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TWO–WEIGHT ENTROPY BOUNDEDNESS OF MULTILINEAR FRACTIONAL TYPE OPERATORS

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Abstract. This paper will be devoted to study the two-weight norm inequalities of the multilinear fractional maximal operator $\mathscr{M}_\alpha$ and the multilinear fractional integral operator $\mathscr{I}_\alpha$. The entropy conditions in the multilinear setting will be introduced and the entropy bounds for $\mathscr{M}_\alpha$ and $\mathscr{I}_\alpha$ will be given.

1. Introduction

1.1. Background

Let M_α and I_α be the fractional maximal operator and fractional integral operator defined by

$$M_\alpha f(x) := \sup_Q |Q|^{\frac{\alpha}{n}} \langle f \rangle_Q \cdot \mathbf{1}_Q(x), \quad I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \quad 0 \leqslant \alpha < n.$$

In 1982, Saywer [22] first showed that $M_\alpha(\cdot\sigma) : L^p(\sigma) \to L^q(w)$ holds if and only if (w,σ) satisfies the following testing condition

$$[w,\sigma]_{S_{(p,q)}} := \sup_Q \sigma(Q)^{-\frac{1}{p}} \left\| \mathbf{1}_Q M_\alpha(\mathbf{1}_Q \sigma) \right\|_{L^q(w)} < \infty.$$

Subsequently, using the similar testing conditions, Saywer [23, 24] gave some characterizations of two-weight weak and strong type inequalities of I_α.

After the works of Saywer, many works have been done in the characterizations of two weighed boundedness of continuous operators. Among such achievements are the celebrated works of Hytönen [7], Lacey [10, 11], Lacey et al [15], which demonstrated the characterizations of the two weighted L^2 inequality of Hilbert transform in terms of Saywer type testing conditions and two-weight A_2 condition. Recently, Lacey and Li [12] gave a characterization of two-weight norm inequalities for the classical

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Littlewood-Paley g-function. Still more recently, Cao, Li and Xue [1] obtained the characterization of two weighted inequalities for the g_λ^*-function with more general fractional type of Poisson kernels. As for the discrete operators, on the one hand, two-weight characterizations of martingale transforms and dyadic shifts were presented by Nazarov et al [20] and Hytönen [8]. The two weighted $L^p(\sigma) \to L^q(w)$-type inequalities of positive dyadic operators were established by Nazarov et al [20] with $p = q = 2$, Lacey et al [14] with $p < q$ and Hytönen [6] with $p, q \in (1, \infty)$. On the other hand, in order to study the sufficient conditionS for the two weight inequalities of the singular integral operators, Treil and Volberg [26] introduced the entropy conditions. Later on, the entropy conditions were used to obtain the two-weight norm inequalities of intrinsic square functions and fractional maximal and integral operators by Lacey, Li [13] and Rahm, Spencer [21], respectively.

In the multilinear setting, several works also have already been done for the mulitilinear fractional maximal operator $\mathscr{M}_\alpha$ and fractional integral operators $\mathscr{I}_\alpha$ $(0 \leqslant \alpha < mn)$, which are defined by

$$\mathscr{M}_\alpha(\vec{f})(x) = \sup_Q |Q|^{\frac{\alpha}{n}} \prod_{i=1}^m \langle |f_i| \rangle_Q \cdot \mathbf{1}_Q(x), \quad \mathscr{I}_\alpha(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} \frac{\prod_{i=1}^m f_i(x - y_i)}{|(y_1, \dots, y_m)|^{mn-\alpha}} d\vec{y}.$$

In 2013, Chen and Damián [4] first gave some sufficient conditions for the two-weight inequalities of the multilinear maximal operator $\mathscr{M}_0$. In 2015, Li and Sun [18] considered the problem of two weighted inequalities of multilinear fractional maximal operator $\mathscr{M}_\alpha$. But it is worth pointing out that their method is not valid for the case $0 \leqslant \alpha < n(1/p - 1/\max\{p_i\})$. In 2016, Cao and Xue [2] extended the ranges of exponents to $0 \leqslant \alpha < mn$ by applying the atomic decomposition of tent space. Moreover, Cao, Xue and Yabuta [3] defined and studied the multilinear fractional strong maximal operator and the corresponding multiple weights associated with rectangles. Under the dyadic reverse doubling condition, a necessary and sufficient condition for two-weight inequalities of the multilinear fractional strong maximal operator was given.

It is well known that it is difficult to give a two-weight characterization of $\mathscr{M}_\alpha$ and $\mathscr{I}_\alpha$ with respect to Saywer-type testing condition. Even if we make it, it is generally very hard to verify Saywer-type testing condition in practice. This leads us to quest some sufficient conditions for two-weight norm inequalities of $\mathscr{M}_\alpha$ and $\mathscr{I}_\alpha$. This kind of conditions should mainly concerned with A_p like conditions.

In this paper, we are mainly concerned with A_p like conditions that are sufficient for two-weight norm inequalities of $\mathscr{M}_\alpha$ and $\mathscr{I}_\alpha$. We will work with the multiple version of entropy conditions and try to obtain the entropy bounds of $\mathscr{M}_\alpha$ and $\mathscr{I}_\alpha$. For simplicity, we only give the results and the proofs in the case m=2, although our results still hold for general $m \geqslant 2$.

1.2. Main results

First, we give one definition related to multiple weights.

