Journal of Mathematical Inequalities
Volume 11, Number 2 (2017), 511–525 doi:10.7153/jmi-11-42

PROPERTIES OF GENERALIZED SHARP HÖLDER'S INEQUALITIES

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(*Communicated by J. Pečarić*)

Abstract. Hölder's inequality and its various refinements are playing very important in mathematical analysis. In this paper, we give some new properties of generalized sharp Hölder's inequalities.

1. Introduction

The classical Hölder's inequality states that if $a_k \geqslant 0$, $b_k \geqslant 0$ $(k = 1,2,\cdots,n)$, $p > 0$, $q > 0$ and $\frac{1}{p}+\frac{1}{q} = 1$, then

$$\sum_{k=1}^{n} a_k b_k \leqslant \left(\sum_{k=1}^{n} a_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} b_k^q\right)^{\frac{1}{q}}. \tag{1}$$

The sign of inequality is reversed for $p < 0$ (for $p < 0$, we assume that a_k, $b_k > 0$).

As is well know, Hölder's inequality is one of the most important inequalities in mathematical analysis. Various generalizations, improvements, properties and applications of Hölder's inequality have been investigated by many authors. For example, Abramovich, Pečarić and Varošanec [1] gave an interesting refinement of Hölder's inequality. Matkowski [7] presented a new converse of Hölder's inequality. Tian [11] obtained some new improvements of Hölder's inequality by using a property of Hölder's inequality. For detailed expositions, the interested reader may consult [2], [6], [8] and [12] the references therein.

In 1981, Hu [4] established a new sharp version of Hölder's inequality as follows.

THEOREM A. *Let $a_k \geqslant 0$, $b_k \geqslant 0$ $(k=1,2,\cdots,n)$, $1-e_i+e_j \geqslant 0$ $(i,j=1,2,\cdots,n)$, and let $p \geqslant q > 0$ such that $\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\sum_{k=1}^{n} a_k b_k \leqslant \left(\sum_{k=1}^{n} b_k^q\right)^{\frac{1}{q}-\frac{1}{p}} \Bigg\{\left[\left(\sum_{k=1}^{n} b_k^q\right)\left(\sum_{k=1}^{n} a_k^p\right)\right]^2 - \left[\left(\sum_{k=1}^{n} b_k^q e_k\right)\left(\sum_{k=1}^{n} a_k^p\right) - \left(\sum_{k=1}^{n} b_k^q\right)\left(\sum_{k=1}^{n} a_k^p e_k\right)\right]^2\Bigg\}^{\frac{1}{2p}}. \tag{2}$$

The integral form is as follows:

Mathematics subject classification (2010): 26D15, 26D10.

Keywords and phrases: Hölder's inequality, generalized Hölder's inequality, property.

This work was supported by the NNSF of China (No. 61073121), the Application Basic Research Plan Key Basic Research Project of Hebei Province of China (No. 16964213D), the Fundamental Research Funds for the Central Universities (No. 2015ZD29, 13ZD19) and the Higher School Science Research Funds of Hebei Province of China (No. Z2015137).

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THEOREM B. *Let $f(x)$, $g(x)$, $e(x)$ be integrable functions defined on $[a,b]$ and $f(x) \geqslant 0$, $g(x) \geqslant 0$, $1-e(x)+e(y) \geqslant 0$ for all $x,y \in [a,b]$, and let $p \geqslant q > 0$ such that $\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\int_a^b f(x)g(x)dx \leqslant \left(\int_a^b g^q(x)dx\right)^{\frac{1}{q}-\frac{1}{p}} \left[\left(\int_a^b g^q(x)dx \int_a^b f^p(x)dx\right)^2 - \left(\int_a^b g^q(x)e(x)dx \int_a^b f^p(x)dx - \int_a^b g^q(x)dx \int_a^b f^p(x)e(x)dx\right)^2\right]^{\frac{1}{2p}}. \quad (3)$$

In 2011, Tian [10] presented the reversed versions of inequalities (2) and (3) as follows.

