SOME INEQUALITIES ON POWER BOUNDED
OPERATORS ACTING ON LOCALLY CONVEX SPACES

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Abstract. In this paper are presented some evaluations regarding characterizations of power boundedness of an operator from $\mathscr{B}_{\mathscr{P}}(X)$ algebra. This represents a generalization from the Banach spaces framework to the locally convex spaces where the operators acting on them are universally bounded. The authors extend some Ritt type theorems according to Kreiss conditions in the context of $\mathscr{B}_{\mathscr{P}}(X)$ algebra.

1. Introduction

In the following, an extended version of theorem 4.5.4 from [14] for universally bounded operators is presented. This will be preceded by a lemma first appeared in [13], to improve the second part of theorem 4.5.4 in the specific way that is sufficient to suppose that Ritt type condition [19] is true outside the unit ball, all studied in the context of universally or locally bounded operators, acting on locally convex spaces. The inequalities proven within the theorems broaden the known Banach setting bringing upfront similar yet more general results and adapted proofs for the theorems containing them (see also [9, 18]).

The main difficulty encountered when developing a spectral theory on locally convex spaces consists in defining the notion of bounded operator. This is due to so many non-equivalent definitions previously appeared in literature. First attempts in consolidating a spectral theory for the elements of a locally convex algebra are those of L. Waelbroeck [21]. Papers dedicated to the study of the class of universally bounded operators acting on a locally convex space have been elaborated by R.T. Moore [11, 12] in 1969, A. Chilana in 1970 [3], G. Joseph in 1977 [6] and more recently in 2010 and 2011 in [15] and [16]. This class of operators was also taken into consideration by A. Michael in [10]. Based on some known results obtained by Allan [1], F. G. Bonales and R. V. Mendoza [2], develop a version of spectral theory for bounded linear operators on barreled spaces. A special attention requires the calibration of $\mathscr{B}_{\mathscr{P}}(X)$. Similar issues were dealt with in [7] and [8].

This framework represents a generalization of the Banach setting used in [13], where Ritt type theorems dealing with Kreiss type conditions were proved (see also [20]).

2. Preliminaries

Let $(X, \mathscr{P})$ be a locally convex Hausdorff space with a calibration $\mathscr{P}$. A locally convex algebra is an algebra with a locally convex topology in which the multiplication is separately continuous. Such an algebra is locally m-convex (l.m.c.) if it has a neighborhood base $\mathscr{U}$ at 0 such that $U \in \mathscr{U}$ is convex, balanced ($\lambda U \subseteq U$ for $|\lambda| < 1$) and satisfies the semigroup property $U^2 \subseteq U$. Any algebra with identity will be called unital. A unital l.m.c. algebra A is characterized by the existence of a calibration $\mathscr{P}$ such that each $p \in \mathscr{P}$ is sub-multiplicative ($p(xy) \leqslant p(x)p(y)$) and satisfies $p(e) = 1$, where e is the unit element ([15], [16]). By G. R. Allan [1], an element $x \in X$, where X is a locally convex algebra, is said to be bounded in X if there exists $\alpha \in \mathbb{C}$ such that the set $\{(\alpha x)^n\}_{n \geqslant 1}$ is bounded in X. The set of all bounded elements in X will be denoted by X_0.

Following E. A. Michael [10] (see also [6], [12]), for a locally convex space $(X, \mathscr{P})$, a linear operator $T : X \to X$ is called quotient-bounded (q-bounded) with respect to $\mathscr{P}$ if for any $p \in \mathscr{P}$ there exists $c_p > 0$ such that $p(Tx) \leqslant c_p p(x)$, for all $x \in X$. The class of q-bounded operators with respect to a calibration $\mathscr{P}$ will be denoted by $Q_{\mathscr{P}}(X)$. Also, for a seminorm $p \in \mathscr{P}$, the application $\hat{p} : Q_{\mathscr{P}}(X) \to \mathbb{R}$ is defined as $\hat{p}(T) = \inf\{r > 0 : p(Tx) \leqslant r p(x), (\forall)x \in X\}$. We denote by $\hat{\mathscr{P}}$ the family of seminorms $\{\hat{p} : p \in \mathscr{P}\}$. The space X, respectively $Q_{\mathscr{P}}(X)$ will be endowed with a topology $\tau_{\mathscr{P}}$ generated by $\mathscr{P}$, respectively the topology $\tau_{\hat{p}}$ generated by $\hat{\mathscr{P}}$. If $T \in Q_{\mathscr{P}}(X)$, the $\mathscr{P}$-spectral radius, denoted by $r_{\mathscr{P}}(T)$, is considered as the boundedness radius in the sense of Allan [1] (see also [2]):