DEFINITION 1.1. (Multiple weights class) Let $0 \leqslant \alpha < mn$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ with $1 < p_1, \cdots, p_m < \infty$, and $0 < p \leqslant q < \infty$. Let w, σ_i $(i = 1, \dots, m)$ be nonnegative and locally integrable functions on $\mathbb{R}^n$, and $\nu_{\vec{\sigma}} = \prod_{i=1}^m \sigma_i^{p/p_i}$. We define

$$
\begin{aligned}
[w,\vec{\sigma}]_{A_{(\vec{p},q)}} &:= \sup_Q \mathscr{A}_{\vec{p},q}(w,\vec{\sigma};Q) < \infty,\\
[w,\vec{\sigma}]_{A_{(\vec{p},q)}A_\infty^{\exp}} &:= \sup_Q \mathscr{A}_{\vec{p},q}(w,\vec{\sigma};Q) A_\infty^{\exp}(\nu_{\vec{\sigma}};Q)^{\frac{1}{p}} < \infty,\\
[w,\vec{\sigma}]_{A_{(\vec{p},q)}H_{\vec{p}}^\infty} &:= \sup_Q \mathscr{A}_{\vec{p},q}(w,\vec{\sigma};Q) \prod_{i=1}^m A_\infty^{\exp}(\sigma_i;Q)^{\frac{1}{p_i}} < \infty,
\end{aligned}
$$

where

$$
A_\infty^{\exp}(w;Q) := \langle w\rangle_Q \exp\big(\langle \log w^{-1}\rangle_Q\big),\ \ \mathscr{A}_{\vec{p},q}(w,\vec{\sigma};Q) := |Q|^{\frac{1}{q}-\frac{1}{p}+\frac{\alpha}{n}} \langle w\rangle_Q^{\frac{1}{q}} \prod_{i=1}^m \langle \sigma_i\rangle_Q^{\frac{1}{p_i'}}.
$$

REMARK 1.2. If we denote

$$
[\vec{\sigma}]_{H_{\vec{p}}^\infty} := \sup_Q \prod_{i=1}^m A_\infty^{\exp}(\sigma_i;Q)^{\frac{p}{p_i}},\ \ [\vec{\sigma}]_{RH_{\vec{p}}} := \sup_Q \nu_{\vec{\sigma}}(Q)^{-1} \prod_{i=1}^m \sigma_i(Q)^{\frac{p}{p_i}},
$$

it is easy to check that

$$
\begin{aligned}
&[\vec{\sigma}]_{H_{\vec{p}}^\infty} \leqslant [\vec{\sigma}]_{RH_{\vec{p}}} [\nu_{\vec{\sigma}}]_{A_\infty^{\exp}},\ \ [w,\vec{\sigma}]_{A_{(\vec{p},q)A_\infty^{\exp}}} \leqslant [w,\vec{\sigma}]_{A_{(\vec{p},q)}} [\nu_{\vec{\sigma}}]_{A_\infty^{\exp}}^{1/p},\\
&[w,\vec{\sigma}]_{A_{(\vec{p},q)}A_\infty^{\exp}} \leqslant [w,\vec{\sigma}]_{A_{(\vec{p},q)}H_{\vec{p}}^\infty} \leqslant [\vec{\sigma}]_{RH_{\vec{p}}}^{1/p} [w,\vec{\sigma}]_{A_{(\vec{p},q)}A_\infty^{\exp}}.
\end{aligned}
$$

Now, we give the definition of multilinear version of entropy conditions.

DEFINITION 1.3. (Multilinear version of entropy conditions) Let $0 \leqslant \alpha < mn$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ with $1 < p_1, \cdots, p_m < \infty$, and $0 < p \leqslant q < \infty$. Let w, σ_i $(i = 1, \ldots, m)$ be nonnegative and locally integrable functions on $\mathbb{R}^n$. We define

$$
\begin{aligned}
\lceil w,\vec{\sigma}\rceil_{\vec{p},q,\varepsilon} &:= \sup_Q \mathscr{A}_{\vec{p},q}(w,\vec{\sigma};Q) \rho_{\nu_{\vec{\sigma}}}(Q)^{\frac{1}{p}} \varepsilon(\rho_{\nu_{\vec{\sigma}}}(Q))\\
\lfloor w,\vec{\sigma}\rfloor_{\vec{p},q,\vec{\varepsilon},\eta} &:= \sup_Q \mathscr{A}_{\vec{p},q}(w,\vec{\sigma};Q) \rho_{w,\eta}(Q)^{\frac{1}{q'}} \prod_{i=1}^m \rho_{\sigma_i,\varepsilon_i}(Q)^{\frac{1}{p_i}}\\
[[\vec{\sigma}]]_{(i,j,k),\varepsilon_i} &:= \sup_Q \Big(|Q|^{\frac{\alpha}{n}} \prod_{i=1}^2 \langle \sigma_i\rangle_Q\Big)^{\frac{p_k'}{p_{ij}'}} \cdot \langle \sigma_3\rangle_Q \gamma_{(i,j,k)}(Q) \varepsilon_i(\gamma_{(i,j,k)}(Q)).
\end{aligned}
$$

where $\varepsilon, \eta, \varepsilon_i$ are monotonic increasing functions on $(1,\infty)$, and

$$
\rho_w(Q) := w(Q)^{-1} \int_Q M(\mathbf{1}_Q w)(x)dx \ \text{ and } \ \rho_{w,\varepsilon}(Q) = \rho_w(Q)\varepsilon(\rho_w(Q)),
$$

$$
\gamma_{(i,j,k)}(Q) := \frac{\int_Q \mathscr{M}_\alpha(\mathbf{1}_Q\sigma_i, \mathbf{1}_Q\sigma_j)(x)^{\frac{p_k'}{p_{ij}}} dx}{\big(\int_Q \sigma_i^{\frac{p_{ij}}{p_i}} \sigma_j^{\frac{p_{ij}}{p_j}} dx\big)^{\frac{p_k'}{p_{ij}}}} \ \text{ and } \ \frac{1}{p_{ij}} = \frac{1}{p_i} + \frac{1}{p_j}.
$$

We now state the main results of this paper as follows.

