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CLASSES OF ANALYTIC FUNCTIONS RELATED TO A COMBINATION OF TWO CONVEX FUNCTIONS

J. DZIOK AND K. I. NOOR

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Abstract. In the paper we introduce classes of functions related to a class of linear combinations of two convex functions. By using properties of multivalent prestarlike functions we obtain various inclusion relationships between defined classes of functions. Some applications of the main results are also considered.

Let $\mathscr{A}$ denote the class of functions which are *analytic* in $\mathscr{U} := \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathscr{A}_p$ $(p \in \mathscr{N}_0 := \{0,1,2,\ldots\})$ denote the class of functions $f \in \mathscr{A}$ of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (z \in \mathscr{U}). \tag{1}$$

By $\mathscr{S}_0^c$ we denote the class of functions $f \in \mathscr{A}_0$ which are univalent in $\mathscr{U}$ and $f(\mathscr{U})$ is the convex domain.

We say that $f \in \mathscr{A}$ is *subordinate* to $F \in \mathscr{A}$, and we write $f(z) \prec F(z)$ (or simply $f \prec F$), if there exists a function

$$\omega \in \Omega := \{\omega \in \mathscr{A} : \ |\omega(z)| \leqslant |z| \ (z \in \mathscr{U})\},$$

such that $f(z) = F(\omega(z))$ $(z \in \mathscr{U})$. In particular, if F is univalent in $\mathscr{U}$, we have the following equivalence

$$f(z) \prec F(z) \Longleftrightarrow [f(0) = F(0) \text{ and } f(\mathscr{U}) \subset F(\mathscr{U})].$$

Let $\alpha, \delta, \mu, \nu, k$ be real parameters, $\alpha < 1$, $\mu \geqslant 0$, $k \geqslant 2$, and let $\Phi = (\phi, \varphi) \in \mathscr{A}_p \times \mathscr{A}_p$, $\xi \in \mathscr{A}_p$, $H = (h_1, h_2)$, $G = (g_1, g_2) \in \mathscr{S}_0^c \times \mathscr{S}_0^c$. Then, we denote

$$\begin{aligned}
\mathscr{K}(h_1) &:= \{q \in \mathscr{A}_0 : q \prec h_1\}, \\
\mathscr{K}_\mu(H) &:= \mu\mathscr{K}(h_1) + (1-\mu)\mathscr{K}(h_2) \\
&= \{\mu q_1 + (1-\mu) q_2 : q_1 \in \mathscr{K}(h_1),\ q_2 \in \mathscr{K}(h_2)\}, \\
\mathscr{K}_\mu(h) &:= \mathscr{K}_\mu((h,h)), \quad \mathscr{P}_k(h) := \mathscr{K}_\mu(h), \ (\mu = k/4 + 1/2).
\end{aligned}$$

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In particular, the classes $P_k(\rho) := \mathscr{P}_k\left(\frac{1+(1-2\rho)z}{1-z}\right)$ $(0 \leqslant \rho < 1)$ and $P_k := \mathscr{P}_k\left(\frac{1+z}{1-z}\right)$ were investigated by Padmanabhan and Parvatham [20] and Pinchuk [22], respectively. The class $\mathscr{P} := P_2$ is the well-known class of Caratheodory functions.

We denote by $\mathscr{M}_\mu^\delta(\Phi,\xi,H)$ the class of functions $f \in \mathscr{A}_p$ such that

$$J_\delta(f)(z) := (1-\delta)\frac{(\xi * \phi) * f}{(\xi * \varphi) * f} + \delta\frac{\phi * f}{\varphi * f} \in \mathscr{K}_\mu(H),$$

where $*$ denote the Hadamard product (or convolution). Moreover, let us denote

$$\begin{aligned}
&\mathscr{M}^\delta(\Phi,\xi,h) := \mathscr{M}_1^\delta(\Phi,\xi,(h,h)), \quad \mathscr{M}_\mu^\delta(\Phi,H) := \mathscr{M}_\mu^\delta(\Phi,\xi_1,H),\\
&\mathscr{M}_\mu^\delta(\varphi,H) := \mathscr{M}_\mu^\delta((\widetilde{\varphi}_1,\widetilde{\varphi}_2),H), \quad \mathscr{M}^\delta(\varphi,h) := \mathscr{M}_1^\delta(\varphi,(h,h)),\\
&\mathscr{W}_\mu(\Phi,H) := \mathscr{M}_\mu^1(\Phi,\xi,H), \quad \mathscr{W}_\mu(\varphi,H) := \mathscr{W}_\mu\left(\left(p^{-1}z\varphi',\varphi\right),H\right),\\
&\mathscr{W}_\mu(H) := \mathscr{W}_\mu\left(z^p/(1-z),H\right), \quad \mathscr{W}(\varphi,h) := \mathscr{W}_1(\varphi,(h,h)),\\
&\mathscr{S}_p^*(\varphi,\alpha) := \mathscr{W}\left(\varphi,[1+(1-2\alpha/p)z](1-z)^{-1}\right),
\end{aligned}$$

where

$$\xi_1(z) = z^p + \sum_{n=p+1}^{\infty}\frac{p}{n}z^n,\ \widetilde{\varphi}_1(z) = \frac{z}{p}\varphi'(z), \quad \widetilde{\varphi}_2(z) = \frac{z}{p}\varphi_1'(z) \quad (z \in \mathscr{U}).$$

It is clear that

$$f \in \mathscr{W}_\mu(\varphi,H) \Leftrightarrow \varphi * f \in \mathscr{W}_\mu(H). \tag{2}$$

We say that a function $f \in \mathscr{A}_p$ belongs to the class $\mathscr{C}\mathscr{M}_{\mu,\nu}^\delta(\Phi,\xi,G,H)$, if there exists a function $g \in \mathscr{W}_\nu(\varphi,G)$ such that

$$(1-\delta)\frac{(\xi * \phi) * f}{(\xi * \varphi) * g} + \delta\frac{\phi * f}{\varphi * g} \in \mathscr{K}_\mu(H).$$