$$r_{\mathscr{P}}(T) = \inf\{\alpha > 0 : \alpha^{-1}T \text{ generates a bounded semigroup in } Q_{\mathscr{P}}(X)\}.$$

Taking into account the particularities of the algebra $Q_{\mathscr{P}}(X)$, the $\mathscr{P}$-spectral radius satisfies:

$$r_{\mathscr{P}}(T) = \sup\{\limsup_{n \to \infty}(\hat{p}(T^n))^{1/n} : p \in \mathscr{P}\} =$$

$$\sup\{\lim_{n \to \infty}(\hat{p}(T^n))^{1/n} : p \in \mathscr{P}\} = \sup\{\inf_{n \geqslant 1}(\hat{p}(T^n))^{1/n} : p \in \mathscr{P}\}.$$

We call resolvent set in the Waelbroeck sense of an element x from a locally convex unital algebra $(X, \mathscr{P})$ the set of all elements $\lambda_0 \in C_\infty$, $(C \cup \{\infty\})$ for which there exists $V \in \mathscr{V}_{\lambda_i}$ such that the following conditions hold:

(a) the element $\lambda e - x$ is invertible in X, for any $\lambda \in V \setminus \{\infty\}$;

(b) the set $\{(\lambda e - x)^{-1} : \lambda \in V \setminus \{\infty\}\}$ is bounded in $(X, \mathscr{P})$.

We will use the Waelbroeck spectrum throughout this paper mainly because of the following two statements (see also [21]).

If $T \in \mathscr{LB}(X)$ and $\mathscr{P} \in \mathscr{P}_{\mathscr{C}}(X)$ such that $T \in \mathscr{B}_{\mathscr{P}}(X)$ and $\sigma(Q_{\mathscr{P}}, T)$ is closed, then $\sigma(Q_{\mathscr{P}}, T) = \sigma_W(Q_{\mathscr{P}}, T)$.

If $(X, \mathscr{P})$ is a locally convex space such that $\mathscr{P} \in \mathscr{C}_0(X)$ and $T \in \mathscr{LB}(X)$ then

$$\sigma(T) = \sigma_{lb}(T) = \sigma(Q_{\mathscr{P}}, T) = \sigma(Q_{\mathscr{P}}^0, T) = \sigma_W(Q_{\mathscr{P}}, T)$$

The resolvent set will be denoted by $\rho(x)$. We also denote by $R(\lambda, x) = (\lambda e - x)^{-1}$ for $\lambda e - x$ invertible in X. The function $\lambda \to R(\lambda, x)$ is called the resolvent function of x.

Following R. T. Moore [12] and A. Chilana in [3], for a locally convex space $(X, \mathscr{P})$, a linear operator $T : X \to X$ is called universally bounded (u-bounded) with respect to $\mathscr{P}$ if there exists $r > 0$ such that $p(Tx) \leqslant r p(x)$, for all $x \in X$ and $p \in \mathscr{P}$. The class of u-bounded operators with respect to a calibration $\mathscr{P}$ will be denoted by $B_{\mathscr{P}}(X)$. For $T \in \mathscr{B}_{\mathscr{P}}(X)$, we will use the norm of T defined as:

$$\|T\|_{\mathscr{P}} = \inf\{r > 0 : p(Tx) \leqslant r p(x) \text{ for all } p \in \mathscr{P}\},$$

that makes $\mathscr{B}_{\mathscr{P}}(X)$ a unital normed algebra and $\mathscr{B}_{\mathscr{P}}(X) \subset Q_{\mathscr{P}}(X)$ (see [6]).

