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SOME NEW VOLTERRA–FREDHOLM TYPE DISCRETE INEQUALITIES WITH FOUR ITERATED INFINITE SUMS WITH APPLICATIONS

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Abstract. Some generalized Volterra-Fredholm type nonlinear discrete inequalities involving four iterated infinite sums are discussed. Some new explicit bounds for solutions of certain difference equations are derived. To illustrate the significance of results, some new applications are given.

1. Introduction

Gronwall-Bellman inequality [1, 2] plays a significant role in the field of integral inequalities. Many authors studied its various generalizations [3]–[13], such as Ou-Iang's inequality [3], which are considered as handy tools in the study of qualitative properties of solutions of differential and integral equations. Q. H. Ma [6] gave the discrete form of Ou-Iang's inequality for the first time, which is considered to be a milestone in the field of Volterra-Fredholm type difference equations. During the last decade, B. Zheng [15], Q. H. Ma et al. [4, 5, 6], B. Zheng and Q. H. Feng [16] and B. Zheng and Bosheng Fu [17] have great contributions in this domain. B. Zheng [15] considered a class of Volterra-Fredholm type difference equation:

$$
\begin{aligned}
u^p(m,n) = g_1(m,n) &+ \sum_{s=m+1}^{\infty} g_2(s,n)u^p(s,n) + \sum_{i=1}^{l_1}\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}[F_{1i}(s,t,m,n,u(s,t)) \\
&+ \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty} F_{2i}(\xi,\eta,m,n,u(\xi,\eta))] + \sum_{j=1}^{l_2}\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}[G_{1j}(s,t,m,n,u(s,t)) \\
&+ \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty} G_{2j}(\xi,\eta,m,n,u(\xi,\eta))], \qquad (1)
\end{aligned}
$$

where $u(m,n), g_1(m,n), g_2(m,n)$ are real-valued functions defined on $\Omega := ([m_0,\infty)\times[n_0,\infty))\cap\mathbb{Z}^2$, $m_0,n_0\in\mathbb{Z}$, $F_{1i},F_{2i},G_{1j},G_{2j}$; $1\leqslant i\leqslant l_1$; $1\leqslant j\leqslant l_2$ are real-valued functions defined on Ω^2 and l_1,l_2 are scalars from $\mathbb{Z}$. Our aim is to establish some

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new more generalized Volterra-Fredholm type difference inequalities and then use them to find the new estimates of the solution of (1). In this paper, $\mathbb{R}$ represents the set of real numbers, $\mathbb{R}_+ = [0,\infty)$ while $\mathbb{Z}$ is set of integers and $\mathfrak{U}$ denotes a lattice. Then $\mathfrak{I}\{\mathfrak{U}\}$ and $\mathfrak{I}_+\{\mathfrak{U}\}$ represent the set of all real-valued functions and positive real-valued functions on $\mathfrak{U}$ respectively. Also, $\sum_{s=M_0}^{M_1} y(s) = 0$ for a function $y \in \mathfrak{I}_+\{\mathfrak{U}\}$ provided $M_0 > M_1$; Δ_{ij} denotes the forward difference in ith and jth components respectively.

LEMMA 1. [15] *Suppose* $u,a,b \in \mathfrak{I}_+(\Omega)$. *If* $a(m,n)$ *is non-increasing in the first variable, then*

$$u(m,n) \leqslant a(m,n) + \sum_{s=m+1}^{\infty} b(s,n)u(s,n),\ \forall (m,n) \in \Omega,$$

implies

$$u(m,n) \leqslant a(m,n) \sum_{s=m+1}^{\infty} [1 + b(s,n)].$$

LEMMA 2. *Suppose* $u,a,H_1 \in \mathfrak{I}_+(\Omega)$, $b \in \mathfrak{I}_+(\Omega^2)$; *let* a,H_1 *be non-increasing in every variable with* $H_1(m,n) > 0$, *while* b *is non-increasing in the third variable. Let* $\varphi,\phi \in C(\mathbb{R}_+,\mathbb{R}_+)$ *be strictly increasing with* $\varphi(r)$, $\phi(r) > 0$ *for* $r > 0$. *such that:*

$$u(m,n) \leqslant H_1(m,n) + \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} b(s,t,m,n)\varphi\{\phi^{-1}(u(s,t) + a(s,t))\}, \tag{2}$$

then

$$u(m,n) \leqslant G^{-1}[G(H_1(m-1,n-1)) + H_2(m-1,n-1) + \sum_{s=m}^{\infty} \sum_{t=n}^{\infty} b(s,t,m,n)], \tag{3}$$

provided that

$$G(z) = \int_{z_0}^{z} \frac{1}{\varphi[\phi^{-1}(z + a(m+1,n+1))]} dz; \quad z \geqslant z_0 > 0. \tag{4}$$

$$\begin{aligned} H_2(m,n) = 2 \sum_{s=m+1}^{\infty} b(s,n+1,m+1,n+1) + \sum_{t=n+1}^{\infty} \triangle_4 b(m+1,t,m+1,n) \\ -b(m+1,n+1,m+1,n+1). \end{aligned} \tag{5}$$