THEOREM C. *Let $a_k \geqslant 0$, $b_k > 0$ $(k=1,2,\cdots,n)$, $1-e_i+e_j \geqslant 0$ $(i,j=1,2,\cdots,n)$, and let $p>0$, $q<0$ such that $\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\sum_{k=1}^n a_k b_k \geqslant \left(\sum_{k=1}^n a_k^p\right)^{\frac{1}{p}-\frac{1}{q}} \left\{\left[\left(\sum_{k=1}^n a_k^p\right)\left(\sum_{k=1}^n b_k^q\right)\right]^2 - \left[\left(\sum_{k=1}^n a_k^p e_k\right)\left(\sum_{k=1}^n b_k^q\right) - \left(\sum_{k=1}^n a_k^p\right)\left(\sum_{k=1}^n b_k^q e_k\right)\right]^2\right\}^{\frac{1}{2q}}. \quad (4)$$

THEOREM D. *Let $f(x)$, $g(x)$, $e(x)$ be integrable functions defined on $[a,b]$ and $f(x) \geqslant 0$, $g(x) > 0$, $1-e(x)+e(y) \geqslant 0$ for all $x,y \in [a,b]$, and let $p>0$, $q<0$ such that $\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\int_a^b f(x)g(x)dx \geqslant \left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}-\frac{1}{q}} \left[\left(\int_a^b f^p(x)dx \int_a^b g^q(x)dx\right)^2 - \left(\int_a^b f^p(x)e(x)dx \int_a^b g^q(x)dx - \int_a^b f^p(x)dx \int_a^b g^q(x)e(x)dx\right)^2\right]^{\frac{1}{2q}}. \quad (5)$$

In 2007, Wu [13] established the following generalizations of (2) and (3).

THEOREM E. *Let $a_k \geqslant 0$, $b_k > 0$ $(k=1,2,\cdots,n)$, $1-e_i+e_j \geqslant 0$ $(i,j=1,2,\cdots,n)$, and let $p \geqslant q > 0$, $\rho = \min\{\frac{1}{p}+\frac{1}{q},1\}$. Then*

$$\sum_{k=1}^n a_k b_k \leqslant n^{1-\rho}\left(\sum_{k=1}^n b_k^q\right)^{\frac{1}{q}-\frac{1}{p}} \left\{\left[\left(\sum_{k=1}^n b_k^q\right)\left(\sum_{k=1}^n a_k^p\right)\right]^2 - \left[\left(\sum_{k=1}^n b_k^q e_k\right)\left(\sum_{k=1}^n a_k^p\right) - \left(\sum_{k=1}^n b_k^q\right)\left(\sum_{k=1}^n a_k^p e_k\right)\right]^2\right\}^{\frac{1}{2p}}. \quad (6)$$

The integral form is as follows:

THEOREM F. *Let* $f(x)$, $g(x)$, $e(x)$ *be integrable functions defined on* $[a,b]$ *and* $f(x) \geqslant 0$, $g(x) > 0$, $1-e(x)+e(y) \geqslant 0$ *for all* $x,y \in [a,b]$, *and let* $p \geqslant q > 0$, $\frac{1}{p}+\frac{1}{q} \leqslant 1$. *Then*

$$\begin{aligned}
&\int_a^b f(x)g(x)dx \\
&\leqslant (b-a)^{1-\frac{1}{p}-\frac{1}{q}} \left(\int_a^b g^q(x)dx\right)^{\frac{1}{q}-\frac{1}{p}} \left[\left(\int_a^b g^q(x)dx \int_a^b f^p(x)dx\right)^2 \right. \\
&\quad \left. -\left(\int_a^b g^q(x)e(x)dx \int_a^b f^p(x)dx - \int_a^b g^q(x)dx \int_a^b f^p(x)e(x)dx\right)^2\right]^{\frac{1}{2p}}. \qquad (7)
\end{aligned}$$

Later, Tian and Hu [9] presented the following reversed versions of inequalities (6) and (7).

THEOREM G. *Let* $a_k \geqslant 0$, $b_k > 0$ $(k=1,2,\cdots,n)$, $1-e_i+e_j \geqslant 0$ $(i,j=1,2,\cdots,n)$, *and let* $p > 0$, $q < 0$, $\vartheta = \max\{\frac{1}{p}+\frac{1}{q},1\}$. *Then*

$$\begin{aligned}
\sum_{k=1}^n a_k b_k \geqslant n^{1-\vartheta} \left(\sum_{k=1}^n a_k^p\right)^{\frac{1}{p}-\frac{1}{q}} &\left\{\left[\left(\sum_{k=1}^n a_k^p\right)\left(\sum_{k=1}^n b_k^q\right)\right]^2 \right. \\
&\left. -\left[\left(\sum_{k=1}^n a_k^p e_k\right)\left(\sum_{k=1}^n b_k^q\right) - \left(\sum_{k=1}^n a_k^p\right)\left(\sum_{k=1}^n b_k^q e_k\right)\right]^2\right\}^{\frac{1}{2q}}. \qquad (8)
\end{aligned}$$