THEOREM 1.1. *Let* $0 \leqslant \alpha < 2n$, $0 < p \leqslant q < \infty$ *and* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ *with* $1 < p_1, p_2 < \infty$. *Suppose that* σ_1, σ_2, w *are weights on* $\mathbb{R}^n$. *Let* ε *be a monotonic increasing function on* $(1,\infty)$ *that satisfies* $\int_1^\infty \frac{dt}{t\varepsilon(t)^q} < \infty$. *Then the following inequality holds*

$$\big\|\mathscr{M}_\alpha(f_1\sigma_1, f_2\sigma_2)\big\|_{L^q(w)} \lesssim \lceil w, \vec{\sigma} \rceil_{\vec{p},q,\varepsilon} \prod_{i=1}^{2} \big\|f_i\big\|_{L^{p_i}(\sigma_i)}.$$

THEOREM 1.2. *Let* $0 \leqslant \alpha < 2n$, $0 < p \leqslant q < \infty$ *and* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ *with* $1 < p_1, p_2, q < \infty$. *Suppose that* σ_1, σ_2, w *are weights on* $\mathbb{R}^n$. *Let* $\varepsilon_1, \varepsilon_2, \eta$ *be monotonic increasing functions on* $(1,\infty)$ *that satisfy* $\int_1^\infty \frac{dt}{t\varepsilon_i(t)^{p_i}} < \infty$ *and* $\int_1^\infty \frac{dt}{t\eta(t)^{q'}} < \infty$. *Then there holds that*

$$\big\|\mathscr{I}_\alpha(f_1\sigma_1, f_2\sigma_2)\big\|_{L^q(w)} \lesssim \lfloor w, \vec{\sigma} \rfloor_{\vec{p},q,\vec{\varepsilon},\eta} \prod_{i=1}^{2} \big\|f_i\big\|_{L^{p_i}(\sigma_i)}.$$

THEOREM 1.3. *Let* $0 \leqslant \alpha < 2n$, *and* $1 < p_i < \infty$ $(i = 1,2,3)$ *with* $\frac{1}{p_i} + \frac{1}{p_j} \geqslant 1$ *for* $i \neq j$. *Suppose that* $\sigma_1, \sigma_2, \sigma_3$ *are weights on* $\mathbb{R}^n$. *Let* ε_i *be a monotonic increasing function on* $(1,\infty)$ *such that* $\int_1^\infty \frac{dt}{t\varepsilon_i(t)^{1/p_i'}} < \infty$, $i = 1,2,3$. *Then the following inequalities hold*

$$\big\|\mathscr{I}_\alpha(f_1\sigma_1, f_2\sigma_2)\big\|_{L^{p_3'}(\sigma_3)} \lesssim \sum_{(i,j,k)\in\Omega} [[\vec{\sigma}]]_{(i,j,k),\varepsilon_i}^{1/p_k'} \prod_{i=1}^{2} \big\|f_i\big\|_{L^{p_i}(\sigma_i)}; \tag{1.1}$$

$$\big\|\mathscr{I}_\alpha(f_1\sigma_1, f_2\sigma_2)\big\|_{L^{p_3',\infty}(\sigma_3)} \lesssim \sum_{\substack{i\neq 3\\ (i,j,k)\in\Omega}} [[\vec{\sigma}]]_{(i,j,k),\varepsilon_i}^{1/p_k'} \prod_{i=1}^{2} \big\|f_i\big\|_{L^{p_i}(\sigma_i)}; \tag{1.2}$$

where Ω *is the set of all permutations of* $(1,2,3)$.

The article is organized as follows: In Section 2, some notations and lemmas will be given. In Section 3, we will demonstrate Theorem 1.1 and Theorem 1.2. Section 4 will be devoted to complete the proofs of Theorem 1.3.

2. Preliminaries

First, we present some definitions and lemmas, which will be used later.

DEFINITION 2.1. A collection, $\mathscr{D}$ of cubes is said to be a dyadic grid if it satisfies

(1) The side length of every $Q \in \mathscr{D}$ equals 2^k for some $k \in \mathbb{Z}$.

(2) For any $Q, R \in \mathscr{D}$, $Q \cap R = \{Q, R, \emptyset\}$.

(3) $\mathbb{R}^n = \bigcup_{Q\in\mathscr{D}_k} Q$, $\mathscr{D}_k = \{Q\in\mathscr{D};\ell(Q)=2^k\}$ for any $k\in\mathbb{Z}$.

DEFINITION 2.2. A subset $\mathscr{S}$ of a dyadic grid is said to be spare, if for every $Q\in\mathscr{S}$ there holds that

$$\Big|\bigcup_{\substack{Q'\in\mathscr{S}\\ Q'\subsetneq Q}} Q'\Big| \leqslant \frac{1}{2}|Q|.$$

Equivalently, if $E(Q) = Q\setminus\bigcup_{\substack{Q'\in\mathscr{S}\\ Q'\subsetneq Q}} Q'$, then the sets $\{E(Q)\}_{Q\in\mathscr{S}}$ are pairwise disjoint and $|Q|\leqslant 2|E(Q)|$.

DEFINITION 2.3. Let $0\leqslant\alpha<mn$ and $\mathscr{D},\mathscr{S}$ be a given dyadic grid and a spare set. The dyadic versions of multilinear fractional maximal and fractional integral operators are defined by

$$\mathscr{M}_\alpha^{\mathscr{D}}(\vec{f})(x) := \sup_{Q\in\mathscr{D}} |Q|^{\frac{\alpha}{n}}\prod_{i=1}^m\langle|f_i|\rangle_Q\cdot\mathbf{1}_Q(x),$$

$$\mathscr{I}_\alpha^{\mathscr{D}}(\vec{f})(x) := \sum_{Q\in\mathscr{D}} |Q|^{\frac{\alpha}{n}}\prod_{i=1}^m\langle f_i\rangle_Q\cdot\mathbf{1}_Q(x),$$

$$\mathscr{T}_{\mathscr{S},\alpha}(\vec{f})(x) := \sum_{Q\in\mathscr{S}} |Q|^{\frac{\alpha}{n}}\prod_{i=1}^m\langle f_i\rangle_Q\cdot\mathbf{1}_{E(Q)}(x).$$

We will need the following lemma given by Hytönen and Pérez in [9].

LEMMA 2.1. *There are 2^n dyadic grids $\mathscr{D}_t$, $t\in\{0,1/3\}^n$ such that for any cube $Q\subset\mathbb{R}^n$ there exists a cube $Q_t\in\mathscr{D}_t$ satisfying $Q\subset Q_t$ and $\ell(Q_t)\leqslant 6\ell(Q)$, where the dyadic grid $\mathscr{D}_t$ is defined by*

$$\mathscr{D}_t := \big\{2^{-k}([0,1)^n+m+(-1)^k t) : k\in\mathbb{Z}, m\in\mathbb{Z}^n\big\},\ \ t\in\{0,1/3\}^n.$$

We also need the following lemma.