Proof. For some fixed m_1 and n_1, (2) is rewritten as:

$$u(m,n) \leqslant H_1(m_1,n_1) + \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} b(s,t,m,n)\varphi\{\phi^{-1}(u(s,t) + a(s,t))\} \tag{6}$$

Denote the right hand side of (6) by $v(m,n)$. Then (6) reduces to

$$u(m,n) \leqslant v(m,n) \tag{7}$$

$$\begin{aligned}
\triangle_{12}v(m,n) &= b(m+1,n+1,m+1,n+1)\varphi\{\phi^{-1}(u(m+1,n+1)+a(m+1,n+1))\} \\
&\quad - \sum_{t=n+1}^{\infty} b(m+1,t,m+1,n)\varphi\{\phi^{-1}(u(m+1,t)+a(m+1,t))\} \\
&\quad - \sum_{s=m+1}^{\infty} \triangle_3 b(s,n+1,m,n+1)\varphi\{\phi^{-1}(u(s,n+1)+a(s,n+1))\} \\
&\quad + \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} \triangle_{34} b(s,t,m,n)\varphi\{\phi^{-1}(u(s,t)+a(s,t))\} \\
&\leqslant b(m+1,n+1,m+1,n+1)\varphi\{\phi^{-1}(v(m+1,n+1)+a(m+1,n+1))\} \\
&\quad - \sum_{t=n+1}^{\infty} b(m+1,t,m+1,n)\varphi\{\phi^{-1}(v(m+1,t)+a(m+1,t))\} \\
&\quad - \sum_{s=m+1}^{\infty} \triangle_3 b(s,n+1,m,n+1)\varphi\{\phi^{-1}(v(s,n+1)+a(s,n+1))\} \\
&\quad + \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} \triangle_{34} b(s,t,m,n)\varphi\{\phi^{-1}(v(s,t)+a(s,t))\} \qquad (8) \\
&\leqslant \{b(m+1,n+1,m+1,n+1) - \sum_{t=n+1}^{\infty} b(m+1,t,m+1,n) \\
&\quad - \sum_{s=m+1}^{\infty} \triangle_3 b(s,n+1,m,n+1) + \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} \triangle_{34} b(s,t,m,n)\}. \\
&\quad \varphi\{\phi^{-1}(v(m+1,n+1)+a(m+1,n+1))\}
\end{aligned}$$

Equivalently:

$$\begin{aligned}
&\frac{v(m+1,n)-v(m+1,n+1)}{\varphi\{\phi^{-1}(v(m+1,n+1)+a(m+1,n+1))\}} \\
&\leqslant H_2(m,n) + \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} b(s,t,m+1,n+1). \qquad (9)
\end{aligned}$$

On the other hand, by the mean-value theorem for integrals there exist ξ such that $v(m+1,n+1) \leqslant \xi \leqslant v(m+1,n)$, we have

$$\begin{aligned}
&G(v(m+1,n+1) - G(H_1(m_1,n_1)) \\
&= \int_{v(m+1,n+1)}^{v(m+1,n)} \frac{1}{\varphi\{\phi^{-1}(z+a(m+1,n+1))\}} dz \\
&= \frac{v(m+1,n)-v(m+1,n+1)}{\varphi\{\phi^{-1}(\xi+a(m+1,n+1))\}} \\
&\leqslant \frac{v(m+1,n)-v(m+1,n+1)}{\varphi\{\phi^{-1}(v(m+1,n+1)+a(m+1,n+1))\}} \qquad (10)
\end{aligned}$$

As G is increasing, combination of (5), (9) and (10) yields:

$$v(m+1,n+1) \leqslant G^{-1}[G(H(m_1,n_1)) + H_2(m,n) + \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} b(s,t,m+1,n+1)]$$

As (m_1,n_1) was selected arbitrarily therefore we have

$$u(m+1,n+1) \leqslant G^{-1}[G(H_1(m,n))+H_2(m,n)+\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} b(s,t,m+1,n+1)]. \quad \square$$

THEOREM 1. *Suppose $u \in \Im_+(\Omega)$; let $b_i, c_i, d_j, e_j \in \Im_+(\Omega^2)$; $1 \leqslant i \leqslant l_1$; $1 \leqslant j \leqslant l_2$ be non-increasing functions in last two variables and one of d_j, e_j is not equivalent to zero and ϕ, φ, a are defined in Lemma 2 such that:*

$$\begin{aligned}\phi(u(m,n)) \leqslant a(m,n)&+\sum_{i=1}^{l_1}\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}[b_i(s,t,m,n)\varphi(u(s,t))\\&+\sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}c_i(\xi,\eta,m,n)\varphi(u(\xi,\eta))]+\sum_{i=1}^{l_2}\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}[d_i(s,t,m,n)\phi(u(s,t))\\&+\sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}e_i(\xi,\eta,m,n)\phi(u(\xi,\eta))],\end{aligned} \tag{11}$$

then

$$\begin{aligned}u(m,n) \leqslant \phi^{-1}\{a(m,n)+G^{-1}\{G(T^{-1}[H_2(M-1,N-1)\\+\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}B(s,t,M,N)])+\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}B(s,t,m,n)\}\},\end{aligned} \tag{12}$$