THEOREM H. *Let* $f(x)$, $g(x)$, $e(x)$ *be integrable functions defined on* $[a,b]$ *and* $f(x) \geqslant 0$, $g(x) > 0$, $1-e(x)+e(y) \geqslant 0$ *for all* $x,y \in [a,b]$, *and let* $q < 0$, $\frac{1}{p}+\frac{1}{q} \geqslant 1$. *Then*

$$\begin{aligned}
&\int_a^b f(x)g(x)dx \\
&\geqslant (b-a)^{1-\frac{1}{p}-\frac{1}{q}} \left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}-\frac{1}{q}} \left[\left(\int_a^b f^p(x)dx \int_a^b g^q(x)dx\right)^2 \right. \\
&\quad \left. -\left(\int_a^b f^p(x)e(x)dx \int_a^b g^q(x)dx - \int_a^b f^p(x)dx \int_a^b g^q(x)e(x)dx\right)^2\right]^{\frac{1}{2q}}. \qquad (9)
\end{aligned}$$

In this paper, the above inequalities (6), (7), (8) and (9) are called as generalized sharp Hölder's inequalities.

In 1994, Hu [5] gave the following properties of inequalities (2) and (3)

THEOREM I. *Let* $a_k \geqslant 0,\ b_k \geqslant 0\ (k=1,2,\cdots),\ 1-e_i+e_j \geqslant 0\ (i,j=1,2,\cdots)$, *let* $p \geqslant q > 0$ *such that* $\frac{1}{p}+\frac{1}{q}=1$, *and let*

$$F_s(n) = \left(\sum_{k=1}^{n} b_k^q\right)^{\frac{2s}{q}-\frac{2s}{p}} \left\{\left[\left(\sum_{k=1}^{n} b_k^q\right)\left(\sum_{k=1}^{n} a_k^p\right)\right]^2 - \left[\left(\sum_{k=1}^{n} b_k^q e_k\right)\left(\sum_{k=1}^{n} a_k^p\right) - \left(\sum_{k=1}^{n} b_k^q\right)\left(\sum_{k=1}^{n} a_k^p e_k\right)\right]^2\right\}^{\frac{s}{p}} - \left(\sum_{k=1}^{n} a_k b_k\right)^{2s}, \quad s=1,2,\cdots. \tag{10}$$

Then

$$F_s(n) \leqslant F_s(n+1). \tag{11}$$

THEOREM J. *Let* $f(x),\ g(x),\ e(x)$ *be integrable functions defined on* $[a,b]$ *and* $f(x) \geqslant 0,\ g(x) \geqslant 0,\ 1-e(x)+e(y) \geqslant 0$ *for all* $x,y \in [a,b]$, *let* $p \geqslant q > 0$ *such that* $\frac{1}{p}+\frac{1}{q}=1$, *and let*

$$G_s(t) = \left(\int_a^t g^q(x)dx\right)^{\frac{2s}{q}-\frac{2s}{p}} \left[\left(\int_a^t g^q(x)dx \int_a^t f^p(x)dx\right)^2 - \left(\int_a^t g^q(x)e(x)dx \int_a^t f^p(x)dx - \int_a^t g^q(x)dx \int_a^t f^p(x)e(x)dx\right)^2\right]^{\frac{s}{p}} - \left(\int_a^t f(x)g(x)dx\right)^{2s}, \quad s=1,2,\cdots. \tag{12}$$

Then we have

$$G_s(t_1) \leqslant G_s(t_2), \quad a \leqslant t_1 \leqslant t_2 \leqslant b. \tag{13}$$

In 2013, Tian [11] gave the similar properties of inequalities (4) and (5).

THEOREM K. *Let* $a_k \geqslant 0,\ b_k > 0\ (k=1,2,\cdots),\ 1-e_i+e_j \geqslant 0\ (i,j=1,2,\cdots)$, *let* $p>0,\ q<0$ *such that* $\frac{1}{p}+\frac{1}{q}=1$, *and let*

$$F(n) = \left(\sum_{k=1}^{n} a_k^p\right)^{\frac{2}{p}-\frac{2}{q}} \left\{\left[\left(\sum_{k=1}^{n} a_k^p\right)\left(\sum_{k=1}^{n} b_k^q\right)\right]^2 - \left[\left(\sum_{k=1}^{n} a_k^p e_k\right)\left(\sum_{k=1}^{n} b_k^q\right) - \left(\sum_{k=1}^{n} a_k^p\right)\left(\sum_{k=1}^{n} b_k^q e_k\right)\right]^2\right\}^{\frac{1}{q}} - \left(\sum_{k=1}^{n} a_k b_k\right)^2. \tag{14}$$