The following results were proved in [16]. Consider the family of semi-norms $\mathscr{P} \in \mathscr{P}_{\mathscr{C}}(X)$ such that $T \in \mathscr{B}_{\mathscr{P}}(X)$. Let $\delta > 0$. Consider the set

$$K_{\delta} := \left\{ \lambda = 1 + re^{i\theta}, \, r > 0, \, |\theta| < \frac{\pi}{2} + \delta \right\}$$

LEMMA 1. [16] *Supose the condition*

$$\sup_{\hat{p} \in \hat{\mathscr{P}}} \hat{p}\left[(T - \lambda I)^{-1}\right] \leqslant \frac{C}{|\lambda - 1|}, \quad |\lambda| > 1 \tag{1}$$

is true. Then,

$$\|(T - \lambda I)^{-1}\|_{\mathscr{P}} \leqslant \frac{M}{|\lambda - 1|}, \quad \lambda \in K_{\delta}, \quad \delta > 0, \quad M > 0$$

DEFINITION 1. [16] Peripheral spectrum of an operator $T \in \mathscr{B}_{\mathscr{P}}(X)$ is the set:

$$\pi\sigma(T) := \{\lambda \,|\, \lambda \in \sigma_W(T), \, |\lambda| = r(T)\}.$$

THEOREM 1. [16] *Let T be an operator from $\mathscr{B}_{\mathscr{P}}(X)$ with $\|T^n\|_{\mathscr{P}} \leqslant C$, $n \in \mathbb{N}^*$ and suppose that $r(T) = 1$. Then*

$$\lim_{k \to \infty} \|T^k(T - I)\|_{\mathscr{P}} = 0 \tag{2}$$

iff $\pi\sigma(T) = \{1\}$.

THEOREM 2. [16] *Let $T \in \mathscr{B}_{\mathscr{P}}(X)$ with $\|T^n\|_{\mathscr{P}} \leqslant C_1$, $n \in \mathbb{N}^*$ and it exists M such that*

$$\|T^k(T - I)\|_{\mathscr{P}} \leqslant \frac{M}{k}, \quad k \in \mathbb{N}^* \tag{3}$$

$$\|R(\lambda, T)\|_{\mathscr{P}} \leqslant \frac{C}{|\lambda - 1|}, \quad \text{for } \lambda \in K_{\delta}. \tag{4}$$

Then it exists $\delta > 0$ such that

$$\sigma(T) \cap K_{\delta} \subset \emptyset. \tag{5}$$

Conversely suppose that $\pi\sigma(T) = \{1\}$ or $\sigma(T) < 1$ and for some $\delta > 0$ we have

$$\|R(\lambda, T)\| \leqslant \frac{C}{|\lambda - 1|}, \quad \text{for } \lambda \in K_\delta. \tag{6}$$

Then T is power bounded and it exists M such that (3) is true.

3. Power boundedness for operators from $\mathscr{B}_{\mathscr{P}}(X)$ algebra

For an operator $T \in \mathscr{B}_{\mathscr{P}}(X)$ we will study the condition

$$\hat{p}\left(T^n - T^{n+1}\right) \to 0, \quad n \to \infty \tag{7}$$

for any $\hat{p} \in \hat{\mathscr{P}}$. Condition

$$\sigma_W(T) \subset \{\lambda \,|\, |\lambda| < 1\} \cup \{1\} \tag{8}$$

is necessary but not sufficient for (7). If (8) is satisfied, the differences $T^n - T^{n+1}$ are Laurent coefficients from the series:

$$(\lambda - 1)(T - \lambda I)^{-1} = -I + \sum_{n=1}^{\infty}(T^{n-1} - T^n)\lambda^{-n}, \quad |\lambda| > 1.$$

In fact, if $|\lambda| > 1$ and (8) is satisfied, we have:

$$(\lambda - 1)(T - \lambda I)^{-1} = (\lambda - 1)\sum_{n=0}^{\infty} -\frac{1}{\lambda^{n+1}}T^n = -I + \sum_{n=0}^{\infty} -\frac{1}{\lambda^{n+1}}T^{n+1} + \sum_{n=0}^{\infty}\frac{1}{\lambda^{n+1}}T^n$$

$$= -I + \sum_{n=1}^{\infty}\lambda^{-n}(T^{n-1} - T^n).$$

Hence, from the boundedness of the resolvent, it follows:

$$\sup_{\hat{p} \in \hat{\mathscr{P}}} \hat{p}[(T - \lambda I)^{-1}] \leqslant \frac{C}{|\lambda - 1|}, \quad |\lambda| > 1 \tag{9}$$

implying that (7) is true.