provided that T is increasing for

$$T(x) := G\left(\frac{x-1-\mu_1}{\mu_2}\right)-G(x);\ x \geqslant 0 \tag{13}$$

$$B(s,t,m,n) := \sum_{i=1}^{l_1}[b_i(s,t,m,n)+\sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}c_i(\xi,\eta,m,n)]$$

$$\begin{aligned}J(m+1,n+1) := \sum_{i=1}^{l_2}\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}&[d_i(s,t,m+1,n+1)a(s,t)\\&+\sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}e_i(\xi,\eta,m+1,n+1)a(\xi,\eta)]\end{aligned} \tag{14}$$

$$\mu_1 := J(M,N)$$

$$\mu_2 := \sum_{i=1}^{l_2}\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}[d_i(s,t,M+1,N+1)+\sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}e_i(\xi,\eta,M+1,N+1)].$$

Proof. Consider

$$v(m,n) = \sum_{i=1}^{l_1} \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} [b_i(s,t,m,n)\varphi(u(s,t)) + \sum_{\xi=s}^{\infty} \sum_{\eta=t}^{\infty} c_i(\xi,\eta,m,n)\varphi(u(\xi,\eta))]$$
$$+ \sum_{i=1}^{l_2} \sum_{s=M+1}^{\infty} \sum_{t=N+1}^{\infty} [d_i(s,t,m,n)\phi(u(s,t)) + \sum_{\xi=s}^{\infty} \sum_{\eta=t}^{\infty} e_i(\xi,\eta,m,n)\phi(u(\xi,\eta))], \tag{15}$$

then (11) is written as

$$u(m,n) \leqslant \phi^{-1}(a(m,n)+v(m,n)). \tag{16}$$

From (15) and (16)

$$\begin{aligned} &\triangle_{12}v(m,n) \\ \leqslant\ & H(m+1,n+1) - \sum_{i=1}^{l_1} \sum_{t=n+1}^{\infty} [\triangle_4 b_i(m+1,t,m+1,n)\varphi\{\phi^{-1}(a(m+1,t)+v(m+1,t))\} \\ &+ \sum_{\xi=m+1}^{\infty} \sum_{\eta=t}^{\infty} \triangle_4 c_i(\xi,\eta,m+1,n)\varphi\{\phi^{-1}(a(\xi,\eta)+v(\xi,\eta))\}] \\ &+ \sum_{i=1}^{l_1} \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} [\triangle_{43} b_i(s,t,m,n)\varphi\{\phi^{-1}(a(s,t)+v(s,t))\} \\ &+ \sum_{\xi=s}^{\infty} \sum_{\eta=t}^{\infty} \triangle_{43} c_i(\xi,\eta,m,n)\varphi\{\phi^{-1}(a(\xi,\eta)+v(\xi,\eta))\}], \end{aligned} \tag{17}$$

where

$$H(m+1,n+1) := \sum_{i=1}^{l_2} \sum_{s=M+1}^{\infty} \sum_{t=N+1}^{\infty} [\triangle_{43} d_i(s,t,m,n)(a(s,t)+v(s,t))$$
$$+ \sum_{\xi=s}^{\infty} \sum_{\eta=t}^{\infty} \triangle_{43} e_i(\xi,\eta,m,n)(a(\xi,\eta)+v(\xi,\eta))]. \tag{18}$$

Setting $m \mapsto p$ and $n \mapsto q$ in (17) and summing over p and q from $m+1$ to ∞ and $n+1$ to ∞ respectively yields:

$$\begin{aligned} &v(m+1,n+1) \\ \leqslant\ & H(m+1,n+1) - \sum_{i=1}^{l_1} \sum_{t=n+1}^{\infty} [b_i(m+1,t,m+1,n+1)\varphi\{\phi^{-1}(a(m+1,t)+v(m+1,t))\} \\ &+ \sum_{\xi=m+1}^{\infty} \sum_{\eta=t}^{\infty} c_i(\xi,\eta,m+1,n+1)\varphi\{\phi^{-1}(a(\xi,\eta)+v(\xi,\eta))\}] \end{aligned}$$

$$+\sum_{i=1}^{l_1}\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}[b_i(s,t,m+1,n+1)\varphi\{\phi^{-1}(a(s,t)+v(s,t))\}$$
$$+\sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}c_i(\xi,\eta,m+1,n+1)\varphi\{\phi^{-1}(a(\xi,\eta)+v(\xi,\eta))\}], \tag{19}$$

$$H(m+1,n+1)=J(m+1,n+1)+\sum_{i=1}^{l_2}\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}[d_i(s,t,m+1,n+1)v(s,t)$$
$$+\sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}e_i(\xi,\eta,m+1,n+1)v(\xi,\eta)],$$

$$v(m+1,n+1)\leqslant H'(M+1,N+1)+[2\sum_{i=1}^{l_1}\sum_{s=m+2}^{\infty}\sum_{t=n+1}^{\infty}\{b_i(s,t,m+1,n+1)$$
$$+\sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}c_i(\xi,\eta,m+1,n+1)\}]\varphi\{\phi^{-1}(a(s,t)+v(s,t))\} \tag{20}$$

for

$$[2\sum_{i=1}^{l_1}\sum_{t=n+1}^{\infty}\{b_i(m+1,t,m+1,n+1)+\sum_{\xi=m+1}^{\infty}\sum_{\eta=t}^{\infty}c_i(\xi,\eta,m+1,n+1)$$
$$+\sum_{i=1}^{l_1}\sum_{s=m+2}^{\infty}\{b_i(s,n+1,m+1,n+1)+\sum_{\xi=s}^{\infty}\sum_{\eta=n+1}^{\infty}c_i(\xi,\eta,m+1,n+1)\}]$$
$$\times\varphi\{\phi^{-1}(a(m+1,n+1)+v(m+1,n+1))\}\leqslant 1$$