Moreover, let us denote

$$\begin{aligned}
&\mathscr{C}\mathscr{M}_\nu^\delta(\Phi,\xi,G,h) := \mathscr{C}\mathscr{M}_{1,\nu}^\delta(\Phi,\xi,G,(h,h)),\\
&\mathscr{C}\mathscr{W}_{\mu,\nu}(\Phi,G,H) := \mathscr{C}\mathscr{M}_{\mu,\nu}^1(\Phi,\xi,G,H),\\
&\mathscr{C}\mathscr{W}_{\mu,\nu}(\Phi,G,H) := \mathscr{C}\mathscr{W}_{\mu,\nu}(\Phi,G,H).
\end{aligned}$$

We note that (2) defines the linear operator $J : \mathscr{W}_\nu(\varphi,G) \to \mathscr{W}_\nu(G)$, $J(f) = \varphi * f$. If $\varphi^{(k)}(0) \neq 0$ $(k = p, p+1, \ldots)$, then the operator J is one-to-one and the class $\mathscr{C}\mathscr{M}_{\mu,\nu}^\delta(\Phi,\xi,G,H)$ contains the functions $f \in \mathscr{A}_p$ for which there exists a function $g \in \mathscr{W}_\nu(G)$ such that

$$(1-\delta)\frac{(\xi * \phi) * f}{\xi * g} + \delta\frac{\phi * f}{g} \in \mathscr{K}_\mu(H).$$

Then, we can omit the function φ as the parameter of the class and we can denote it by $\mathscr{CM}^{\delta}_{\mu,\nu}(\phi,\xi,G,H)$ i.e.

$$\mathscr{CM}^{\delta}_{\mu,\nu}(\phi,\xi,G,H) := \mathscr{CM}^{\delta}_{\mu,\nu}\left(\left(\phi,\frac{z^p}{1-z}\right),\xi,G,H\right). \tag{3}$$

These general classes reduced to well-known subclasses by judicious choices of the parameters, see for example [2]–[12], [14]–[23] and [26]–[27]. In particular, the class $\mathscr{M}^{\delta}_{\mu}(\varphi,H)$ contains the functions $f \in \mathscr{A}_p$ such that

$$\delta\left(1+\frac{z(\varphi*f)''(z)}{(\varphi*f)'(z)}\right)+(1-\delta)\frac{z(\varphi*f)'(z)}{(\varphi*f)(z)} \in \mathscr{K}_{\mu}(H).$$

It is related to the class of functions with the bounded Mocanu variation defined by Coonce and Ziegler [4] and investigated by Dziok [5, 6] and Noor *et al.* [15, 16, 17, 18] and others. Choosig paremeters $\varphi(z)=\frac{z}{(1-z)^2}$, $h(z)=\frac{1+z}{1-z}$, $p=1$ in the class $\mathscr{W}_{\mu}(\varphi,(h,h))$ we obtain the well-known class of functions of bounded boundary rotation (see [7, 12, 20, 22, 23]).

The classes

$$\mathscr{S}^{*}_{p}(\alpha) := \mathscr{S}^{*}_{p}\left(\frac{z^p}{1-z},\alpha\right), \quad \mathscr{S}^{c}_{p} := \mathscr{S}^{*}_{p}\left(\frac{z^p\left[1+(1-p)z\right]}{(1-z)^2},0\right)$$

are the classes of *multivalent starlike* functions of order α $(p>0)$ and *multivalent convex* functions, respectively. If we put

$$\Phi(z)=\left(\frac{z}{(1-z)^2},\frac{z}{1-z}\right), \quad H(z)=\left(\frac{1+z}{1-z},\frac{1+z}{1-z}\right) \quad (z\in\mathscr{U}),$$

then $CC := \mathscr{CW}_{1,1}(\Phi,z,H,H)$ is the well-known class of *close-to-convex* functions with parameter $\beta=0$.

It is clear that

$$\begin{aligned}\mathscr{M}^{\delta}_{\mu}((\phi,z),\xi,H) &= \mathscr{CM}^{\delta}_{\mu,\nu}((\phi,z),\xi,G,H)\\ &\subset \mathscr{CM}^{\delta}_{\mu,\nu}(\Phi,\xi,G,H) \subset \mathscr{CM}^{\delta}_{\mu,\nu}(\phi,\xi,G,H).\end{aligned} \tag{4}$$

The main object of this paper is to investigate convolution properties related to the prestarlike functions and various inclusion relationships between defined classes of functions. Some characterizations of the class $\mathscr{K}_{\mu}(h)$ are also given.

1. Preliminary results

THEOREM 1. *The class* $\mathscr{K}_\mu(H)$ *is convex.*

Proof. Let $q,r \in \mathscr{K}_\mu(H)$, $\alpha \in [0,1]$. Then there exist $q_j, r_j \in \mathscr{K}(h_j)$ $(j=1,2)$ such that

$$q = \mu q_1 + (1-\mu)q_2, \quad r = \mu r_1 + (1-\mu) r_2.$$

It follows that

$$\alpha q + (1-\alpha) r = \mu [\alpha q_1 + (1-\alpha) r_1] + (1-\mu)[\alpha q_2 + (1-\alpha) r_2].$$

Since $\alpha q_j + (1-\alpha) r_j \in \mathscr{K}(h_j)$ $(j=1,2)$, we conclude that $\alpha q + (1-\alpha) r \in \mathscr{K}_\mu(H)$. Hence, the class $\mathscr{K}_\mu(H)$ is convex. $\square$

THEOREM 2.

$$\mathscr{K}_\mu(H) \subset \mathscr{K}_\lambda(H) \quad (h_2 \prec h_1,\ 0 \leqslant \mu < \lambda \leqslant 1), \tag{5}$$

$$\mathscr{K}_\mu(H) \subset \mathscr{K}_\lambda(H) \quad (h_1 \prec h_2,\ 1 \leqslant \mu < \lambda), \tag{6}$$

$$\mathscr{K}_\mu(h) \subset \mathscr{K}_\lambda(h),\ \mathscr{P}_k(h) \subset \mathscr{P}_\lambda(h) \quad (k < \lambda,\ 0 \leqslant \mu < \lambda). \tag{7}$$

Proof. Let $q \in \mathscr{K}_\mu(H)$. Then there exist $q_1 \in \mathscr{K}(h_1)$, $q_2 \in \mathscr{K}(h_2)$ such that $q = \mu q_1 + (1-\mu) q_2$. Thus, we have

$$q = \lambda q_1 + (1-\lambda)\widetilde{q}_2 \quad \left(\widetilde{q}_2 = \frac{\lambda-\mu}{\lambda-1} q_1 + \frac{\mu-1}{\lambda-1} q_2, \lambda \neq 1\right).$$