R. K. Ritt ([19]) showed that in the case of Banach spaces, the two conditions (1) and (8) yield $\hat{p}(T^n) = O_{(n)}$ for $n \to \infty$. Let $M_n(T) = \frac{I + T + \ldots + T^{n-1}}{n}$, for $n \in \mathbb{N}^*$. In the following, our goal is to obtain the results of Ritt for the case of universally bounded operators acting on a locally convex algebra X. We consider the family of semi-norms $\mathscr{P} \in \mathscr{P}_{\mathscr{C}}(X)$ such that $T \in \mathscr{B}_{\mathscr{P}}(X)$.

THEOREM 3. *Let T be an operator satisfying (1). Then, for $n \to \infty$ we have:*

$$\|T^n\|_{\mathscr{P}} = O(\log n) \tag{10}$$

$$\|M_n(T)\|_{\mathscr{P}} = O(1) \tag{11}$$

$$\|T^n - T^{n+1}\|_{\mathscr{P}} \to 0 \tag{12}$$

Proof. One can easily observe that $(1) \Rightarrow (8)$. Indeed, the resolvent set $\rho_W(T) = C \backslash \sigma_W(T)$ is open. Moreover, if $\lambda \in \rho_W(T)$, then $\mu : |\mu - \lambda| < |R(\lambda, T)^{-1}|_{\mathscr{P}} \subset \rho_W(T)$ that comes from the formula $R(\mu, T) = \sum_{k=0}^{\infty} R(\lambda, T)^{k+1} (\mu - \lambda)^k$, where the series is uniformly convergent on the above disc. Then, the lower margin $dist[\lambda, \sigma_W(T)] \geqslant \|R(\lambda, T)^{-1}\|_{\mathscr{P}}$ for $\lambda \in \rho_W(T)$. It will result that if $\mu \in \partial \sigma_W(T)$, we have $\|R(\lambda, T)\|_{\mathscr{P}} \to \infty$ when $\lambda \to \mu$, $\lambda \in \rho_W(T)$.

If we assume now (1) true, we obtain $\sigma_W(T) \subset \{\mu : |\mu| < 1\} \cup \{1\}$. To prove (10) consider the curve $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ which surrounds $\sigma_W(T)$, where Γ_1 is contained in the open unit disc (with the exception of the extremities), Γ_2 is closed by two arcs on the unit disc, for a distance greater or equal than $\frac{1}{n}$ of 1 and $\Gamma_3 = \{|\lambda - 1| = \frac{1}{n}, |\lambda| \geqslant 1\}$. Choose Γ_2 of length $\varepsilon > 0$. Denote by Γ_2^+ the portion of Γ_2 which lies underneath real axis and Γ_2^- the other symmetrical portion. We also denote Γ_δ the portion of curve that is centered in 1 and that surrounds Γ_3, such that the intersection with Γ_1 is given by the same points z_0^+, z_0^-. Consider N, such that for $n > N$, we get:

$$\left\| \log^{-1} n \int_{\Gamma_1} z^n (zI - T)^{-1} dz \right\|_{\mathscr{P}} < 2\pi c; \qquad \left\| \log^{-1} n \int_{\Gamma_\delta} (zI - T)^{-1} dz \right\|_{\mathscr{P}} < 2\pi c;$$

Fix n. By using the functional calculus for the operators from $\mathscr{B}_{\mathscr{P}}(X)$ (see [4]), we obtain: $T^n = \int_\Gamma z^n (zI - T)^{-1} dz$. Then,

$$\frac{1}{\ln n} \int_{\Gamma_\delta} z^n (zI - T)^{-1} dz = \left(\int_{\Gamma_3} + \int_{\Gamma_2^-} + \int_{\Gamma_2^+} \right) \frac{1}{\ln n} z^n (zI - T)^{-1} dz$$