And $H'(M+1,N+1)=H(M+1,N+1)+1\ >0$.
Setting $m\mapsto m-1$, $n\mapsto n-1$, $M\mapsto M-1$, $N\mapsto N-1$

$$v(m,n)\leqslant H'(M,N)+\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}B(s,t,m,n)\varphi\{\phi^{-1}(a(s,t)+v(s,t))\}.$$

Obviously $B(s,t,m,n)$ is non-increasing in last two variables and by application of Lemma 2 yields:

$$v(m,n)\leqslant G^{-1}[G(H'(M-1,N-1))+H_2(M-1,N-1)+\sum_{s=m}^{\infty}\sum_{t=n}^{\infty}B(s,t,m,n)]. \tag{21}$$

Furthermore, by definition of $H'(M,N)$, μ_1,μ_2 and (21), we have

$$H'(M,N)\leqslant 1+J(M,N)+v(M+1,N+1)\sum_{i=1}^{l_2}\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}[d_i(s,t,m+1,n+1)$$
$$+\sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}e_i(\xi,\eta,m+1,n+1)].$$

Equivalently

$$G\left(\frac{H'(M-1,N-1)-1-\mu_1}{\mu_2}\right)$$
$$\leqslant G(H'(M-1,N-1))+H_2(M-1,N-1)+\sum_{s=M}^{\infty}\sum_{t=N}^{\infty}B(s,t,M,N)],$$

or

$$T(H'(M-1,N-1))\leqslant H_2(M-1,N-1)+\sum_{s=M}^{\infty}\sum_{t=N}^{\infty}B(s,t,M,N)$$

where T is defined in (13). By T is increasing, we have

$$H'(M-1,N-1))\leqslant T^{-1}[H_2(M-1,N-1)+\sum_{s=M}^{\infty}\sum_{t=N}^{\infty}B(s,t,M,N)]. \tag{22}$$

Combination of (16),(21) and (22), yields the desired result. □

THEOREM 2. *Let the conditions of Theorem 1 hold. Moreover, if $w\in\Im_+(\Omega)$ and assume $\varphi o\phi^{-1}$ is sub-multiplicative, that is, $\varphi(\phi^{-1}(\alpha\beta))\leqslant\varphi(\phi^{-1}(\alpha))\varphi(\phi^{-1}(\beta))$ $\forall\,\alpha,\beta\in\mathbb{R}_+$ such that*

$$\begin{aligned}\phi(u(m,n))\leqslant a(m,n)&+\sum_{s=m+1}^{\infty}w(s,n)\phi(u(s,n))+\sum_{i=1}^{l_1}\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}[b_i(s,t,m,n)\varphi(u(s,t))\\&+\sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}c_i(\xi,\eta,m,n)\varphi(u(\xi,\eta))]+\sum_{i=1}^{l_2}\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}[d_i(s,t,m,n)\phi(u(s,t))\\&+\sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}e_i(\xi,\eta,m,n)\phi(u(\xi,\eta))],\end{aligned} \tag{23}$$

then

$$\begin{aligned}u(m,n)\leqslant\phi^{-1}\{\{a(m,n)&+G^{-1}\{G(T^{-1}[\bar{H_2}(M-1,N-1)+\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}\bar{B}(s,t,M,N)])\\&+\sum_{i=1}^{l_1}\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\bar{B}(s,t,m,n)\}\}\bar{w}(m,n)\},\end{aligned} \tag{24}$$

where G is as defined in Lemma 2 and

$$T(x)=G\left(\frac{x-1-\bar{\mu_1}}{\bar{\mu_2}}\right)-G(x);\;\;x\geqslant 0, \tag{25}$$

$$\bar{B}(s,t,m,n)=\sum_{i=1}^{l_1}[\bar{b_i}(s,t,m,n)+\sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}\bar{c_i}(\xi,\eta,m,n)]$$

$$\bar{J}(m,n)=\sum_{i=1}^{l_2}\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}[\bar{d_i}(s,t,m,n)a(s,t)+\sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}\bar{e_i}(\xi,\eta,m,n)a(\xi,\eta)] \tag{26}$$

$$\bar{b_i}(s,t,m,n) = b_i(s,t,m,n)\varphi[\phi^{-1}(\bar{w}(s,t))]$$
$$\bar{c_i}(s,t,m,n) = c_i(s,t,m,n)\varphi[\phi^{-1}(\bar{w}(s,t))]; \quad 1 \leqslant i \leqslant l_1$$
$$\bar{d_j}(s,t,m,n) = d_j(s,t,m,n)\varphi[\phi^{-1}(\bar{w}(s,t))]$$
$$\bar{e_j}(s,t,m,n) = e_j(s,t,m,n)\varphi[\phi^{-1}(\bar{w}(s,t))]; \quad 1 \leqslant j \leqslant l_2$$
$$\bar{w}(m,n) = \prod_{s=m+1}^{\infty}[1+w(s,n)] \tag{27}$$
$$\bar{\mu_1} = \bar{J}(M,N)$$
$$\bar{\mu_2} = \sum_{j=1}^{l_2}\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}[\bar{d_j}(s,t,M+1,N+1) + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}\bar{e_j}(\xi,\eta,M+1,N+1)]. \tag{28}$$