If $h_2 \prec h_1$, $0 \leqslant \mu < \lambda \leqslant 1$, then $\mathscr{K}(h_2) \subset \mathscr{K}(h_1)$ and

$$q = \lambda \widetilde{q}_1 + (1-\lambda) q_2 \quad \left(\widetilde{q}_1 = \frac{\mu}{\lambda} q_1 + \frac{\lambda-\mu}{\lambda} q_2\right)$$

Thus, by Theorem 1 $\widetilde{q}_1 \in \mathscr{K}(h_1)$ and consequently $q \in \mathscr{K}_\lambda(H)$. If $h_1 \prec h_2$, $1 \leqslant \mu < \lambda$, then $\mathscr{K}(h_1) \subset \mathscr{K}(h_2)$ and

$$q = \lambda q_1 + (1-\lambda)\widetilde{q}_2 \quad \left(\widetilde{q}_2 = \frac{\lambda-\mu}{\lambda-1} q_1 + \frac{\mu-1}{\lambda-1} q_2\right).$$

Hence, by Theorem 1 $\widetilde{q}_2 \in \mathscr{K}(h_2)$ and, in consequence, $q \in \mathscr{K}_\lambda(H)$. From (5) and (6) we get (7). $\square$

The class $\mathscr{P}_k(h)$ is related to the class M_k of real-valued functions m of bounded variation on $[0,2\pi]$ which satisfy the conditions

$$\int_0^{2\pi} \mathrm{d}m(t) = 2, \quad \int_0^{2\pi} |\mathrm{d}m(t)| \leqslant k. \tag{8}$$

It is clear that $M_2 =$ is the class of nondecreasing functions on $[0,2\pi]$ satisfying (8) or equivalently $\int_0^{2\pi} \mathrm{d}m(t) = 2$.

LEMMA 1. [7] *Let* $|2a-1| \leqslant 1,\ a \neq 1,\ h(z) = \frac{1+(1-2a)z}{1-z}\ (z \in \mathscr{U})$. *Then* $q \in \mathscr{P}_k(h)$ *if and only if there exists* $m \in M_k$ *such that*

$$q(z) = \frac{1}{2}\int_0^{2\pi} h\left(ze^{-it}\right) \mathrm{d}m(t) \quad (z \in \mathscr{U}). \tag{9}$$

LEMMA 2. [5] *Let* $|2a-1| \leqslant 1,\ a \neq 1,\ h(z) = \frac{1+(1-2a)z}{1-z}\ (z \in \mathscr{U})$. *Then* $q \in \mathscr{P}_k(h)$ *if and only if* $q \in \mathscr{A}_0$ *and*

$$\int_0^{2\pi} \left| \operatorname{Re} \frac{q\left(re^{it}\right) - a}{1-a} \right| \mathrm{d}t \leqslant k\pi \quad (0 < r < 1). \tag{10}$$

REMARK 1. If we put $a = 0$ in Lemma 1, then we obtain the definition of the class $P_k := \mathscr{P}_k\left(\frac{1+z}{1-z}\right)$ introduced by Pinchuk [22]. Putting $a = \rho\ (0 \leqslant \rho < 1)$ in Lemma 2, we obtain the definition of the class $P_k(\rho) := \mathscr{P}_k\left(\frac{1+(1-2\rho)z}{1-z}\right)$ introduced by Padmanabhan and Parvatham [20].

Let $p, h \in \mathscr{A}_0$. The first-order differential subordination

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z) \tag{11}$$

is called the Briot-Bouquet differential subordination. This particular differential subordination has a surprising number of important applications in the theory of analytic functions (for details see [13]). In particular, Eenigenburg, Miller, Mocanu and Reade [9] proved the following result.

LEMMA 3. [9] *Let* $h \in \mathscr{S}_0^c$, $\operatorname{Re}(\beta h(z) + \gamma) \geqslant 0\ (z \in \mathscr{U})$. *If* $p \in \mathscr{A}_0$ *satisfies the Briot-Bouquet differential subordination* (11), *then* $p \prec h$.

For $\beta = 0$ we can extend this result.

THEOREM 3. *Let* $h_1, h_2 \in \mathscr{S}_0^c$, $\operatorname{Re}\gamma \geqslant 0$. *If* $p \in \mathscr{A}_0$ *satisfies*

$$p(z) + \gamma z p'(z) \in K_\mu(H), \tag{12}$$

then $p \in K_\mu(H)$.

Proof. From (12) there exist $q_1 \prec h_1$, $q_2 \prec h_2$ such that

$$p(z) + \gamma z p'(z) = \mu q_1(z) + (1-\mu) q_2(z) \quad (z \in \mathscr{U}). \tag{13}$$

Let p_1, p_2 be the solutions of the Cauchy problems

$$\begin{aligned} p(z) + \gamma z p'(z) &= q_1(z), \quad p(0) = 1, \\ p(z) + \gamma z p'(z) &= q_2(z), \quad p(0) = 1. \end{aligned}$$

respectively. Then the function $p=\mu p_1+(1-\mu)p_2$ is the solution of the first-order differential equation (13). Moreover, by (13) and Lemma 3 we have $p_1 \prec h_1$ and $p_2 \prec h_2$. Therefore, $p \in K_\mu(H)$ and the proof is completed. □

A more general problem can be formulated as the following problem.

PROBLEM 1. Let $h_1,h_2 \in \mathscr{S}_0^c$, $\operatorname{Re}(\beta h_j(z)+\gamma) \geqslant 0$ $(z \in \mathscr{U},\ j=1,2)$. To verify the following result: if $q \in \mathscr{A}_0$ satisfies

$$q(z)+\frac{zq'(z)}{\beta q(z)+\gamma} \in \mathscr{K}_\mu(H),$$

then $q \in \mathscr{K}_\mu(H)$.

REMARK 2. By Theorem 3 the result is true for $\beta=0$. For $\beta \neq 0$ the problem is open, but it seems to be false.