By evaluating the three integrals it follows:

$$\int_{\Gamma_2^-} \frac{1}{\ln n} z^n (zI - T)^{-1} dz = \frac{1}{\ln n} \sum_{j=0}^{n-1} \int_{\Gamma_2^-} z^j (z-1)(zI-T)^{-1} dz + \frac{1}{\ln n} \int_{\Gamma_2^-} (zI-T)^{-1} dz$$

$$\int_{\Gamma_2^+} \frac{1}{\ln n} z^n (zI - T)^{-1} dz = \frac{1}{\ln n} \sum_{j=0}^{n-1} \int_{\Gamma_2^+} z^j (z-1)(zI-T)^{-1} dz + \frac{1}{\ln n} \int_{\Gamma_2^-} (zI-T)^{-1} dz$$

$$\int_{\Gamma_3} \frac{1}{\ln n} z^n (zI - T)^{-1} dz = \frac{1}{\ln n} \sum_{j=1}^{n} \binom{j}{n} \int_{\Gamma_3} (z-1)^{j-1}(z-1)(zI-T)^{-1} dz$$

$$+ \frac{1}{\ln n} \int_{\Gamma_3} (zI-T)^{-1} dz$$

The sum of the last terms of the right members is:

$$\ln^{-1} n \int_{\Gamma_\delta} (zI - T)^{-1} dz \quad \text{where} \quad \left\| \ln^{-1} n \int_{\Gamma_\delta} (zI-T)^{-1} dz \right\|_{\mathscr{P}} < 2\pi C$$

If we choose $\varepsilon = 2\pi \frac{\ln n}{n} > \frac{1}{n}$, we obtain

$$\left\| \frac{1}{\ln n} \sum_{j=0}^{n-1} \int_{\Gamma_2^+} z^j (z-1)(zI-T)^{-1} dz \right\|_{\mathscr{P}} \leqslant \frac{1}{\ln n} \cdot n \cdot C \cdot \varepsilon \leqslant 2\pi C$$

In the same way we proceed for Γ_2^-. Hence

$$\left\| \frac{1}{\ln n} \sum_{j=1}^{n} \binom{j}{n} \int_{\Gamma_3} (z-1)^{j-1}(z-1)(zI-T)^{-1} dz \right\|_{\mathscr{P}}$$

$$\leqslant \frac{1}{\ln n} \cdot C \cdot 2\pi \sum_{j=1}^{n} \binom{j}{n}\left(\frac{1}{n}\right)^{j} = \frac{1}{\ln n} C \cdot 2\pi \left[\left(1+\frac{1}{n}\right)^{n} - 1 \right] \leqslant \frac{2\pi C}{\ln n}(e-1) \leqslant 2\pi C.$$

$$\left\| \frac{1}{\ln n} T^{n} \right\|_{\mathscr{P}} = \left\| (2\pi i)^{-1} \left(\int_{\Gamma_1} + \int_{\Gamma_\delta} \right) \frac{1}{\ln n} z^{n} (zI-T)^{-1} dz \right\|_{\mathscr{P}} \leqslant 5C.$$

To prove (11) we choose the same contour but we take $\varepsilon = \frac{2\pi C}{n}$. By using the same functional calculus we obtain:

$$M_n(T) = \int_{\Gamma} \frac{(1+z+\cdots+z^{n})}{n}(zI-T)^{-1} dz.$$

Let N be such that for each $n > N$ one has:

$$\left\| n^{-1} \int_{\Gamma_1} z^{n}(zI-T)^{-1} dz \right\|_{\mathscr{P}} < 2\pi C \quad \text{and} \quad \left\| n^{-1} \int_{\Gamma_\delta} (zI-T)^{-1} dz \right\|_{\mathscr{P}} < 2\pi C$$

$$\int_{\Gamma_2^-} \frac{1+\ldots+z^{n-1}}{n}(zI-T)^{-1} dz = \int_{\Gamma_2^-} \left(\frac{1+z+\ldots+z^{n-1}-n}{n} +1 \right)(zI-T)^{-1} dz$$