Proof. Consider

$$z(m,n) = a(m,n) + \sum_{i=1}^{l_1}\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}[b_i(s,t,m,n)\varphi(u(s,t)) + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}c_i(\xi,\eta,m,n)\varphi(u(\xi,\eta))] + \sum_{i=1}^{l_2}\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}[d_i(s,t,m,n)\phi(u(s,t)) + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}e_i(\xi,\eta,m,n)\phi(u(\xi,\eta))].$$

Then, (23) is written as

$$\phi(u(m,n)) \leqslant z(m,n) + \sum_{s=m+1}^{\infty} w(s,n)\phi(u(s,n)). \tag{29}$$

Obviously, $z(m,n)$ is non-increasing in the first variable. By Lemma 1,

$$\phi(u(m,n)) \leqslant z(m,n)\prod_{s=m+1}^{\infty}[1+w(s,n)] = z(m,n)\bar{w}(m,n). \tag{30}$$

Consider

$$v(m,n) = \sum_{i=1}^{l_1}\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}[b_i(s,t,m,n)\varphi(u(s,t)) + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}c_i(\xi,\eta,m,n)\varphi(u(\xi,\eta))] + \sum_{i=1}^{l_2}\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}[d_i(s,t,m,n)\phi(u(s,t)) + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}e_i(\xi,\eta,m,n)\phi(u(\xi,\eta))]. \tag{31}$$

Then a combination of (27) and (30) yields:

$$u(m,n) \leqslant \phi^{-1}[(a(m,n)+v(m,n))\bar{w}(m,n)]. \tag{32}$$

Applying forward differences on m and n, and using (31). then same procedure as in Theorem 1 from (17) to (19), we have

$$\begin{aligned}
&v(m+1,n+1)\\
&\leqslant H(m+1,n+1)+\sum_{i=1}^{l_1}\sum_{t=n+1}^{\infty}[b_i(m+1,t,m+1,n+1)\varphi\{\phi^{-1}((a(m+1,t)\\
&+v(m+1,t))\bar{w}(m+1,t))\}+\sum_{\xi=m+1}^{\infty}\sum_{\eta=t}^{\infty}c_i(\xi,\eta,m+1,n+1)\varphi\{\phi^{-1}((a(\xi,\eta)\\
&+v(\xi,\eta))\bar{w}(\xi,\eta))\}]+\sum_{i=1}^{l_1}\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}[b_i(s,t,m+1,n+1)\varphi(u(s,t))\\
&\times\varphi\{\phi^{-1}((a(s,t)+v(s,t))\bar{w}(s,t))\}+\sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}c_i(\xi,\eta,m+1,n+1)\\
&\times\varphi\{\phi^{-1}((a(\xi,\eta)+v(\xi,\eta))\bar{w}(\xi,\eta))\}],
\end{aligned}$$

provided that

$$\begin{aligned}
H(m+1,n+1):=J(m+1,n+1)+\sum_{i=1}^{l_2}\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}[d_i(s,t,m+1,n+1)v(s,t)\bar{w}(s,t)\\
+\sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}e_i(\xi,\eta,m+1,n+1)v(\xi,\eta)\bar{w}(\xi,\eta)],
\end{aligned}$$

$$\begin{aligned}
J(m+1,n+1):=\sum_{i=1}^{l_2}\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}[d_i(s,t,m+1,n+1)a(s,t)\bar{w}(s,t)\\
+\sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}e_i(\xi,\eta,m+1,n+1)a(\xi,\eta)\bar{w}(\xi,\eta)].
\end{aligned}$$