Then

$$F(n) \geqslant F(n+1). \tag{15}$$

THEOREM L. *Let* $f(x)$, $g(x)$, $e(x)$ *be integrable functions defined on* $[a,b]$ *and* $f(x) \geqslant 0$, $g(x) > 0$, $1-e(x)+e(y) \geqslant 0$ *for all* $x,y \in [a,b]$, *let* $p > 0$, $q < 0$ *such that* $\frac{1}{p}+\frac{1}{q}=1$, *and let*

$$G(t) = \left(\int_a^t f^p(x)dx\right)^{\frac{2}{p}-\frac{2}{q}} \left[\left(\int_a^t f^p(x)dx \int_a^t g^q(x)dx\right)^2 - \left(\int_a^t f^p(x)e(x)dx \int_a^t g^q(x)dx - \int_a^t f^p(x)dx \int_a^t g^q(x)e(x)dx\right)^2\right]^{\frac{1}{q}} - \left(\int_a^t f(x)g(x)dx\right)^2. \tag{16}$$

Then we have

$$G(t_1) \geqslant G(t_2), \quad a \leqslant t_1 \leqslant t_2 \leqslant b. \tag{17}$$

Stimulated by the works of Hu [5] and Tian [11], in this paper, some similar properties of generalized sharp Hölder's inequalities (6), (7), (8) and (9) are given.

2. Main results

We begin this section with a lemma, which will be used in the sequel.

LEMMA 2.1. [3] (Power means inequality) *If* $x_i \geqslant 0$, $\lambda_i > 0$, $i=1,2,\cdots,n$, $0 < p \leqslant 1$, *then*

$$\sum_{i=1}^n \lambda_i x_i^p \leqslant \left(\sum_{i=1}^n \lambda_i\right)^{1-p} \left(\sum_{i=1}^n \lambda_i x_i\right)^p. \tag{18}$$

The sign of inequality is reversed for $p \geqslant 1$ *or* $p < 0$.

Next, we give some new properties of generalized sharp Hölder's inequalities (6) and (7).

THEOREM 2.2. *Let* $a_k \geqslant 0$, $b_k > 0$ $(k=1,2,\cdots)$, $1-e_i+e_j \geqslant 0$ $(i,j=1,2,\cdots)$, *let* $p \geqslant q > 0$, $\rho = \min\{\frac{1}{p}+\frac{1}{q},1\}$, *and let*

$$F_s(n) = n^{2s(1-\rho)}\left(\sum_{k=1}^n b_k^q\right)^{\frac{2s}{q}-\frac{2s}{p}} \left\{\left[\left(\sum_{k=1}^n b_k^q\right)\left(\sum_{k=1}^n a_k^p\right)\right]^2 - \left[\left(\sum_{k=1}^n b_k^q e_k\right)\left(\sum_{k=1}^n a_k^p\right) - \left(\sum_{k=1}^n b_k^q\right)\left(\sum_{k=1}^n a_k^p e_k\right)\right]^2\right\}^{\frac{s}{p}} - \left(\sum_{k=1}^n a_k b_k\right)^{2s}, \quad s=1,2,\cdots. \tag{19}$$

Then

$$F_s(n) \leqslant F_s(n+1). \tag{20}$$

The integral form is as follows:

THEOREM 2.3. *Let $f(x)$, $g(x)$, $e(x)$ be integrable functions defined on $[a,b]$ and $f(x) \geqslant 0$, $g(x) \geqslant 0$, $1-e(x)+e(y) \geqslant 0$ for all $x,y \in [a,b]$, let $p \geqslant q > 0$, $\frac{1}{p}+\frac{1}{q} \leqslant 1$, and let*

$$\begin{aligned} G_s(t) = (t-a)^{2s-\frac{2s}{p}-\frac{2s}{q}} \left(\int_a^t g^q(x)dx \right)^{\frac{2s}{q}-\frac{2s}{p}} \Bigg[\left(\int_a^t g^q(x)dx \int_a^t f^p(x)dx \right)^2 \\ - \left(\int_a^t g^q(x)e(x)dx \int_a^t f^p(x)dx - \int_a^t g^q(x)dx \int_a^t f^p(x)e(x)dx \right)^2 \Bigg]^{\frac{s}{p}} \\ - \left(\int_a^t f(x)g(x)dx \right)^{2s}, \quad s = 1,2,\cdots. \end{aligned} \tag{21}$$

