+(2v-w)^2]^2[(v+4w)u+4(v+w)^3]} \\ \sigma_3 = \dfrac{vw^2}{u+(2v-w)^2} \\ \sigma_1\sigma_2 = \dfrac{w[u+(v+w)(4v+w)]}{u+(2v-w)^2} \end{cases} \tag{3.5}$$

Substitute (3.2) into (2.7) with:

$$p = (y_0+z_0)^2, \quad q = (z_0+x_0)^2, \quad r = (x_0+y_0)^2.$$

Transform the result into the expression of u, v, w by the equations (3.4) and (3.5), and then we can achieve an equivalent inequality of (2.7) as following:

$$D = c_0u^7 + c_1u^6 + c_2u^5 + c_3u^4 + c_4u^3 + c_5u^2 + c_6u + c_7 \geqslant 0, \tag{3.6}$$

in which

$$\begin{aligned} c_0 &= 2(\lambda+2)^5v^8 + 48w(\lambda+2)^5v^7 + 4w^2(5\lambda^3+144\lambda^2+552\lambda \\ &\quad +504)(\lambda+2)^3v^6 + \cdots + 162w^8(\lambda+2)(5\lambda+6)^4, \\ c_1 &= 56(\lambda+2)^5v^{10} + 1288w(\lambda+2)^5v^9 + 2w^2(240\lambda^3+7399\lambda^2 \\ &\quad +26980\lambda+26092)(\lambda+2)^3v^8 + \cdots + 1134w^{10}(\lambda+2)(5\lambda+6)^4, \\ c_2 &= 672(\lambda+2)^5v^{12} + 14784w(\lambda+2)^5v^{11} + 48w^2(100\lambda^3+3381\lambda^2 \\ &\quad +12420\lambda+12036)(\lambda+2)^3v^{10} + \cdots + 3402w^{12}(\lambda+2)(5\lambda+6)^4, \\ c_3 &= 4480(\lambda+2)^5v^{14} + 94080w(\lambda+2)^5v^{13} + 160w^2(160\lambda^3+6153\lambda^2 \\ &\quad +22812\lambda+22164)(\lambda+2)^3v^{12} + \cdots + 5670w^{14}(\lambda+2)(5\lambda+6)^4, \end{aligned}$$

$$\begin{aligned}
c_4 &= 17920(\lambda+2)^5v^{16}+358400w(\lambda+2)^5v^{15}+2560w^2(30\lambda^3+1393\lambda^2\\
&\quad +5224\lambda+5092)(\lambda+2)^3v^{14}+\cdots+5670w^{16}(\lambda+2)(5\lambda+6)^4,\\
c_5 &= 43008(\lambda+2)^5v^{18}+817152w(\lambda+2)^5v^{17}+1536w^2(80\lambda^3+5019\lambda^2\\
&\quad +19092\lambda+18684)(\lambda+2)^3v^{16}+\cdots+3402w^{18}(\lambda+2)(5\lambda+6)^4,\\
c_6 &= 4vw^2(5w^2+4wv+4v^2)(9w^2+20wv+20v^2)(2v-w)^2\\
&\quad (2v+w)^2(2v+3w)^4(v+w)^5\lambda^6+\cdots+64(3w+v)(63w^3+\\
&\quad 141vw^2+112wv^2+28v^3)(3w^2+4wv+4v^2)^5(v+w)^6,\\
c_7 &= 32768(\lambda+2)^5v^{22}+557056w(\lambda+2)^5v^{21}+8192w^2(571\lambda\\
&\quad +1118)(\lambda+2)^4v^{20}+\cdots+162w^{22}(\lambda+2)(5\lambda+6)^4.
\end{aligned}$$

It's easy to find every c_i $(i=0,1,2,\ldots,7)$ is positive semidefinite.
So formula (2.7) holds, and this completes the proof of Proposition 2.

4. Several inequalities with parameters

By applying similar method presented in this article, we found the following interesting inequalities with parameters about any point in the plane of $\triangle ABC$:

PROPOSITION 3. *Let $\lambda \geqslant 1$ be a real number, so for any point P in the plane of $\triangle ABC$, the following inequalities*

$$\frac{2R_1^2-r_2^2-r_3^2}{b^2+c^2+\lambda a^2}+\frac{2R_2^2-r_3^2-r_1^2}{c^2+a^2+\lambda b^2}+\frac{2R_3^2-r_1^2-r_2^2}{a^2+b^2+\lambda c^2}\geqslant\frac{3}{2(\lambda+2)}, \tag{4.1}$$

$$\frac{R_2^2+R_3^2-r_2^2-r_3^2}{b^2+c^2+\lambda a^2}+\frac{R_3^2+R_1^2-r_3^2-r_1^2}{c^2+a^2+\lambda b^2}+\frac{R_1^2+R_2^2-r_1^2-r_2^2}{a^2+b^2+\lambda c^2}\geqslant\frac{3}{2(\lambda+2)} \tag{4.2}$$

hold.