$$= \int_{\Gamma_2^-} \left[\frac{(z-1)+(z^2-1)+\ldots+(z^{n-1}-1)}{n} +1 \right](zI-T)^{-1} dz$$

$$= \int_{\Gamma_2^-} \left[\frac{(z-1)(1+z+1+\ldots+z^{n-2}+\ldots+1)}{n} +1 \right](zI-T)^{-1} dz$$

$$= \int_{\Gamma_2^-} \left[(z-1)\frac{(n+1)+(n+2)z+\ldots+(n-(n-1))z^{n-2}}{n} +1 \right](zI-T)^{-1} dz$$

$$= n^{-1}\int_{\Gamma_2^-} \left[(n-1)+(n-2)z+\ldots+z^{n-2} \right](z-1)(zI-T)^{-1} dz + \int_{\Gamma_2^-}(zI-T)^{-1} dz.$$

Analogously, for Γ_2^+:

$$\int_{\Gamma_2^+} \frac{1+z+\cdots+z^{n-1}}{n}(zI-T)^{-1} dz$$

$$= n^{-1}\int_{\Gamma_2^+} \left[(n-1)+(n-2)z+\cdots+z^{n-2} \right](z-1)(zI-T)^{-1} dz + \int_{\Gamma_2^-}(zI-T)^{-1} dz$$

$$\int_{\Gamma_3} \frac{1 + z + \cdots + z^{n-1}}{n} (zI - T)^{-1} dz = \int_{\Gamma_3} \frac{z^n - 1}{n(z-1)} (zI - T)^{-1} dz$$

$$= \int_{\Gamma_3} \frac{\left[(z-1) + 1\right]^n - 1}{n(z-1)} (zI - T)^{-1} dz = \int_{\Gamma_3} \frac{\sum_{j=0}^{n} \binom{j}{n} (z-1)^j - 1}{n(z-1)} (zI - T)^{-1} dz$$

$$= \frac{1}{n} \sum_{j=1}^{n} \binom{j}{n} \int_{\Gamma_3} \frac{(z-1)^{j-1}}{z-1} (z-1)(zI - T)^{-1} dz.$$

The sum of the last terms of the right members is:

$$\frac{1}{n} \int_{\Gamma_\delta} (zI - T)^{-1} dz, \quad \text{where} \quad \left\| \frac{1}{n} \int_{\Gamma_\partial} (zI - T)^{-1} dz \right\|_{\mathscr{P}} \leqslant 2\pi C.$$

It follows the evaluates

$$\left\| \frac{1}{n} \int_{\Gamma_2^-} \left[(n-1) + (n-2)z + \ldots + z^{n-2} \right] (z-1)(zI - T)^{-1} dz \right\|_{\mathscr{P}}$$

$$\leqslant \frac{1}{n} \frac{(n-1)n}{2} l\left(\Gamma_2^-\right) = \frac{n-1}{2n} 2\pi C \leqslant \pi C$$

$$\left\| \frac{1}{n} \sum_{j=1}^{n} \binom{j}{n} \int_{\Gamma_3} \frac{(z-1)^{j-1}}{z-1} (z-1)(zI - T)^{-1} dz \right\|_{\mathscr{P}}$$

$$\leqslant \frac{2\pi C}{n} n \left(\sum_{j=0}^{n} \binom{j}{n} \frac{1}{n} - 1 \right) = 2\pi C \left[\left(1 + \frac{1}{n}\right)^n - 1 \right] < 2\pi C(e-1).$$

Then

$$\|M_n(T)\|_{\mathscr{P}} = \left\| (2\pi i)^{-1} \left(\int_{\Gamma_1} + \int_{\Gamma_\delta} \right) n^{-1} \sum_{k=0}^{n-1} z^k (zI - T)^{-1} dz \right\|_{\mathscr{P}} \leqslant 5\pi Ce.$$

Relation (12) follows immediately pursuing a similar reasoning. $\square$

One can observe that from (1) one can obtain (7). It is interesting to compare (1) with the weaker Kreiss type condition:

$$\left\| (T - \lambda I)^{-1} \right\|_{\mathscr{P}} \leqslant \frac{C}{|\lambda| - 1}, \quad |\lambda| > 1 \tag{13}$$

implying the weaker conclusions:

$$\|T^n\|_{\mathscr{P}} = O(n) \tag{14}$$

$$\|M_n(T)\|_{\mathscr{P}} = O(\log n) \tag{15}$$

for $n \in \mathbb{N}^*$, both of them resulting easily from evaluations like above.