Then we have

$$G_s(t_1) \leqslant G_s(t_2), \quad a \leqslant t_1 \leqslant t_2 \leqslant b. \tag{22}$$

Proof of Theorem 2.2. We first consider the case $s=1$. Denote

$$\begin{aligned} x_n = n^{2(1-\rho)} \left(\sum_{k=1}^n b_k^q \right)^{\frac{2}{q}-\frac{2}{p}} \Bigg\{ \left[\left(\sum_{k=1}^n b_k^q \right) \left(\sum_{k=1}^n a_k^p \right) \right]^2 \\ - \left[\left(\sum_{k=1}^n b_k^q e_k \right) \left(\sum_{k=1}^n a_k^p \right) - \left(\sum_{k=1}^n b_k^q \right) \left(\sum_{k=1}^n a_k^p e_k \right) \right]^2 \Bigg\}^{\frac{1}{p}} \end{aligned}$$

We first consider the case $s=1$. Let $\frac{1}{p}+\frac{1}{q}=t$ $(0<t\leqslant 1)$, which implies $\frac{1}{pt}+\frac{1}{qt}=1$. By Theorem E since $\rho = t$ we have

$$\begin{aligned} y_n = \left(\sum_{k=1}^n a_k b_k \right)^2 \leqslant n^{2(1-t)} \left(\sum_{k=1}^n b_k^q \right)^{\frac{2}{q}-\frac{2}{p}} \Bigg\{ \left[\left(\sum_{k=1}^n b_k^q \right) \left(\sum_{k=1}^n a_k^p \right) \right]^2 \\ - \left[\left(\sum_{k=1}^n b_k^q e_k \right) \left(\sum_{k=1}^n a_k^p \right) - \left(\sum_{k=1}^n b_k^q \right) \left(\sum_{k=1}^n a_k^p e_k \right) \right]^2 \Bigg\}^{\frac{1}{p}} = x_n, \end{aligned} \tag{23}$$

that is $y_n \leqslant x_n$. A simple calculation gives

$$\begin{aligned} & \sum_{k=1}^n a_k b_k \sum_{r=1}^n a_r b_r (1-e_k+e_r) \\ & = \sum_{k=1}^n \sum_{r=1}^n a_k b_k a_r b_r - \sum_{k=1}^n \sum_{r=1}^n a_r b_r a_k b_k e_k + \sum_{k=1}^n \sum_{r=1}^n a_k b_k a_r b_r e_r \\ & = \left(\sum_{k=1}^n a_k b_k \right)^2, \end{aligned} \tag{24}$$

and so

$$\begin{aligned} x_n &= n^{2(1-t)}\left[\sum_{k=1}^{n} b_k^q \sum_{r=1}^{n} b_r^q (1-e_k+e_r)\right]^{\frac{1}{q}-\frac{1}{p}} \\ &\quad\times\left[\sum_{k=1}^{n} a_k^p \sum_{r=1}^{n} b_r^q - \sum_{k=1}^{n} a_k^p e_k \sum_{r=1}^{n} b_r^q + \sum_{k=1}^{n} a_k^p \sum_{r=1}^{n} b_r^q e_r\right]^{\frac{1}{p}} \\ &\quad\times\left[\sum_{k=1}^{n} b_k^q \sum_{r=1}^{n} a_r^p - \sum_{k=1}^{n} b_k^q e_k \sum_{r=1}^{n} a_r^p + \sum_{k=1}^{n} b_k^q \sum_{r=1}^{n} a_r^p e_r\right]^{\frac{1}{p}} \\ &= n^{2(1-t)}\left[\sum_{k=1}^{n} b_k^q \sum_{r=1}^{n} b_r^q (1-e_k+e_r)\right]^{\frac{1}{q}-\frac{1}{p}} \\ &\quad\times\left[\sum_{k=1}^{n}\sum_{r=1}^{n} a_k^p b_r^q (1-e_k+e_r)\right]^{\frac{1}{p}}\left[\sum_{k=1}^{n}\sum_{r=1}^{n} b_k^q a_r^p (1-e_k+e_r)\right]^{\frac{1}{p}}. \end{aligned} \tag{25}$$