The class

$$\mathscr{R}_p(\alpha) := \begin{cases} \mathscr{W}\left(\frac{z^p}{(1-z)^{2(p-\alpha)}}, \frac{1+(1-2\alpha/p)z}{1-z}\right) & \text{for } \alpha<p \\ \mathscr{W}\left(\left(\frac{z^p}{1-z}, z^p\right), \frac{1}{1-z}\right) & \text{for } \alpha=p \end{cases}$$

will be called the class of *multivalent prestarlike* functions of order α. The class $\mathscr{R}(\alpha):=\mathscr{R}_1(\alpha)$ is the well-know class of *prestarlike* functions of order α introduced by Ruscheweyh [25]. Simple calculations show that $f \in \mathscr{R}_p(\alpha)$ if and only if

$$\begin{cases} f(z) * \frac{z^p}{(1-z)^{2(p-\alpha)}} \in \mathscr{S}_p^*(\alpha) & \text{for } \alpha<p \\ \operatorname{Re}\left(\frac{f(z)}{z^p}\right) > \frac{1}{2} & \text{for } \alpha=p \end{cases}.$$

LEMMA 4. [5] *If $f,g \in \mathscr{R}_p(\alpha)$, then $f*g \in \mathscr{R}_p(\alpha)$.*

LEMMA 5. [5] *If either*

$$\operatorname{Re} a \leqslant \operatorname{Re} c, \quad \operatorname{Im} a = \operatorname{Im} c \ \text{ and } \ (2p+1-a-\overline{c})/2 \leqslant \alpha < p \tag{14}$$

or

$$0<a \leqslant c \ \text{ and } \ \left(p-\frac{c}{2}\right) \leqslant \alpha < p, \tag{15}$$

then the multivalent incomplete Beta function

$$l_p(a,c)(z) := z^p {}_2F_1(a,1;c;z) = \sum_{n=p}^{\infty} \frac{(a)_{n-p}}{(c)_{n-p}} z^n \quad (z \in \mathscr{U}) \tag{16}$$

belongs to the class $\mathscr{R}_p(\alpha)$.

LEMMA 6. [5] *Let $f \in \mathscr{R}_p(\alpha)$, $g \in \mathscr{S}_p^*(\alpha)$, $h \in \mathscr{A}$. Then*

$$\frac{f*(hg)}{f*g}(\mathscr{U}) \subseteq \overline{co}\{h(\mathscr{U})\}, \tag{17}$$

where $\overline{co}\{h(\mathscr{U})\}$ denotes the closed convex hull of $h(\mathscr{U})$.

2. The main results

From now on we make the assumptions: $h_j, g_j \in \mathscr{S}_0^c \ (j=1,2)$ and

$$\operatorname{Re}\{\mu h_1(z) + (1-\mu) h_2(z)\} > \alpha, \ \operatorname{Re}\{\nu g_1(z) + (1-\nu) g_2(z)\} > \alpha \quad (z \in \mathscr{U}). \tag{18}$$

Then we have

$$\mathscr{W}_\mu(H) \subset \mathscr{S}_p^*(\alpha) \text{ and } \mathscr{W}_\mu(\varphi, H) \subset \mathscr{S}_p^*(\varphi, \alpha). \tag{19}$$

THEOREM 4. *If* $\psi \in \mathscr{R}_p(\alpha)$, *then*

$$\left[\mathscr{W}_\mu(\Phi, H) \cap \mathscr{S}_p^*(\varphi, \alpha)\right] \subset \mathscr{W}_\mu(\psi * \Phi, H), \tag{20}$$

$$\left[\mathscr{W}_\mu(\Phi, H) \cap \mathscr{W}_\mu(\varphi, H)\right] \subset \mathscr{W}_\mu(\psi * \Phi, H), \tag{21}$$

$$\mathscr{W}_\mu(\varphi, H) \subset \mathscr{W}_\mu(\psi * \varphi, H). \tag{22}$$

Proof. Let $f \in \left[\mathscr{W}_\mu(\Phi, H) \cap \mathscr{S}_p^*(\varphi, \alpha)\right]$. Thus there exist $\omega_1, \omega_2 \in \Omega$ such that

$$\frac{\phi * f}{\varphi * f} = \mu h_1 \circ \omega_1 + (1-\mu) h_2 \circ \omega_2$$

and $F = \varphi * f \in \mathscr{S}_p^*(\alpha)$. Thus, applying properties of the convolution, we get

$$\frac{(\psi * \phi) * f}{(\psi * \varphi) * f} = \mu \frac{\psi * [(h_1 \circ \omega_1) F]}{\psi * F} + (1-\mu) \frac{\psi * [(h_2 \circ \omega_2) F]}{\psi * F}. \tag{23}$$

By Lemma 6 we conclude that

$$q_j(z) := \frac{\psi * [(h_j \circ \omega_j) F]}{\psi * F}(z) \in \overline{co}\left\{h_j(\omega(\mathscr{U}))\right\} \subset h_j(\mathscr{U}) \quad (z \in \mathscr{U}, \ j = 1, 2).$$

Therefore, $q_j \in \mathscr{K}_\mu(h_j)$ and by (23) we have that $f \in \mathscr{W}_p^\mu(\psi * \Phi, H)$, which proves the inclusion (20). If we apply the relationship (19) in (20), then we obtain (21).

Let now $f \in \mathscr{W}_\mu(\varphi, H)$. Then $f \in \mathscr{W}_\mu((\phi, \varphi), H)$, where $\phi(z) = \frac{z}{p}\varphi'(z) \ (z \in \mathscr{U})$. Thus, by (21) we obtain that $f \in \mathscr{W}_\mu((\psi * \phi, \psi * \varphi), H)$. Since

$$(\psi * \phi)(z) = \frac{z}{p}(\psi * \varphi)'(z) \quad (z \in \mathscr{U}),$$

by (18) we have $f \in \mathscr{W}_\mu(\psi * \varphi, H)$, which proves (22) and completes the proof. □

THEOREM 5. *Let* $\psi, \xi \in \mathscr{R}_p(\alpha)$, $0 \leqslant \delta \leqslant 1$. *Then*

$$\mathscr{M}_\mu^\delta(\Phi, \xi, H) \cap \mathscr{W}_\mu(\Phi, H) \subset \mathscr{M}_\mu^\delta(\psi * \Phi, \xi, H). \tag{24}$$