4. Characterizations for resolvent growth

Consider now the family of semi-norms $\mathscr{P} \in \mathscr{P}_{\mathscr{C}}(X)$ such that $T \in \mathscr{B}_{\mathscr{P}}(X)$. Let $\delta > 0$. We consider the set

$$K_\delta := \left\{ \lambda = 1 + re^{i\theta}, \; r > 0, \; |\theta| < \frac{\pi}{2} + \delta \right\}$$

As defined above, the set K_δ is a partial circle from the complex plane centered at 1 of radius r. It is partial because a sector of δ radians is "missing". This set K_δ represents a generalization for the next lemma's hypothesis. Previously the lemma was proved true by taking instead of K_δ the whole circle described above.

LEMMA 2. *Suppose we have the condition (1) true. Then,*

$$\|(T - \lambda I)^{-1}\|_{\mathscr{P}} \leqslant \frac{M}{|\lambda - 1|}, \quad \lambda \in K_\delta, \quad \delta > 0, \quad M > 0$$

Proof. Let $A = T - I$. If we denote by C the constant from inequality (1), we obtain:

$$\sigma(A) \subset \{\lambda : |\lambda + 1| < 1\} \cup \{0\} \text{ and } \|(A - \lambda I)^{-1}\|_{\mathscr{P}} \leqslant \frac{C}{|\lambda|}, \quad \text{for} \;\; |\lambda + 1| > 1.$$

In particular, the above estimate is true for each λ_0 with $\text{Re}\lambda_0 = 0$ and $\text{Im}\lambda_0 \neq 0$.

$$(A - \lambda I)^{-1} = \sum_{n=0}^{\infty} (A - \lambda_0 I)^{-n-1} (\lambda - \lambda_0)^n, \text{ if } |\lambda - \lambda_0| \, \|(A - \lambda_0 I)^{-1}\|_{\mathscr{P}} < 1.$$

One can observe that $(A - \lambda I)^{-1}$ exists for all λ with $\text{Im}\lambda = \text{Im}\lambda_0$ whence $|\text{Re}\lambda| < \frac{|\lambda_0|}{C}$.

Indeed, $|\lambda - \lambda_0| \, \|(A - \lambda_0 I)^{-1}\|_{\mathscr{P}} < |\lambda - \lambda_0| \cdot \frac{C}{|\lambda_0|}$. If $|\lambda - \lambda_0| \cdot \frac{C}{|\lambda_0|} < 1$ then $|\lambda - \lambda_0| < \frac{|\lambda_0|}{C}$ which for $\text{Im}\lambda = \text{Im}\lambda_0$, we get $|\text{Re}\lambda| < \frac{\lambda_0}{C}$. If we choose ε such that $\tan \varepsilon = \frac{1}{e}$, then $(A - \lambda T)^{-1}$ exists for all $\lambda \in K_\varepsilon - 1$. To obtain a better estimate, fix $\delta \in (0, \varepsilon)$ such that $\tan \delta = \frac{q}{C}$, for all $q \in (0, 1)$. Consider $\lambda \in K_\delta - 1$ with $\text{Re}\lambda < 1$. Let $\lambda_0 = i\text{Im}\lambda$. Then $\frac{|\lambda - \lambda_0|}{|\lambda_0|} \cdot \frac{C}{q} < 1$, which implies $|\lambda - \lambda_0| \, \|(A - \lambda_0 I)^{-1}\|_{\mathscr{P}} < q < 1$, hence

$$\|(A - \lambda I)^{-1}\|_{\mathscr{P}} \leqslant \|(A - \lambda_0 I)^{-1}\|_{\mathscr{P}} \sum_{n=0}^{\infty} q^n \leqslant \frac{C}{|\lambda_0|(1 - q)} < \frac{C}{|\lambda_0|(1 - q)\cos\delta}$$

Choosing $M = \frac{C}{(1-q)\cos\delta} = \frac{C}{(1-q)\sqrt{\frac{C^2}{C^2+q^2}}} = \frac{\sqrt{C^2+q^2}}{1-q} \geqslant C$, it results the conclusion for T.