LEMMA 2.2. *Let $\mathscr{D}$ be a dyadic grid. For any non-negative integrable f_i $(i=1,\ldots,m)$, there exist sparse families $\mathscr{S}\subset\mathscr{D}$ such that for all $x\in\mathbb{R}^n$, it holds that*

$$\mathscr{M}_\alpha(\vec{f})(x) \simeq \sup_{t\in\{0,1/3\}^n}\mathscr{M}_\alpha^{\mathscr{D}_t}(\vec{f})(x),\ \ \mathscr{M}_\alpha^{\mathscr{D}}(\vec{f})(x)\simeq\mathscr{T}_{\mathscr{S},\alpha}(\vec{f})(x); \tag{2.1}$$

$$\mathscr{I}_\alpha(\vec{f})(x) \simeq \sup_{t\in\{0,1/3\}^n}\mathscr{I}_\alpha^{\mathscr{D}_t}(\vec{f})(x),\ \ \mathscr{I}_\alpha^{\mathscr{D}}(\vec{f})(x)\simeq\mathscr{T}_{\mathscr{S},\alpha}(\vec{f})(x). \tag{2.2}$$

The proof of (2.1) can be found in [17] and (2.2) was shown in [19].

We will apply the following multilinear version of Carleson embedding theorem [25] at certain key points in the proofs of our results.

LEMMA 2.3. (Carleson embedding theorem) *Let* $0 < p \leqslant q < \infty$ *and* $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ *satisfying* $1 < p_1, \ldots, p_m < \infty$*. Suppose that the nonnegative numbers* $\{c_Q\}_Q$ *satisfy*

$$\sum_{Q \subset Q'} c_Q \leqslant A\, \nu_{\vec{\sigma}}(Q')^{q/p}, \text{ for any } Q' \in \mathscr{D},$$

where σ_i $(i = 1, \cdots, m)$ *are weights and* $\nu_{\vec{\sigma}} = \prod_{i=1}^m \sigma_i^{p/p_i}$*. Then for all nonnegative functions* $f_i \in L^{p_i}(\sigma_i)$*, we have*

$$\sum_{Q \in \mathscr{D}} c_Q \prod_{i=1}^m \big(\langle f_i \rangle_Q^{\sigma_i}\big)^q \lesssim A \big\| \mathscr{M}_{\vec{\sigma}}^d(\vec{f}) \big\|_{L^{p,q}(\nu_{\vec{\sigma}})}^q \lesssim A \prod_{i=1}^m \|f_i\|_{L^{p_i}(\sigma_i)}^q,$$

where $L^{p,q}(w)$ *is the Lorentz space defined by*

$$\|f\|_{L^{p,q}(w)} = \left[\int_0^\infty \Big(\lambda w\big(\{x \in \mathbb{R}^n; |f(x)| > \lambda\}\big)^{1/p} \Big)^q \frac{d\lambda}{\lambda} \right]^{1/q} < \infty.$$

3. Proofs of Theorems 1.1–1.2

In this section, our aim is to demonstrate Theorem 1.1 and Theorem 1.2 by making use of dyadic techniques (see for examples, [8] and [16]).

3.1. Proof of Theorem 1.1

Let $\mathscr{S}$ be any sparse set of $\mathscr{D}$. By Lemma 2.2, it suffices to show that

$$\big\| \mathscr{T}_{\mathscr{S},\alpha}(f_1\sigma_1, f_2\sigma_2) \big\|_{L^q(w)} \lesssim \lceil w, \vec{\sigma} \rceil_{\vec{p},q,\varepsilon} \prod_{i=1}^2 \|f_i\|_{L^{p_i}(\sigma_i)}. \tag{3.1}$$

We may assume that each f_i is a non-negative function for $i = 1, 2$. Denote

$$\mathscr{S}_k := \Big\{ Q \in \mathscr{S};\ 2^{-k} \lceil w, \vec{\sigma} \rceil_{\vec{p},q,\varepsilon} \leqslant \Gamma(Q) \leqslant 2^{-k+1} \lceil w, \vec{\sigma} \rceil_{\vec{p},q,\varepsilon} \Big\},$$

where $\Gamma(Q) := |Q|^{\frac{1}{q} - \frac{1}{p} + \frac{\alpha}{n}} \langle w \rangle_Q^{\frac{1}{q}} \prod_{i=1}^2 \langle \sigma_i \rangle_Q^{\frac{1}{p_i'}} \cdot \rho_{\nu_{\vec{\sigma}}}(Q)^{\frac{1}{p}} \varepsilon(\rho_{\nu_{\vec{\sigma}}}(Q))$. Using the pairwise disjointness of the sets $\{E(Q)\}_{Q \in \mathscr{S}}$, we deduce that

$$\big\| \mathscr{T}_{\mathscr{S},\alpha}(f_1\sigma_1, f_2\sigma_2) \big\|_{L^q(w)}^q = \sum_{k=1}^\infty \sum_{Q \in \mathscr{S}_k} \Big(|Q|^{\frac{\alpha}{n}} \prod_{i=1}^2 \langle f_i \sigma_i \rangle_Q \Big)^q w(E(Q)) := \sum_{k=1}^\infty \Delta_k.$$

To obtain the bound of Δ_k, we need to introduce the notion

$$c_Q = \big(|Q|^{\alpha/n} \langle \sigma_1 \rangle_Q \langle \sigma_2 \rangle_Q \big)^q w(E(Q)).$$

Then, it is easy to see that

$$\Delta_k = \sum_{Q \in \mathscr{S}_k} c_Q \Big(\langle f_1 \rangle_Q^{\sigma_1} \langle f_2 \rangle_Q^{\sigma_2} \Big)^q.$$