Proof. Let $f \in \mathcal{M}_{\mu}^{\delta}(\Phi, \xi, H) \cap \mathcal{W}_{\mu}(\Phi, H)$. Then, applying Lemma 4 and Theorem 4, we obtain $f \in \mathcal{W}_{\mu}(\psi * \Phi, H)$ and $f \in \mathcal{W}_{\mu}((\xi * \psi) * \Phi, H)$ or equivalently

$$q_1 := \frac{\phi * \psi * f}{\varphi * \psi * f} \in \mathcal{K}_{\mu}(H), \quad q_2 := \frac{\xi * \phi * \psi * f}{\xi * \varphi * \psi * f} \in \mathcal{K}_{\mu}(H).$$

Since the class $\mathcal{K}_{\mu}(H)$ is convex by Theorem 1, we conclude that

$$(1-\delta)\frac{\xi * \psi * \phi * f}{\xi * \psi * \varphi * f} + \delta \frac{\psi * \phi * f}{\psi * \varphi * f} \in \mathcal{K}_{\mu}(H).$$

Thus, we have $f \in \mathcal{M}^{\delta}(\psi * \Phi, \xi, h)$ and, in consequence, we get (24). □

LEMMA 7. [5] *If* $0 \leqslant \lambda < \delta$, *then*

$$\mathcal{M}^{\delta}(\varphi, h) \subset \mathcal{M}^{\lambda}(\varphi, h) \subset \mathcal{W}(\varphi, h).$$

From Theorem 5 and Lemma 7 we have the following corollary.

COROLLARY 1. *Let* $\psi \in \mathcal{R}_p(\alpha)$, $0 \leqslant \delta \leqslant 1$. *Then*

$$\mathcal{M}^{\delta}(\varphi, h) \subset \mathcal{M}^{\delta}(\psi * \varphi, h).$$

THEOREM 6. *Let* $\psi, \xi \in \mathcal{R}_p(\alpha)$, $0 \leqslant \delta \leqslant 1$. *Then*

$$\mathcal{CW}_{\mu,\nu}(\Phi, G, H) \subset \mathcal{CW}_{\mu,\nu}(\psi * \Phi, G, H), \ \mathcal{CW}_{\mu}(\Phi, H) \subset \mathcal{CW}_{\mu}(\psi * \Phi, H), \tag{25}$$

$$\mathcal{CM}_{\mu,\nu}^{\delta}(\Phi, \xi, G, H) \cap \mathcal{CW}_{\mu,\nu}(\Phi, G, H) \subset \mathcal{CM}_{\mu,\nu}^{\delta}(\psi * \Phi, \xi, G, H). \tag{26}$$

Proof. Let $f \in \mathcal{CW}_{\mu,\nu}(\Phi, G, H)$. Then there exists $g \in \mathcal{W}_{\nu}(\varphi, G)$ and $\omega_1, \omega_2 \in \Omega$ such that

$$\frac{\phi * f}{\varphi * g} = \mu h_1 \circ \omega_1 + (1-\mu) h_2 \circ \omega_2$$

Since $F = \varphi * g \in \mathcal{W}_{\nu}(G) \subset S_p^*(\alpha)$ by (19), applying properties of the convolution, we obtain

$$\frac{(\psi * \phi) * f}{(\psi * \varphi) * g} = \mu \frac{\psi * [(h_1 \circ \omega_1) F]}{\psi * F} + (1-\mu) \frac{\psi * [(h_2 \circ \omega_2) F]}{\psi * F}. \tag{27}$$

Analysis similarly to that in the proof of Theorem 4 gives

$$\frac{(\psi * \phi) * f}{(\psi * \varphi) * g} \in \mathcal{K}_{\mu}(H).$$

Moreover, by Theorem 4 we have $g \in \mathcal{W}_{\nu}(\psi * \varphi, G)$ and, in consequence, $f \in \mathcal{CW}_{\mu,\nu}(\psi * \Phi, G, H)$. This proves the first inclusion in (25). Puting e.g.

$$\varphi(z) = \frac{z^p}{1-z} \quad (z \in \mathcal{U})$$

in (25), by (3) and (4) we obtain the second inclusion in (25).

Let now $f \in \mathscr{CM}^{\delta}_{\mu,\nu}(\Phi,\xi,G,H) \cap \mathscr{CW}_{\mu,\nu}(\Phi,G,H)$. Thus, applying (25) and Theorem 4, we obtain $f \in \mathscr{CW}_{\mu}(\psi * \Phi,H)$ and $f \in \mathscr{CW}_{\mu}((\xi * \psi) * \Phi,H)$ or equivalently

$$q_1 := \frac{\psi * \phi * f}{\psi * \varphi * g} \in \mathscr{K}_{\mu}(H), \quad q_2 := \frac{\xi * \psi * \phi * f}{\xi * \psi * \varphi * g} \in \mathscr{K}_{\mu}(H).$$

Since the class $\mathscr{K}_{\mu}(H)$ is convex by Theorem 1, we conclude that

$$(1-\delta)\frac{\xi * \psi * \phi * f}{\xi * \psi * \varphi * g} + \delta\frac{\psi * \phi * f}{\psi * \varphi * g} \in \mathscr{K}_{\mu}(H).$$

This gives (26) and completes the proof. $\square$

Combining Theorems 4-6 with Lemma 5 we obtain the following theorem.

THEOREM 7. *If either (14) or (15), then*

$$\begin{aligned}
&\left[\mathscr{W}_{\mu}(\Phi,H) \cap \mathscr{S}^*_p(\varphi,\alpha)\right] \subset \mathscr{W}_{\mu}(l_p(a,c,z) * \Phi,H),\\
&\left[\mathscr{W}_{\mu}(\Phi,H) \cap \mathscr{W}_{\mu}(\varphi,H)\right] \subset \mathscr{W}_{\mu}(l_p(a,c,z) * \Phi,H),\\
&\mathscr{W}_{\mu}(\varphi,H) \subset \mathscr{W}_{\mu}(l_p(a,c) * \varphi,H).
\end{aligned}$$

Moreover, if $\xi \in \mathscr{R}_p(\alpha)$ *and* $0 \leqslant \delta \leqslant 1$, *then*

$$\begin{aligned}
&\mathscr{M}^{\delta}_{\mu}(\Phi,\xi,H) \cap \mathscr{W}_{\mu}(\Phi,H) \subset \mathscr{M}^{\delta}_{\mu}(l_p(a,c) * \Phi,\xi,H),\\
&\mathscr{CW}_{\mu,\nu}(\Phi,G,H) \subset \mathscr{CW}_{\mu,\nu}(l_p(a,c) * \Phi,G,H),\\
&\mathscr{M}^{\delta}(\varphi,h) \subset \mathscr{M}^{\delta}(l_p(a,c) * \varphi,h),\ \mathscr{CW}_{\mu}(\Phi,H) \subset \mathscr{CW}_{\mu}(l_p(a,c) * \Phi,H),\\
&\mathscr{CM}^{\delta}_{\mu,\nu}(\Phi,\xi,G,H) \cap \mathscr{CW}_{\mu,\nu}(\Phi,G,H) \subset \mathscr{CM}^{\delta}_{\mu,\nu}(l_p(a,c) * \Phi,\xi,G,H).
\end{aligned}$$

Since $l_p(a,c) * l_p(c,a) * \phi = \phi$, by Theorem 7 we obtain the next result.