Equivalently, by using $\varphi o\phi^{-1}$ is sub-multiplicative,

$$\begin{aligned}
v(m+1,n+1)\leqslant H(m+1,n+1)+2\sum_{i=1}^{l_1}\sum_{t=n+1}^{\infty}[b_i(s,t,m+1,n+1)\varphi\{\phi^{-1}(a(s,t)+v(s,t))\}\\
\times\varphi\{\phi^{-1}\bar{w}(s,t)\}+\sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}c_i(\xi,\eta,m+1,n+1)\varphi\{\phi^{-1}(a(\xi,\eta)+v(\xi,\eta))\}\\
\times\varphi\{\phi^{-1}\bar{w}(\xi,\eta)\}]-\sum_{i=1}^{l_1}\sum_{s=m+2}^{\infty}\sum_{t=n+1}^{\infty}[b_i(s,t,m+1,n+1)\\
\times\varphi\{\phi^{-1}(a(s,t)+v(s,t))\}\varphi\{\phi^{-1}\bar{w}(s,t)\}+\sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}c_i(\xi,\eta,m+1,n+1)\\
\times\varphi\{\phi^{-1}(a(\xi,\eta)+v(\xi,\eta))\}\varphi\{\phi^{-1}\bar{w}(\xi,\eta)\}],
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \bar{H}(m+1,n+1)+2\sum_{i=1}^{l_1}\sum_{t=n+1}^{\infty}[\bar{b_i}(s,t,m+1,n+1)\\
&\quad+\sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}c_i(\xi,\eta,m+1,n+1)]\varphi\{\phi^{-1}(a(s,t)+v(s,t))\}\\
&\quad-\sum_{i=1}^{l_1}\sum_{s=m+2}^{\infty}\sum_{t=n+1}^{\infty}[b_i(s,t,m+1,n+1)\\
&\quad+\sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}c_i(\xi,\eta,m+1,n+1)]\varphi\{\phi^{-1}(a(\xi,\eta)+v(\xi,\eta))\},
\end{aligned}
$$

provided that

$$
\begin{aligned}
\bar{H}(m+1,n+1) = \bar{J}(m+1,n+1)&+\sum_{i=1}^{l_2}\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}[\bar{d_i}(s,t,m+1,n+1)v(s,t)\\
&+\sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}\bar{e_i}(\xi,\eta,m+1,n+1)v(\xi,\eta)].
\end{aligned}
$$

Then similar to the process (20)–(22) obtained in Theorem 1,

$$
v(m,n)\leqslant G^{-1}[G(\bar{H}'(M-1,N-1))+\bar{H_2}(M-1,N-1)+\sum_{s=m}^{\infty}\sum_{t=n}^{\infty}\bar{B}(s,t,m,n)] \tag{33}
$$

and

$$
\bar{H}'(M-1,N-1)\leqslant T^{-1}[\bar{H_2}(M-1,N-1)+\sum_{s=M}^{\infty}\sum_{t=N}^{\infty}\bar{B}(s,t,M,N)] \tag{34}
$$

Combination of (32)–(34),yields the desired result. □

THEOREM 3. *Let* $L_{1i},L_{2i},T_{1i},T_{2i}:\Omega\times\mathbb{R}_+\to\mathbb{R}_+$*;* $i=1,2,\ldots,l_2$ *be such that*

$$
0\leqslant L_{ji}(m,n,u)-L_{ji}(m,n,v)\leqslant T_{ji}(m,n,v)(u-v);\quad j=1,2 \text{ for } u\geqslant v\geqslant 0,
$$

and all conditions of Theorem 1 hold such that

$$
\begin{aligned}
\phi(u(m,n))\leqslant a(m,n)&+\sum_{i=1}^{l_1}\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}[b_i(s,t,m,n)\varphi(u(s,t))\\
&+\sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}c_i(\xi,\eta,m,n)\varphi(u(\xi,\eta))]\\
&+\sum_{i=1}^{l_2}\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}[d_i(s,t,m,n)L_{1i}(s,t,\phi(u(s,t)))\\
&+\sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}e_i(\xi,\eta,m,n)L_{2i}(\xi,\eta,\phi(u(\xi,\eta)))],
\end{aligned} \tag{35}
$$

then

$$u(m,n) \leqslant \phi^{-1}\{\{a(m,n)+G^{-1}\{G(T^{-1}[\hat{H}_2(M-1,N-1) + \sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}\hat{B}(s,t,M,N)]) + \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\hat{B}(s,t,m,n)\}\}, \quad (36)$$

where G *is defined in Lemma 2 and provided that* T *is increasing,*

$$T(x) = G\left(\frac{x-1-\hat{\mu}_1}{\hat{\mu}_2}\right) - G(x); \quad x \geqslant 0 \quad (37)$$

$$\hat{B}(s,t,m,n) = \sum_{i=1}^{l_1}[\hat{b}_i(s,t,m,n) + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}\hat{c}_i(\xi,\eta,m,n)]$$

$$\hat{J}(m,n) = \sum_{i=1}^{l_2}\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}[\hat{d}_i(s,t,m,n)L_{1i}(s,t,a(s,t)) + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}\bar{e_i}(\xi,\eta,m,n)L_{2i}(\xi,\eta,a(\xi,\eta))] \quad (38)$$

$$\hat{b}_i(s,t,m,n) = b_i(s,t,m,n)$$

$$\hat{c}_i(s,t,m,n) = c_i(s,t,m,n); \quad 1 \leqslant i \leqslant l_1$$

$$\hat{d}_j(s,t,m,n) = d_j(s,t,m,n)T_{1j}(s,t,a(s,t))$$

$$\hat{e}_j(s,t,m,n) = e_j(s,t,m,n)T_{2j}(s,t,a(s,t)); \quad 1 \leqslant j \leqslant l_2$$

$$\hat{\mu}_1 = \hat{J}(M,N)$$

$$\hat{\mu}_2 = \sum_{j=1}^{l_2}\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}[\hat{d}_j(s,t,M+1,N+1) + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}\hat{e}_j(\xi,\eta,M+1,N+1)].$$