In order to apply the Carleson embedding theorem, we need to analyze $\{c_Q\}_{Q\in\mathscr{S}_k}$. Fix $Q'\in\mathscr{S}_k$. Since $\Gamma(Q)\simeq 2^{-k}\lceil w,\vec{\sigma}\rceil_{\vec{p},q,\varepsilon}$ for each $Q\in\mathscr{S}_k$, we get

$$\begin{aligned}\sum_{Q\in\mathscr{S}_k:Q\subset Q'} c_Q &\leqslant \sum_{Q\in\mathscr{S}_k:Q\subset Q'}\Gamma(Q)^q\left(\frac{\sigma_1(Q)^{\frac{1}{p_1}}\sigma_2(Q)^{\frac{1}{p_2}}}{\rho_{v_{\vec{\sigma}}}(Q)^{\frac{1}{p}}\varepsilon(\rho_{v_{\vec{\sigma}}}(Q))}\right)^q\\ &\lesssim 2^{-kq}\lceil w,\vec{\sigma}\rceil_{\vec{p},q,\varepsilon}^q\sum_{Q\in\mathscr{S}_k:Q\subset Q'}\frac{v_{\vec{\sigma}}(Q)^{q/p}}{\rho_{v_{\vec{\sigma}}}(Q)^{q/p}\varepsilon(\rho_{v_{\vec{\sigma}}}(Q))^q}\\ &:= 2^{-kq}\lceil w,\vec{\sigma}\rceil_{\vec{p},q,\varepsilon}^q\Delta_k'.\end{aligned}$$

Now, we tentatively claim that

$$\Delta_k'\lesssim v_{\vec{\sigma}}(Q')^{q/p}. \tag{3.2}$$

Therefore, if the above claim is true, we actually obtain that

$$\sum_{Q\in\mathscr{S}_k:Q\subset Q'} c_Q\lesssim 2^{-kq}\lceil w,\vec{\sigma}\rceil_{\vec{p},q,\varepsilon}^q v_{\vec{\sigma}}(Q')^{q/p},$$

and

$$\Delta_k\lesssim 2^{-kq}\lceil w,\vec{\sigma}\rceil_{\vec{p},q,\varepsilon}^q\prod_{i=1}^2\|f_i\|_{L^{p_i}(\sigma_i)}^q,$$

where we have used Lemma 2.3. Consequently, it yields that

$$\|\mathscr{T}_{\mathscr{S},\alpha}(\vec{f}\cdot\vec{\sigma})\|_{L^q(w)}=\sum_{k=1}^\infty\Delta_k\lesssim\lceil w,\vec{\sigma}\rceil\prod_{i=1}^2\|f_i\|_{L^{p_i}(\sigma_i)}.$$

This shows that inequality (3.1) is true.

Now, we are in the position to demonstrate (3.2). Set

$$\mathscr{S}_{k,j}:=\{Q\in\mathscr{S}_k;\ Q\subset Q',\ 2^{j-1}\leqslant\rho_{v_{\vec{\sigma}}}(Q)<2^j\},$$

and $\mathscr{S}_{k,j}^*$ is the collection of maximal elements in $\mathscr{S}_{k,j}$. Thereby, we have

$$\begin{aligned}\Big(\sum_{Q\in\mathscr{S}_{k,j}}v_{\vec{\sigma}}(Q)^{q/p}\Big)^{p/q} &\leqslant\sum_{Q^*\in\mathscr{S}_{k,j}^*}\sum_{Q\subset Q^*}v_{\vec{\sigma}}(Q)\\ &\lesssim\sum_{Q^*\in\mathscr{S}_{k,j}^*}\sum_{Q\subset Q^*}\int_{E(Q)}\langle\mathbf{1}_{Q^*}v_{\vec{\sigma}}\rangle_Q\mathbf{1}_Q(x)dx\\ &\lesssim\sum_{Q^*\in\mathscr{S}_{k,j}^*}\sum_{Q\subset Q^*}\int_{E(Q)}\sup_{P\in\mathscr{D}}\langle\mathbf{1}_{Q^*}v_{\vec{\sigma}}\rangle_P\mathbf{1}_P(x)dx\\ &\leqslant\sum_{Q^*\in\mathscr{S}_{k,j}^*}\int_{Q^*}\sup_{P\in\mathscr{D}}\langle\mathbf{1}_{Q^*}v_{\vec{\sigma}}\rangle_P\mathbf{1}_P(x)dx\end{aligned}$$

$$\begin{aligned}&\leqslant \sum_{Q^*\in\mathscr{S}^*_{k,j}} \int_{Q^*} M(\mathbf{1}_{Q^*}\nu_{\vec{\sigma}})(x)dx\\&= \sum_{Q^*\in\mathscr{S}^*_{k,j}} \nu_{\vec{\sigma}}(Q^*)\rho_{\nu_{\vec{\sigma}}}(Q^*) \lesssim 2^j \nu_{\vec{\sigma}}(Q').\end{aligned}$$

Accordingly, we deduce that

$$\begin{aligned}\Delta'_k &\leqslant \sum_{j=0}^{\infty} \frac{1}{2^{jq/p}\varepsilon(2^j)^q} \sum_{Q\in\mathscr{S}_{k,j}} \nu_{\vec{\sigma}}(Q)^{q/p}\\&\lesssim \nu_{\vec{\sigma}}(Q')^{q/p}\sum_{j=0}^{\infty}\frac{1}{\varepsilon(2^j)^q}\\&\lesssim \nu_{\vec{\sigma}}(Q')^{q/p}\int_1^{\infty}\frac{dt}{t\varepsilon(t)^q} \lesssim \nu_{\vec{\sigma}}(Q')^{q/p}.\end{aligned}$$

The proof of (3.2) is finished. $\square$

3.2. Proof of Theorem 1.2

By duality, we have

$$\begin{aligned}\left\|\mathscr{T}_{\mathscr{S},\alpha}(\vec{f}\cdot\vec{\sigma})\right\|_{L^q(w)} &= \sup_{\|g\|_{L^{q'}(w)}\leqslant 1} \left|\sum_{Q\in\mathscr{S}} \left(|Q|^{\frac{\alpha}{n}}\langle f_1\sigma_1\rangle_Q\langle f_2\sigma_2\rangle_Q\right)\int_Q g(x)wdx\right|\\&:= \sup_{\|g\|_{L^{q'}(w)}\leqslant 1}\left|\mathscr{V}(g)\right|.\end{aligned}$$