PROPOSITION 4. *Let $\lambda \geqslant 2$ be a real number, so for any point P in the plane of $\triangle ABC$, the following inequalities*

$$\frac{2R_1^2-r_2^2-r_3^2}{b^2+c^2+\lambda bc}+\frac{2R_2^2-r_3^2-r_1^2}{c^2+a^2+\lambda ca}+\frac{2R_3^2-r_1^2-r_2^2}{a^2+b^2+\lambda ab}\geqslant\frac{3}{2(\lambda+2)}, \tag{4.3}$$

$$\frac{R_2^2+R_3^2-r_2^2-r_3^2}{b^2+c^2+\lambda bc}+\frac{R_3^2+R_1^2-r_3^2-r_1^2}{c^2+a^2+\lambda ca}+\frac{R_1^2+R_2^2-r_1^2-r_2^2}{a^2+b^2+\lambda ab}\geqslant\frac{3}{2(\lambda+2)} \tag{4.4}$$

hold.

Due to limited space, the proof of proposition 3 and proposition 4 will be explored in another paper.

REFERENCES

[1] JIAN LIU, *A new inequality for a point in the plane of a triangle*, Journal of Mathematical Inequalities, 2014, **8** (1): 597–611.

[2] WENJUN WU, *Mathematics Mechanization*, Beijing, Science Press, 2003.3.

[3] LU YANG, BICAN XIA, *Inequality proving and automatic discovery*, Beijing, Science Press, 2008.1.

[4] SHENGLI CHEN, YONG YAO, JIA XU, *The method of split Dimension reduction algebraic inequality and Machine proof*, System science and Mathematics, 2009, **29** (1): 26–34.

[5] SHENGLI CHEN, LIANGYU CHEN, *The Degree-Decreasing decomposition of symmetric forms with a program for algebraic inequality decision*, System science and Mathematics, 2013, **33** (2): 179–196.

[6] SHENGLI CHEN, FANGJIAN HUANG, *Schur decomposition for symmetric ternary forms and readable proof to inequalities*, Journal of Mathematics, 2006, **49** (3): 491–502.

[7] F. J. HUANG, S. L. CHEN, *Schur partition for symmetric ternary forms and readable proof to inequalites*, Proceedings of the 2005 International Symposium on Symbolic and Algebraic Computation, ACM press 2005: 185–192.

[8] L. YANG, X. HOU, Z. ZENG, *A complete discrimination symsten for polynomials*, China **E**, 1996, **39** (6): 628–646.

[9] LU YANG, BICAN XIA, *Quantifier Elimination for Quartics*, Artificial Intelligence and Symbolic Computation, Lecture Notes in Computer Science, Vol. 4120, (2006), pp. 131–145

[10] LU YANG, JINGZHONG ZHANG, XIAORONG HOU, *Nonlinear algebraic equations and theorem Machine proving*, Shanghai: Shanghai science and Education Press, 1996.9.

(Received November 5, 2015)

Yong Yang
Chongqing Institute of Green and Intelligent Technology
Chinese Academy of Sciences
Chong Qing, 400714, China
e-mail: yangyong@cigit.ac.cn

Shengli Chen
Chongqing Institute of Green and Intelligent Technology
Chinese Academy of Sciences
Chong Qing, 400714, China
e-mail: fjwxcsl@163.com

Dong Huang
Chongqing Institute of Green and Intelligent Technology
Chinese Academy of Sciences
Chong Qing, 400714, China
e-mail: huangdong@cigit.ac.cn

Xiang Wang
Chongqing Institute of Green and Intelligent Technology
Chinese Academy of Sciences
Chong Qing, 400714, China
e-mail: wangxiang@cigit.ac.cn

Xiaoguang Lin
Chongqing Institute of Green and Intelligent Technology
Chinese Academy of Sciences
Chong Qing, 400714, China
e-mail: lxg@cigit.ac.cn