THEOREM 8. *If either (14) or (15), then*

$$\begin{aligned}
&\left[\mathscr{W}_{\mu}(l_p(c,a) * \Phi,H) \cap \mathscr{S}^*_p(l_p(c,a) * \varphi,\alpha)\right] \subset \mathscr{W}_{\mu}(\Phi,H),\\
&\left[\mathscr{W}_{\mu}(l_p(c,a) * \Phi,H) \cap \mathscr{W}_{\mu}(l_p(c,a) * \varphi,H)\right] \subset \mathscr{W}_{\mu}(\Phi,H),\\
&\mathscr{W}_{\mu}(l_p(c,a) * \varphi,H) \subset \mathscr{W}_{\mu}(\varphi,H).
\end{aligned}$$

Moreover, if $\xi \in \mathscr{R}_p(\alpha)$ *and* $0 \leqslant \delta \leqslant 1$, *then*

$$\begin{aligned}
&\mathscr{M}^{\delta}_{\mu}(l_p(c,a) * \Phi,\xi,H) \cap \mathscr{W}_{\mu}(l_p(c,a) * \Phi,H) \subset \mathscr{M}^{\delta}_{\mu}(\Phi,\xi,H),\\
&\mathscr{CW}_{\mu,\nu}(l_p(c,a) * \Phi,G,H) \subset \mathscr{CW}_{\mu,\nu}(\Phi,G,H),\\
&\mathscr{M}^{\delta}(l_p(c,a) * \varphi,h) \subset \mathscr{M}^{\delta}(\varphi,h),\ \mathscr{CW}_{\mu}(l_p(c,a) * \Phi,H) \subset \mathscr{CW}_{\mu}(\Phi,H),\\
&\mathscr{CM}^{\delta}_{\mu,\nu}(l_p(c,a) * \Phi,\xi,G,H) \cap \mathscr{CW}_{\mu,\nu}(l_p(c,a) * \Phi,G,H) \subset \mathscr{CM}^{\delta}_{\mu,\nu}(\Phi,\xi,G,H).
\end{aligned}$$

Let us define the linear operators $J_\lambda : \mathscr{A}_p \longrightarrow \mathscr{A}_p$, $J_\lambda^\times : \mathscr{A}_p \times \mathscr{A}_p \longrightarrow \mathscr{A}_p \times \mathscr{A}_p$,

$$J_\lambda (f)(z) := \lambda \frac{zf'(z)}{p} + (1-\lambda) f(z), \tag{28}$$
$$J_\lambda^\times (f,g) := (J_\lambda (f), J_\lambda (g)) \quad (z \in \mathscr{U},\ \operatorname{Re} \lambda) > 0).$$

Since $J_\lambda (\phi) = l_p(\frac{p}{\lambda}, \frac{p}{\lambda}+1) * \phi$, putting $a = \frac{p}{\lambda}$, $c = \frac{p}{\lambda}+1$ in Theorem 8, we have the following theorem.

THEOREM 9. *If* $p - \operatorname{Re}\frac{p}{\lambda} \leqslant \alpha < p$, *then*

$$\left[\mathscr{W}_\mu \left(J_\lambda^\times (\Phi), H\right) \cap \mathscr{S}_p^* (J_\lambda (\varphi), \alpha)\right] \subset \mathscr{W}_\mu (\Phi, H),$$
$$\left[\mathscr{W}_\mu \left(J_\lambda^\times (\Phi), H\right) \cap \mathscr{W}_\mu (J_\lambda (\varphi), H)\right] \subset \mathscr{W}_\mu (\Phi, H),$$
$$\mathscr{W}_\mu (J_\lambda (\varphi), H) \subset \mathscr{W}_\mu (\varphi, H).$$

Moreover, if $\xi \in \mathscr{R}_p(\alpha)$ *and* $0 \leqslant \delta \leqslant 1$, *then*

$$\mathscr{M}_\mu^\delta \left(J_\lambda^\times (\Phi), \xi, H\right) \cap \mathscr{W}_\mu \left(J_\lambda^\times (\Phi), H\right) \subset \mathscr{M}_\mu^\delta (\Phi, \xi, H),$$
$$\mathscr{C}\mathscr{W}_{\mu,\nu} \left(J_\lambda^\times (\Phi), G, H\right) \subset \mathscr{C}\mathscr{W}_{\mu,\nu} (\Phi, G, H),$$
$$\mathscr{M}^\delta \left(J_\lambda^\times (\Phi), h\right) \subset \mathscr{M}^\delta (\varphi, h),\ \mathscr{C}\mathscr{W}_\mu \left(J_\lambda^\times (\Phi), H\right) \subset \mathscr{C}\mathscr{W}_\mu (\Phi, H),$$
$$\mathscr{C}\mathscr{M}_{\mu,\nu}^\delta \left(J_\lambda^\times (\Phi), \xi, G, H\right) \cap \mathscr{C}\mathscr{W}_{\mu,\nu} \left(J_\lambda^\times (\Phi), G, H\right) \subset \mathscr{C}\mathscr{M}_{\mu,\nu}^\delta (\Phi, \xi, G, H).$$

In particular, for $\lambda = 1$ we get the following theorem.