Denote

$$\mathscr{S}_k = \left\{Q\in\mathscr{S};\ 2^k<\lambda_Q\leqslant 2^{k+1}\right\},\ \ \lambda_Q = \mathscr{A}_{\vec{p},q}(w,\vec{\sigma};Q)\rho_{w,\eta}(Q)^{\frac{1}{q'}}\prod_{i=1}^{2}\rho_{\sigma_i,\varepsilon_i}(Q)^{\frac{1}{p_i}}.$$

Then, we have $k\leqslant K_0 := \log_2\lfloor w,\vec{\sigma}\rfloor_{\vec{p},q,\vec{\varepsilon},\eta}$. Therefore, by the Hölder inequality, it now follows that

$$\begin{aligned}\mathscr{V}(g) &= \sum_{k=1}^{K_0}\sum_{Q\in\mathscr{S}_k}\lambda_Q\prod_{i=1}^{2}\frac{\langle f_i\rangle_Q^{\sigma_i}\sigma_i(Q)^{\frac{1}{p_i}}}{\rho_{\sigma_i,\varepsilon_i}(Q)^{\frac{1}{p_i}}}\frac{\langle g\rangle_Q^w w(Q)^{\frac{1}{q'}}}{\rho_{w,\eta}(Q)^{\frac{1}{q'}}}\\&\lesssim \sum_{k=1}^{K_0}2^k\left(\sum_{Q\in\mathscr{S}_k}\prod_{i=1}^{2}\frac{\left(\langle f_i\rangle_Q^{\sigma_i}\right)^q\sigma_i(Q)^{\frac{q}{p_i}}}{\rho_{\sigma_i,\varepsilon_i}(Q)^{\frac{q}{p_i}}}\right)^{\frac{1}{q}}\left(\sum_{Q\in\mathscr{S}_k}\left(\langle g\rangle_Q^w\right)^{q'}\frac{w(Q)}{\rho_{w,\eta}(Q)}\right)^{\frac{1}{q'}}\\&\lesssim \lfloor w,\vec{\sigma}\rfloor_{\vec{p},q,\vec{\varepsilon},\eta}\prod_{i=1}^{2}\left(\sum_{Q\in\mathscr{S}_k}\left(\langle f_i\rangle_Q^{\sigma_i}\right)^{p_i}\frac{\sigma_i(Q)}{\rho_{\sigma_i,\varepsilon_i}(Q)}\right)^{\frac{1}{p_i}}\left(\sum_{Q\in\mathscr{S}_k}\left(\langle g\rangle_Q^w\right)^{q'}\frac{w(Q)}{\rho_{w,\eta}(Q)}\right)^{\frac{1}{q'}}.\end{aligned}$$

By the Carleson embedding theorem 2.3, it is enough to show that for each $Q'\in\mathscr{S}_k$

$$\sum_{\substack{Q\in\mathscr{S}_k\\ Q\subset Q'}}\frac{\sigma_i(Q)}{\rho_{\sigma_i,\varepsilon_i}(Q)}\lesssim\sigma_i(Q'),\ \ i=1,2,3,$$

where $\sigma_3 = w$ and $\varepsilon_3 = \eta$. A completely analogous calculation to that of the preceding subsection yields the desired result. $\square$

4. Proof of Theorem 1.3

In this section, we shall give the proof of Theorem 1.3. We need the following two-weight characterization of $\mathscr{I}_\alpha^{\mathscr{S}}$, which was proved in [6] and [19].

LEMMA 4.1. *Let $\mathscr{D}$ be a dyadic grid and $\mathscr{S} \subset \mathscr{D}$ be a sparse family. Suppose that σ_1, σ_2 and σ_3 are positive Borel measures and $1 < p_i < \infty$ $(i = 1,2,3)$ with $\frac{1}{p_i} + \frac{1}{p_j} \geqslant 1$ for $i \neq j$. Then*

(1) The strong type inequality

$$\left\|\mathscr{I}_\alpha^{\mathscr{S}}(f_1\sigma_1, f_2\sigma_2)\right\|_{L^q(\sigma_3)} \leqslant \mathfrak{N} \prod_{i=1}^{2} \|f_i\|_{L^{p_i}(\sigma_i)}$$

holds if and only if the following test conditions hold for any triple $(i,j,k) \in \Omega$,

$$\mathfrak{T}_{\mathscr{S},(i,j,k)} := \sup_{R\in\mathscr{S}} \frac{\left\| \sum_{\substack{Q\in\mathscr{S}\\ Q\subset R}} |Q|^{\frac{\alpha}{n}} \langle\sigma_j\rangle_Q \langle\sigma_k\rangle_Q \mathbf{1}_Q \right\|_{L^{p_i'}(\sigma_i)}}{\sigma_j(R)^{1/p_j}\sigma_k(R)^{1/p_k}} < \infty.$$

(2) The weak type inequality

$$\left\|\mathscr{I}_\alpha^{\mathscr{S}}(f_1\sigma_1, f_2\sigma_2)\right\|_{L^{q,\infty}(\sigma_3)} \leqslant \mathfrak{N}_{weak} \prod_{i=1}^{2} \|f_i\|_{L^{p_i}(\sigma_i)}$$

holds if and only if $\mathfrak{T}_{\mathscr{S},(i,j,k)} < \infty$, for any triple $(i,j,k) \in \Omega$ and $i \neq 3$.