THEOREM 4. *Let* $w \in \Im_+(\Omega)$ *and* $L_{ji}, T_{ji};\ j=1,2;\ i=1,2,\ldots,l_2$ *are defined in Theorem 3 and all other conditions are same as in Theorem 1 such that*

$$\phi(u(m,n)) \leqslant a(m,n) + \sum_{s=m+1}^{\infty} w(s,n)\phi(u(m,n)) + \sum_{i=1}^{l_1}\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}[b_i(s,t,m,n)\varphi(u(s,t)) + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}c_i(\xi,\eta,m,n)\varphi(u(\xi,\eta))] + \sum_{i=1}^{l_2}\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}[d_i(s,t,m,n)L_{1i}(s,t,\phi(u(s,t))) + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}e_i(\xi,\eta,m,n)L_{2i}(\xi,\eta,\phi(u(\xi,\eta)))], \quad (39)$$

then

$$u(m,n) \leqslant \phi^{-1}\{\{a(m,n)+G^{-1}\{G(T^{-1}[\tilde{H}_2(M-1,N-1) + \sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}\tilde{B}(s,t,M,N)])+\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\tilde{B}(s,t,m,n)\}\}, \tag{40}$$

where G *is as defined in Lemma 2 and*

$$T(x) = G\left(\frac{x-1-\tilde{\mu_1}}{\tilde{\mu_2}}\right) - G(x);\ \ x \geqslant 0$$

$$\tilde{B}(s,t,m,n) = \sum_{i=1}^{l_1}[\tilde{b_i}(s,t,m,n)+\sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}\tilde{c_i}(\xi,\eta,m,n)]$$

$$\tilde{J}(m,n) = \sum_{i=1}^{l_2}\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}[d_i(s,t,m,n)L_{1i}(s,t,a(s,t)\tilde{w}(s,t)) + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}e_i(\xi,\eta,m,n)L_{2i}(\xi,\eta,a(\xi,\eta)\tilde{w}(\xi,\eta))]$$

$$\tilde{b_i}(s,t,m,n) = b_i(s,t,m,n)\varphi[\phi^{-1}(w(s,t))]$$

$$\tilde{c_i}(s,t,m,n) = c_i(s,t,m,n)\varphi[\phi^{-1}(w(s,t))];\ \ i=1,2,\ldots,l_1$$

$$\tilde{d_j}(s,t,m,n) = d_j(s,t,m,n)\tilde{w}(s,t)T_{1j}(s,t,a(s,t)\tilde{w}(s,t))$$

$$\tilde{e_j}(s,t,m,n) = e_j(s,t,m,n)\tilde{w}(s,t)T_{2j}(s,t,a(s,t)\tilde{w}(s,t));\ \ j=1,2,\ldots,l_2$$

$$\tilde{\mu_1} = \tilde{J}(M,N),$$

$$\tilde{\mu_2} = \sum_{j=1}^{l_2}\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}[\tilde{d_j}(s,t,M+1,N+1)+\sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}\tilde{e_j}(\xi,\eta,M+1,N+1)].$$

The proof for theorems 3 and 4 are similar to the previous theorems, we omit the details here.

2. Applications

EXAMPLE 1. Consider the following Volterra-Fredholm type infinite sum-difference equation:

$$u^q(m,n) = \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}[F_1(s,t,m,n,u(s,t))+\sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}F_2(\xi,\eta,m,n,u(\xi,\eta))] + \sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}[G_1(s,t,m,n,u(s,t))+\sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}G_2(\xi,\eta,m,n,u(\xi,\eta))], \tag{41}$$

where $u \in \Im_+(\Omega)$, $q \geqslant 1$ is an odd number, $F_i, G_i :\ \Omega^2 \times \mathbb{R} \to \mathbb{R}$, $i=1,2$.

THEOREM 5. *Suppose $u(m,n)$ is a solution of (41), and*

$$|F_1(s,t,m,n,u)| \leqslant f_1(s,t,m,n)|u|^{2q};$$
$$|F_2(s,t,m,n,u)| \leqslant f_2(s,t,m,n)|u|^{2q};$$
$$|G_1(s,t,m,n,u)| \leqslant g_1(s,t,m,n)|u|^{q};$$
$$|G_2(s,t,m,n,u)| \leqslant g_2(s,t,m,n)|u|^{q};$$

$f_i, g_i \in \Im_+(\Omega^2)$; $i = 1,2$; f_i, g_i are nondecreasing in the last two variables.

If there is at least one function among g_1, g_2 which is not equivalent to zero, then

$$u(m,n) \leqslant \left[\frac{z_0(\mu-1)}{z_0\{H_2(M-1,N-1)+D(M,N)+(\mu-1)D(m,n)\}-2(\mu-1)}\right]^{\frac{1}{q}}, \quad (42)$$

provided that denominator is not zero, $\mu < 1$ and

$$D(m,n) := \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} B_1(s,t,m,n), \quad (43)$$

where

$$B_1(s,t,m,n) = f_1(s,t,m,n) + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty} f_2(\xi,\eta,m,n),$$

$$\mu = \sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}\Big[g_1(s,t,M,N) + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty} g_2(\xi,\eta,M,N)\Big].$$