THEOREM 10. *If* $0 \leqslant \alpha < p$, *then*

$$\left[\mathscr{W}_\mu \left(p^{-1} z\Phi'(z), H\right) \cap \mathscr{S}_p^* \left(p^{-1} z\varphi'(z), \alpha\right)\right] \subset \mathscr{W}_\mu (\Phi, H),$$
$$\left[\mathscr{W}_\mu \left(p^{-1} z\Phi'(z), H\right) \cap \mathscr{W}_\mu \left(p^{-1} z\varphi'(z), H\right)\right] \subset \mathscr{W}_\mu (\Phi, H),$$
$$\mathscr{W}_\mu \left(p^{-1} z\varphi'(z), H\right) \subset \mathscr{W}_\mu (\varphi, H).$$

Moreover, if $\xi \in \mathscr{R}_p(\alpha)$ *and* $0 \leqslant \delta \leqslant 1$, *then*

$$\mathscr{M}_\mu^\delta \left(p^{-1} z\Phi'(z), \xi, H\right) \cap \mathscr{W}_\mu \left(p^{-1} z\Phi'(z), H\right) \subset \mathscr{M}_\mu^\delta (\Phi, \xi, H),$$
$$\mathscr{C}\mathscr{W}_{\mu,\nu} \left(p^{-1} z\Phi'(z), G, H\right) \subset \mathscr{C}\mathscr{W}_{\mu,\nu} (\Phi, G, H),$$
$$\mathscr{M}^\delta \left(J_\lambda^\times (\Phi), h\right) \subset \mathscr{M}^\delta (\varphi, h),\ \mathscr{C}\mathscr{W}_\mu \left(p^{-1} z\Phi'(z), H\right) \subset \mathscr{C}\mathscr{W}_\mu (\Phi, H),$$
$$\mathscr{M}_{\mu,\nu}^\delta \left(p^{-1} z\Phi'(z), \xi, G, H\right) \cap \mathscr{C}\mathscr{W}_{\mu,\nu} \left(p^{-1} z\Phi'(z), G, H\right) \subset \mathscr{C}\mathscr{M}_{\mu,\nu}^\delta (\Phi, \xi, G, H).$$

3. Applications and concluding remarks

The classes $\mathscr{M}_{\mu}^{\delta}(\Phi,\xi,H)$ and $\mathscr{C}\mathscr{M}_{\mu,\nu}^{\delta}(\Phi,\xi,G,H)$ generalize well-known important classes, which were investigated in earlier works. Most of these classes were defined by using linear operators and special functions.

Suppose $A_1,\dots,A_q$ and $B_1,\dots,B_s$ $(q,s\in\mathbb{N})$ be positive real numbers such that

$$1+\sum_{k=1}^{s}B_k-\sum_{k=1}^{q}A_k\geqslant 0.$$

For complex parameters $a_1,\dots,a_q$ and $b_1,\dots,b_s$ $(q,s\in\mathscr{N})$ such that

$$\frac{a_k}{A_k},\frac{b_k}{B_k}\neq 0,1,2,\dots$$

we define the Fox-Wright generalization of the hypergeometric ${}_qF_s$ function by

$${}_q\Psi_s\left[\begin{matrix}(a_1,A_1),\dots,(a_q,A_q);\\(b_1,B_1),\dots,(b_s,B_s);\end{matrix}z\right]:=\sum_{n=0}^{\infty}\frac{\Gamma(a_1+A_1n)\cdots\Gamma(a_q+A_qn)}{\Gamma(b_1+B_1n)\cdots\Gamma(b_s+B_sn)}\frac{z^n}{n!}. \tag{29}$$

If $A_n=1$ $(n=1,\dots,q)$ and $B_n=1$ $(n=1,\dots,s)$, we have the obvious relationship:

$$\omega\,{}_q\Psi_s\left[\begin{matrix}(a_1,1),\dots,(a_q,1);\\(b_1,1),\dots,(b_s,1);\end{matrix}z\right]={}_qF_s(a_1,\dots,a_q;b_1,\dots,b_s;z)\ \ (z\in\mathscr{U}).$$

where

$$\omega=\frac{\Gamma(\beta_1)\cdots\Gamma(\beta_s)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_q)}. \tag{30}$$

Moreover, in terms of Fox's H-function, we have

$${}_q\Psi_s\left[\begin{matrix}(a_1,A_1),\dots,(a_q,A_q);\\(b_1,B_1),\dots,(b_s,B_s);\end{matrix}z\right]=H_{q,s+1}^{1,q}\left[-z\,\middle|\,\begin{matrix}(1-a_1,A_1),\dots,(1-a_q,A_q)\\(0,1)\,(1-b_1,B_1),\dots,(1-b_s,B_s)\end{matrix}\right].$$

It should be remarked in passing that a further generalization of Fox's H-function is provided by the $\overline{H}$ -function which was encountered in the physics literature while investigating and illustrating the use of certain Feynman integrals that arise naturally in perturbation calculations of the equilibrium properties of a magnetic model of phase transitions. Other interesting and useful special cases of the Fox-Wright generalized hypergeometric ${}_q\Psi_s$ function defined by (29) include (for example) the generalized Bessel function J_ν^μ defined by

$$J_\nu^\mu(z):=\sum_{n=0}^{\infty}\frac{(-z)^n}{n!\,\Gamma(1+\nu+\mu n)}\ \ (z\in\mathscr{U}).$$

which, for $\mu=1$, corresponds essentially to the classical Bessel function J_ν, and the generalized Mittag-Leffler function $E_{\lambda,\mu}$ defined by

$$E_{\lambda,\mu}(z):=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\nu+\lambda n)}\quad (z\in\mathscr{U}).$$

For real numbers λ, t $(\lambda > -p)$, we define the function

$$\Phi(a_1,b_1,t)(z) = \left(\omega z^p \, {}_q\Psi_s \left[\begin{matrix} (a_1,A_1),\ldots,(a_q,A_q) \\ (b_1,B_1),\ldots,(b_s,B_s) \end{matrix} ;z \right]\right) * f_{\lambda,t}(z), \tag{31}$$

where ω is defined by (30) and

$$f_{\lambda,t}(z) = \sum_{n=p}^{\infty} \left(\frac{n+\lambda}{p+\lambda}\right)^t z^n \ \ (z \in \mathscr{U}).$$

It is easy to verify that

$$\begin{aligned} &a\Phi(a+1,b,t) = Az\Phi'(a;b,t) + (a-pA)\Phi(a,b,t), && (32)\\ &b\Phi(a,b,t) = Bz\Phi'(a,b+1,t) + (b-pB)\Phi(a,b+1,t), &&\\ &(p+\lambda)\Phi(a,b,t+1) = z\Phi'(a,b,t) + \lambda\Phi(a,b,t), &&\\ &\Phi(a,b,t) = l_p(a,c) * \Phi(c,b,t) \quad (A=1), && (33)\\ &\Phi(a,c,t) = l_p(b,c) * \Phi(a,b,t) \quad (B=1), && \end{aligned}$$

where $l_p(a,c)$ is the multivalent incomplete Beta function (16).