THEOREM 4. *Let $T \in \mathscr{B}_{\mathscr{P}}(X)$ be a power bounded operator and $\sigma(T) \cap \Gamma = \{1\}$. The following statements are equivalent:*

(i) it exists $M < \infty$ such that $\left\| T^{n+1} - T^n \right\|_{\mathscr{P}} \leqslant \frac{M}{n+1}$, $n \geqslant 0$;

(i) it exists $K < \infty$ such that $\left\| (T-I)e^{t(T-1)} \right\|_{\mathscr{P}} \leqslant K \frac{1-e^{-t}}{t}$, $t > 0$;

(iii) it exists $K < \infty$: $\left\| (T-I)(\lambda I - T)^{-(n+1)} \right\|_{\mathscr{P}} \leqslant \frac{K}{n} \left[\frac{1}{(\lambda-1)^n} - \frac{1}{\lambda^n} \right]$, $n \geqslant 1$, $\lambda > 1$;

(iv) it exists $B < \infty$, and $\delta > 0$ such that $\left\| (\lambda I - T)^{-1} \right\|_{\mathscr{P}} < \frac{B}{|\lambda-1|}$, for any $\lambda \in K_\delta$.

Proof. "*(i) $\Rightarrow$ (ii)*"   For $t > 0$, we have:

$$\left\| (T-I)e^{tT} \right\|_{\mathscr{P}} \leqslant \sum_0^\infty \left\| T^n(T-I) \right\|_{\mathscr{P}} \frac{t^n}{n!} = M \frac{e^t - 1}{t}$$

"*(ii) $\Rightarrow$ (iii)*"   For $n \geqslant 1$, $\lambda > 0$:

$$(\lambda I - T)^{-(n+1)} = \frac{1}{n!} \int_0^\infty t^n e^{-(\lambda-1)t} e^{t(T-I)} \mathrm{d}t$$

By multiplying with $T - I$ and from relation *(ii)* we get *(iii)*.

"*(iii) $\Rightarrow$ (ii)*"   It is obtained by substitution $\lambda := (n+1)/t$, in

$$\left\| (T-I)(I - T/\lambda)^{-(n+1)} \right\|_{\mathscr{P}} \leqslant \frac{K\lambda}{n} \left[\frac{1}{\left(1 - \frac{1}{\lambda}\right)^n} - 1 \right]$$

and by making $n \to \infty$, we reach *(ii)*.

"*(ii) $\Rightarrow$ (iv)*"   Suppose $\|T^n\|_{\mathscr{P}} \leqslant C$, for any $n \geqslant 0$. Then:

$$\left\| e^{tT} \right\|_{\mathscr{P}} \leqslant C \sum_n \frac{t^n}{n!} = Ce^t$$

and it follows $\left\| e^{t(T-I)} \right\|_{\mathscr{P}} \leqslant C$, for any $t > 0$. By estimating $\left\| e^{z(T-I)} \right\|_{\mathscr{P}}$ uniformly in a sector that surrounds the positive axis $t > 0$, we could change the integration path:

$$(\lambda I - T)^{-1} = \int_0^\infty e^{-(\lambda-1)t} e^{t(T-I)} \mathrm{d}t, \quad \text{for } \lambda > 1$$

with a new path of radius $z = re^{i\theta}$, for θ small enough. This proof is the same as the one in [17] regarding the uniformly bounded analytic semi-groups where it yields more

$$\left\| (T - \lambda I)^{-1} \right\|_{\mathscr{P}} \leqslant \frac{C}{\operatorname{Re}\lambda - 1}, \quad \text{for } \operatorname{Re}\lambda > 1.$$

"*(iv) $\Rightarrow$ (i)*"   This implication is proven by a direct and immediate application of Theorem 2.

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