Moreover, the best constants satisfy

$$\mathfrak{N} \simeq \sum_{(i,j,k)\in\Omega} \mathfrak{T}_{\mathscr{S},(i,j,k)}, \quad \mathfrak{N}_{weak} \simeq \sum_{i\neq 3,(i,j,k)\in\Omega} \mathfrak{T}_{\mathscr{S},(i,j,k)}.$$

Proof of Theorem 1.3. By Lemma 4.1, it suffices to show

$$\mathfrak{T}_{\mathscr{S},(i,j,k)} \lesssim [[\vec{\sigma}]]_{(i,j,k),\varepsilon_i}, \quad \text{for each } (i,j,k) \in \Omega.$$

By symmetry, we only focus on estimating the case $(i,j,k) = (1,2,3)$. For convenience, we write $q = p_3'$, $p = p_{12}$ and $\gamma = \gamma_{(1,2,3)}$. From now on, we fix the cube $R \in \mathscr{D}$ and introduce the notations

$$\mathscr{A}(R) := \left\| \sum_{Q\in\mathscr{S}} |Q|^{\frac{\alpha}{n}} \langle\sigma_1\rangle_Q \langle\sigma_2\rangle_Q \mathbf{1}_Q \right\|_{L^q(R,w)},$$

$$\mathscr{B}(Q) := \left(|Q|^{\frac{\alpha}{n}} \prod_{i=1}^{m} \langle\sigma_i\rangle_Q\right)^{\frac{q}{p'}} \cdot \langle w\rangle_Q \gamma(Q)\varepsilon_1(\gamma(Q)).$$

Then, we make a partition of $\mathscr{S}$ by setting

$$\mathscr{S}_{a,b} := \{Q \in \mathscr{S};\ Q \subset R, 2^a < \mathscr{B}(Q) \leqslant 2^{a+1}, 2^b < \gamma(Q) \leqslant 2^{b+1}\}.$$

Note that $2^a \leqslant [[\vec{\sigma}]]_{(1,2,3),\varepsilon_1}$. Now we construct the stopping cubes $\mathscr{F}$. Let $\mathscr{F}$ be the minimal subset of $\mathscr{S}_{a,b}$ containing the maximal cubes in $\mathscr{S}_{a,b}$ such that whenever $F \in \mathscr{F}$, the maximal cubes $Q \subset F$, $Q \in \mathscr{S}_{a,b}$ with $|Q|^{\frac{\alpha}{n}}\langle\sigma_1\rangle_Q\langle\sigma_1\rangle_Q > 4|F|^{\frac{\alpha}{n}}\langle\sigma_1\rangle_F\langle\sigma_1\rangle_F$ are also in $\mathscr{F}$. Denote by $\pi_{\mathscr{F}}(Q)$ the minimal cube in $\mathscr{F}$ which contains Q. Denote

$$\mathscr{S}_{a,b}^k := \Big\{Q \in \mathscr{S}_{a,b};\ |Q|^{\frac{\alpha}{n}}\langle\sigma_1\rangle_Q\langle\sigma_2\rangle_Q \simeq 2^{-k}|\pi_{\mathscr{F}}(Q)|^{\frac{\alpha}{n}}\langle\sigma_1\rangle_{\pi_{\mathscr{F}}(Q)}\langle\sigma_2\rangle_{\pi_{\mathscr{F}}(Q)}\Big\}.$$

Then Minkowski inequality implies that

$$\mathscr{A}(R) \leqslant \sum_{a,b}\sum_{k=1}^{\infty}\Big\|\sum_{Q\in\mathscr{S}_{a,b}^k}|Q|^{\frac{\alpha}{n}}\langle\sigma_1\rangle_Q\langle\sigma_2\rangle_Q\mathbf{1}_Q\Big\|_{L^q(w)} := \sum_{a,b}\sum_{k=1}^{\infty}\Theta_{a,b}^k. \tag{4.1}$$

For each $F \in \mathscr{F}$, write

$$\Psi_F := \sum_{\substack{Q\in\mathscr{S}_{a,b}^k \\ \pi_{\mathscr{F}}(Q)=F}}|Q|^{\frac{\alpha}{n}}\langle\sigma_1\rangle_Q\langle\sigma_2\rangle_Q,\ \ \Psi_{F,j} := \Psi_F\mathbf{1}_{\{\Psi_F\simeq j2^{-k}|F|^{\frac{\alpha}{n}}\langle\sigma_1\rangle_F\langle\sigma_2\rangle_F\}}.$$

Making use of Hölder inequality, we may obtain that

$$\begin{aligned}
\Theta_{a,b}^k &\leqslant \Big\|\Big(\sum_{j=1}^{\infty}j^{-\frac{2}{q'}}j^{\frac{2}{q'}}\sum_{F\in\mathscr{F}}\Psi_{F,j}\Big)^q\Big\|_{L^1(w)}^{\frac{1}{q}} \\
&\leqslant \Big(\sum_{j=1}^{\infty}j^{-\frac{2}{q'}q'}\Big)^{\frac{1}{q'}}\Big\|\sum_{j=1}^{\infty}j^{\frac{2q}{q'}}\sum_{F\in\mathscr{F}}\Psi_{F,j}^q\Big\|_{L^1(w)}^{\frac{1}{q}} \\
&\lesssim \Big(\sum_{j=1}^{\infty}j^{2(q-1)}\sum_{F\in\mathscr{F}}\int_{Q_0}\Psi_{F,j}(x)^q w\,dx\Big)^{\frac{1}{q}}.
\end{aligned} \tag{4.2}$$

Therefore, we are in a position to consider the contribution of the integral in the above inequality. Before doing that, we first claim that the following estimate is true:

$$w\big(\{x;\Psi_F(x) > \lambda 2^{-k}|F|^{\frac{\alpha}{n}}\langle\sigma_1\rangle_F\langle\sigma_2\rangle_F\}\big) \lesssim 2^{-\lambda}w(F). \tag{4.3}$$

By (4.3) and noticing the fact that the set $\{x;\Psi_F(x) > \lambda 2^{-k}|F|^{\frac{\alpha}{n}}\langle\sigma_1\rangle_F\langle\sigma_2\rangle_F\}$ coincides with F if $0 < \lambda < j/2$ and is empty if $\lambda > j$, it is easy to get that

$$\int_0^{\infty}q\lambda^{q-1}w\big(\{x;\Psi_F(x) > \lambda 2^{-k}|F|^{\frac{\alpha}{n}}\langle\sigma_1\rangle_F\langle\sigma_2\rangle_F\}\big)d\lambda \lesssim j^q 2^{-\frac{j}{2}}w(F).$$

Hence, it now follows that

$$\int_{Q_0}\Psi_{F,j}(x)^q w\,dx \lesssim 2^{-kq}|F|^{\frac{\alpha q}{n}}\langle\sigma_1\rangle_F^q\langle\sigma_2\rangle_F^q\big(j^q 2^{-\frac{j}{2}}w(F)\big). \tag{4.4}$$