Proof. By using the properties of modulus and under the given conditions, from (41) we have

$$\begin{aligned}
|u(m,n)|^q \leqslant & \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\Big[|F_1(s,t,m,n,u(s,t))| + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}|F_2(\xi,\eta,m,n,u(\xi,\eta))\Big] \\
& + \sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\Big[|G_1(s,t,m,n,u(s,t))| + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}|G_2(\xi,\eta,m,n,u(\xi,\eta))\Big], \\
\leqslant & |a(m,n)| + \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\Big[f_1(s,t,m,n)|u(s,t)|^{2q} \\
& + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty} f_2(\xi,\eta,m,n)|u(\xi,\eta)|^{2q}\Big] + \sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}\Big[g_1(s,t,m,n)|u(s,t)|^{q} \\
& + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty} g_2(\xi,\eta,m,n)|u(\xi,\eta)|^{q}\Big]
\end{aligned} \quad (44)$$

Consider $\phi(u) = u^q$, $\varphi(u) = u^{2q}$, then

$$G(z) = \int_{z_0}^{z}\frac{1}{z^2}dz = \frac{1}{z} - \frac{1}{z_0}; \quad z \geqslant z_0 \geqslant 0, \quad (45)$$

$$T(x) = G\left(\frac{x-1}{\mu}\right) - G(x) = \frac{\mu}{x-1} - \frac{1}{x}; \quad x \geqslant 0. \tag{46}$$

Then by $\mu < 1$, it can be observed that T is strictly increasing. Then with $a(m,n) = 0$, $l_1 = l_2 = 1$ and a suitable application of Theorem 1 to (44) combined with (45)–(46) yields the desired result. □

THEOREM 6. *Suppose $u(m,n)$ is a solution of (41), and*

$$|F_1(s,t,m,n,u)| \leqslant f_1(s,t,m,n)|u|^q;$$
$$|F_2(s,t,m,n,u)| \leqslant f_2(s,t,m,n)|u|^q;$$
$$|G_1(s,t,m,n,u)| \leqslant g_1(s,t,m,n)|u|^q L_1(s,t,|u|^q);$$
$$|G_2(s,t,m,n,u)| \leqslant g_2(s,t,m,n)|u|^q L_2(s,t,|u|^q);$$

f_i, g_i, are defined as in Theorem 5, L_1, L_2, T_1, T_2 are defined as in Theorem 3 and $L_i(m,n,0) = 0$; $i = 1,2$, then

$$\begin{aligned} &u(m,n) \\ &\leqslant \Big[\frac{2}{\hat{\mu}^2 - 1}\Big\{-2 + 4\hat{\mu}\sqrt{z_0} - 4\hat{\mu}^2\sqrt{z_0} + \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\hat{B}_2(s,t,m,n) \\ &\quad \pm\sqrt{8\hat{\mu}\sqrt{z_0}(1+\hat{\mu}-2\hat{\mu}^2) + 16\hat{\mu}^2 z_0(1-\hat{\mu}+\hat{\mu}^2) + 16\hat{\mu}^2 H_3(M,N)(1-\hat{\mu}^2) + 4\hat{\mu}^2}\Big\}\Big] \end{aligned} \tag{47}$$

$$\hat{B}_2(s,t,m,n) = f_1(s,t,m,n) + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty} f_2(\xi,\eta,m,n),$$

$$\hat{\mu} = \sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}[g_1(s,t,M,N)T_1(s,t,0) + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty} g_2(\xi,\eta,M,N)T_2(\xi,\eta,0)],$$

provided that $\hat{\mu} < 1$.

Proof. By using the properties of modulus and under the given conditions, from (41) we have

$$\begin{aligned} |u(m,n)|^q \leqslant &\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}[|F_1(s,t,m,n,u(s,t))| + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}|F_2(\xi,\eta,m,n,u(\xi,\eta))] \\ &+ \sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}[|G_1(s,t,m,n,u(s,t))| + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}|G_2(\xi,\eta,m,n,u(\xi,\eta))], \\ \leqslant\ & |a(m,n)| + \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}[f_1(s,t,m,n)|u(s,t)|^{\frac{q}{2}} \\ &+ \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty} f_2(\xi,\eta,m,n)|u(\xi,\eta)|^{\frac{q}{2}}] + \sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}[g_1(s,t,m,n)|u(s,t)|^q \\ &+ \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty} g_2(\xi,\eta,m,n)|u(\xi,\eta)|^q]. \end{aligned} \tag{48}$$

Consider $\phi(u) = u^q$, $\varphi(u) = u^{\frac{q}{2}}$, then

$$G(z) = \int_{z_0}^{z} \frac{1}{\sqrt{z}} dz = 2\sqrt{z} - 2\sqrt{z_0}; \quad z \geqslant z_0 \geqslant 0, \tag{49}$$

$$T(x) = \left(\frac{x-1}{2\hat{\mu}}\right)^2 - \left(\frac{x}{2}\right)^2 + \sqrt{z_0}\left(\frac{x-1}{\hat{\mu}} - x\right); \quad x \geqslant 0. \tag{50}$$

For $\hat{\mu} < 1$, it can be observed that T is strictly increasing. Then with $a(m,n) = 0$, $l_1 = l_2 = 1$ and a suitable application of Theorem 3 to (48) combined with (49)–(50) yields the desired result. □

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