Corresponding to the function $\Phi(a,b,t)$ we consider the following classes of functions:

$$\mathscr{V}(a,b,t) := \mathscr{W}_\mu(\Phi(a,b,t);H), \quad \mathscr{CV}(a,b,t) := \mathscr{CW}_{\mu,\nu}(\Phi(a,b,t),G,H).$$

By using the linear operator

$$\Theta_p[a,b,t]f = \Phi(a,b,t) * f \quad (f \in \mathscr{A}_p) \tag{34}$$

we can define the class $\mathscr{V}(a,b,t)$ alternatively in the following way:

$$f \in \mathscr{V}(a,b,t) \Longleftrightarrow \frac{a}{A}\frac{\Theta_p[a+1,b,t]f(z)}{\Theta_p[a,b,t]f(z)} + p - \frac{a}{A} \in \mathscr{K}_\mu(H).$$

COROLLARY 2. *If* $p - \operatorname{Re} a \leqslant \alpha < p$, $m \in \mathbb{N}$, *then*

$$\mathscr{V}(a+m,b,t) \subset \mathscr{V}(a,b,t), \quad \mathscr{CV}(a+m,b,t) \subset \mathscr{CV}(a,b,t). \tag{35}$$

Proof. It is clear that it is sufficient to prove the corollary for $m=1$. Let J_λ and $\Phi(a,b,t)$ be defined by (28) and (31), respectively. Then, by (32) we have $\Phi(a+1,b,t) = J_{\frac{p}{a}}(\Phi(a,b,t))$. Hence, by using Theorem 9 we conclude that

$$\begin{aligned} &\mathscr{W}_\mu(\Phi(a+1,b,t),H) \subset \mathscr{W}_\mu(\Phi(a,b,t),H),\\ &\mathscr{CW}_{\mu,\nu}(\Phi(a+1,b,t),G,H) \subset \mathscr{CW}_{\mu,\nu}(\Phi(a,b,t),G,H). \end{aligned}$$

This clearly forces the inclusion relations (35) for $m=1$. $\square$

Analogously to Corollary 2, we prove the following corollary.

COROLLARY 3. *Let $m \in \mathbb{N}$. If $p - \operatorname{Re} b \leqslant \alpha < p$, then*

$$\mathscr{V}(a,b,t) \subset \mathscr{V}(a,b+m,t), \quad \mathscr{C}\mathscr{V}(a,b,t) \subset \mathscr{C}\mathscr{V}(a,b+m,t).$$

If $-\operatorname{Re}\lambda \leqslant \alpha < p$, then

$$\mathscr{V}(a,b,t+m) \subset \mathscr{V}(a,b,t), \quad \mathscr{C}\mathscr{V}(a,b,t+m) \subset \mathscr{C}\mathscr{V}(a,b,t).$$

It is natural to ask about the inclusion relations in Corollaries 2 and 3 when m is positive real. Using Theorems 4 and 6, we shall give a partial answer to this question.

COROLLARY 4. *If the multivalent incomplete Beta function $l_p(a,c)$ defined by (16) belongs to the class $\mathscr{R}_p(\alpha)$, then*

$$\mathscr{V}(c,b,t) \subset \mathscr{V}(a,b,t),\ \mathscr{C}\mathscr{V}(c,b,t) \subset \mathscr{C}\mathscr{V}(a,b,t) \quad (A=1), \tag{36}$$
$$\mathscr{V}(b,a,t) \subset \mathscr{V}(b,c,t),\ \mathscr{C}\mathscr{V}(b,a,t) \subset \mathscr{C}\mathscr{V}(b,c,t) \quad (B=1). \tag{37}$$

Proof. Let us put $\psi = l_p(a,c)$, $\varphi = \Phi(c,b,t)$, where $\Phi(a,b,t)$ is defined by (31). Then, by using Theorems 4, 6 and relationship (33) we obtain

$$\mathscr{W}_\mu(\Phi(c,b,t),H) \subset \mathscr{W}_\mu(\Phi(a,b,t),H),$$
$$\mathscr{C}\mathscr{W}_{\mu,\nu}(\Phi(c,b,t),G,H) \subset \mathscr{C}\mathscr{W}_{\mu,\nu}(\Phi(a,b,t),G,H).$$

Thus, we get the inclusion relations (36). Analogously, we prove the inclusions (37). □

Combining Corollary 4 with Lemma 5 we obtain the following result.

COROLLARY 5. *If either (14) or (15), then the inclusion relations (36) and (37) hold true.*

The linear operator $\Theta_p[a,b,t]$ defined by (34) includes (as its special cases) other linear operators of Geometric Function Theory which were considered in earlier works. It contains, as its further special cases, such other linear operators as the Dziok-Srivastava operator, the Hohlov operator, the Carlson-Shaffer operator, the Ruscheweyh derivative operator, the generalized Bernardi-Libera-Livingston operator, the fractional derivative operator, and so on (see, for the precise relationships, Dziok and Srivastava ([8], p. 3–4). Moreover, the linear operator $\Theta_p[a,b,t]$ includes also the Salagean operator, the Noor operator, the Choi-Saigo-Srivastava operator, the Kim-Srivastava operator, and others (see, for the precise relationships, Cho *et al.* [3]). By using these linear operators we can consider several subclasses of the classes $\mathscr{V}(a,b,t)$, $\mathscr{C}\mathscr{V}(a,b,t)$, see for example [1]–[7], [10, 21, 26, 27]. Also, the obtained results generalize several results obtained in these classes of functions.

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J. Dziok
Faculty of Mathematics and Natural Sciences
University of Rzeszów
35-310 Rzeszów, Poland
e-mail: jdziok@ur.edu.pl

K. I. Noor
Mathematics Department
COMSATS Institute of Information Technology
Islamabad, Pakistan
e-mail: khalidanoor@hotmail.com