Collecting the inequalities (4.2) and (4.4), we have

$$\begin{aligned}\Big(\sum_{k=1}^{\infty}\Theta_{a,b}^{k}\Big)^{q} &\lesssim \Big(\sum_{k=1}^{\infty}2^{-k}\Big)^{q}\sum_{j=1}^{\infty}j^{3q-2}2^{-\frac{j}{2}}\sum_{F\in\mathscr{F}}|F|^{\frac{\alpha q}{n}}\langle\sigma_1\rangle_F^q\langle\sigma_2\rangle_F^q w(F)\\ &\lesssim \frac{2^a}{2^b\varepsilon_1(2^b)}\sum_{F\in\mathscr{F}}\Big(|F|^{\frac{\alpha}{n}}\langle\sigma_1\rangle_F\langle\sigma_2\rangle_F\Big)^{\frac{q}{p}}|F|.\end{aligned}$$

Let $\mathscr{F}^*$ be the maximal elements of $\mathscr{F}$, then we obtain

$$\begin{aligned}\Big(\sum_{k=1}^{\infty}\Theta_{a,b}^{k}\Big)^{q} &\lesssim \frac{2^a}{2^b\varepsilon_1(2^b)}\sum_{F^*\in\mathscr{F}^*}\sum_{F^*\supset F\in\mathscr{F}}\int_{E(F)}\mathscr{M}_{\alpha}(\mathbf{1}_{F^*}\sigma_1,\mathbf{1}_{F^*}\sigma_2)(x)^{\frac{q}{p}}\,dx\\ &\leqslant \frac{2^a}{2^b\varepsilon_1(2^b)}\sum_{F^*\in\mathscr{F}^*}\int_{F^*}\mathscr{M}_{\alpha}(\mathbf{1}_{F^*}\sigma_1,\mathbf{1}_{F^*}\sigma_2)(x)^{\frac{q}{p}}\,dx\\ &\leqslant \frac{2^a}{\varepsilon_1(2^b)}\sum_{F^*\in\mathscr{F}^*}\nu_{\vec{\sigma}}(F^*)^{\frac{q}{p}} \leqslant \frac{2^a}{\varepsilon_1(2^b)}\Big(\sum_{F^*\in\mathscr{F}^*}\nu_{\vec{\sigma}}(F^*)\Big)^{\frac{q}{p}}\\ &\leqslant \frac{2^a}{\varepsilon_1(2^b)}\nu_{\vec{\sigma}}(R)^{\frac{q}{p}} \leqslant \frac{2^a}{\varepsilon_1(2^b)}\big[\sigma_1(R)^{\frac{1}{p_1}}\sigma_2(R)^{\frac{1}{p_2}}\big]^q.\end{aligned}$$

Consequently, the equation (4.1) gives that

$$\begin{aligned}\mathscr{A}(R) &\lesssim \sum_{a,b}\frac{2^{a/q}}{\varepsilon_1(2^b)^{1/q}}\sigma_1(R)^{\frac{1}{p_1}}\sigma_2(R)^{\frac{1}{p_2}}\\ &\lesssim [[\vec{\sigma}]]_{(1,2,3),\varepsilon_1}^{1/q}\int_1^{\infty}\frac{dt}{t\varepsilon_1(t)^{1/q}}\sigma_1(R)^{\frac{1}{p_1}}\sigma_2(R)^{\frac{1}{p_2}}.\end{aligned}$$

This shows that

$$\mathfrak{T}_{\mathscr{S},(1,2,3)} \lesssim [[\vec{\sigma}]]_{(1,2,3),\varepsilon_1}^{1/q}.$$

We are left to prove the claim (4.3). If w is the Lebesgue measure, the inequality is obvious. For any $Q\in\mathscr{S}_{a,b}^k$ satisfying $\pi_{\mathscr{F}}(Q)=F$, it holds that

$$2^a \simeq \mathscr{B}(Q) \simeq \big(2^{-k}|F|^{\frac{\alpha}{n}}\langle\sigma_1\rangle_F\langle\sigma_2\rangle_F\big)^{\frac{q}{p'}}\langle w\rangle_Q 2^b\varepsilon_1(2^b).$$

Let $\mathscr{S}_{a,b}^{k,*}$ be the maximal cubes in $\mathscr{S}_{a,b}^k$ and

$$\Lambda_F := \frac{2^a}{2^b\varepsilon_1(2^b)}\big(2^{-k}|F|^{\frac{\alpha}{n}}\langle\sigma_1\rangle_F\langle\sigma_2\rangle_F\big)^{-\frac{q}{p'}}.$$

Note that the set $\{x;\Psi_F(x)>\lambda 2^{-k}|F|^{\frac{\alpha}{n}}\langle\sigma_1\rangle_F\langle\sigma_2\rangle_F\}$ is the union of maximal cubes $P\in\mathscr{S}_{a,b}^k$ with $\pi_{\mathscr{F}}(P)=F$ and $\inf\limits_{x\in P}\Psi_F(x)>\lambda 2^{-k}|F|^{\frac{\alpha}{n}}\langle\sigma_1\rangle_F\langle\sigma_2\rangle_F$. Then, it yields that

$$\begin{aligned}
&w\big(\{x;\Psi_F(x) > \lambda 2^{-k}|F|^{\frac{\alpha}{n}}\langle\sigma_1\rangle_F\langle\sigma_2\rangle_F\}\big)\\
&\simeq \Lambda_F\big|\{x;\Psi_F(x) > \lambda 2^{-k}|F|^{\frac{\alpha}{n}}\langle\sigma_1\rangle_F\langle\sigma_2\rangle_F\}\big|\\
&\lesssim \Lambda_F\, 2^{-\lambda}\sum_{Q^*\in\mathscr{S}_{a,b}^{k,*}}|Q^*| \lesssim 2^{-\lambda}\sum_{Q^*\in\mathscr{S}_{a,b}^{k,*}} w(Q^*)\\
&\lesssim 2^{-\lambda}w(F). \quad \square
\end{aligned}$$

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