Additionally, preforming some simple computations, we have

$$\begin{aligned} y_{n+1}-y_n+x_n &= a_{n+1}b_{n+1}\sum_{k=1}^{n+1} a_k b_k (1-e_{n+1}+e_k) \\ &\quad+\sum_{r=1}^{n} a_r b_r a_{n+1} b_{n+1} (1-e_r+e_{n+1}) + x_n. \end{aligned} \tag{26}$$

By Hölder's inequality we have

$$\begin{aligned} &a_{n+1}b_{n+1}\sum_{k=1}^{n+1} a_k b_k (1-e_{n+1}+e_k) + \sum_{r=1}^{n} a_r b_r a_{n+1} b_{n+1} (1-e_r+e_{n+1}) \\ &\leqslant a_{n+1}b_{n+1}\left[\sum_{k=1}^{n+1} a_k^{pt}(1-e_{n+1}+e_k)\right]^{\frac{1}{pt}}\left[\sum_{k=1}^{n+1} b_k^{qt}(1-e_{n+1}+e_k)\right]^{\frac{1}{qt}} \\ &\quad+\sum_{r=1}^{n} a_r b_r \left[a_{n+1}^{pt}(1-e_r+e_{n+1})\right]^{\frac{1}{pt}}\left[b_{n+1}^{qt}(1-e_r+e_{n+1})\right]^{\frac{1}{qt}}, \end{aligned} \tag{27}$$

and so

$$\begin{aligned} &y_{n+1}-y_n+x_n \\ &\leqslant a_{n+1}b_{n+1}\left[\sum_{k=1}^{n+1} a_k^{pt}(1-e_{n+1}+e_k)\right]^{\frac{1}{pt}}\left[\sum_{k=1}^{n+1} b_k^{qt}(1-e_{n+1}+e_k)\right]^{\frac{1}{qt}} \\ &\quad+\sum_{r=1}^{n} a_r b_r \left[a_{n+1}^{pt}(1-e_r+e_{n+1})\right]^{\frac{1}{pt}}\left[b_{n+1}^{qt}(1-e_r+e_{n+1})\right]^{\frac{1}{qt}} + x_n \\ &= b_{n+1}^{1-\frac{q}{p}}\left[\sum_{k=1}^{n+1} b_k^{qt}(1-e_{n+1}+e_k)\right]^{\frac{1}{qt}-\frac{1}{pt}} \times a_{n+1}\left[\sum_{k=1}^{n+1} b_k^{qt}(1-e_{n+1}+e_k)\right]^{\frac{1}{pt}} \\ &\quad\times b_{n+1}^{\frac{q}{p}}\left[\sum_{k=1}^{n+1} a_k^{pt}(1-e_{n+1}+e_k)\right]^{\frac{1}{pt}} \end{aligned}$$

$$+\sum_{r=1}^{n}\left\{b_r^{1-\frac{q}{p}}\left[b_{n+1}^{qt}(1-e_r+e_{n+1})\right]^{\frac{1}{qt}-\frac{1}{pt}}\times a_r\left[b_{n+1}^{qt}(1-e_r+e_{n+1})\right]^{\frac{1}{pt}}\right.$$
$$\left.\times b_r^{\frac{q}{p}}\left[a_{n+1}^{pt}(1-e_r+e_{n+1})\right]^{\frac{1}{pt}}\right\}+x_n$$
$$=b_{n+1}^{1-\frac{q}{p}}\left[\sum_{k=1}^{n+1}b_k^{qt}(1-e_{n+1}+e_k)\right]^{\frac{1}{qt}-\frac{1}{pt}}\times a_{n+1}\left[\sum_{k=1}^{n+1}b_k^{qt}(1-e_{n+1}+e_k)\right]^{\frac{1}{pt}}$$
$$\times b_{n+1}^{\frac{q}{p}}\left[\sum_{k=1}^{n+1}a_k^{pt}(1-e_{n+1}+e_k)\right]^{\frac{1}{pt}}$$
$$+\sum_{r=1}^{n}\left\{\left[b_r^{qt}b_{n+1}^{qt}(1-e_r+e_{n+1})\right]^{\frac{1}{qt}-\frac{1}{pt}}\times\left[a_r^{pt}b_{n+1}^{qt}(1-e_r+e_{n+1})\right]^{\frac{1}{pt}}\right.$$
$$\left.\times\left[b_r^{qt}a_{n+1}^{pt}(1-e_r+e_{n+1})\right]^{\frac{1}{pt}}\right\}+x_n. \tag{28}$$

By using Hölder's inequality and Power means inequality (18), we obtain

$$b_{n+1}^{1-\frac{q}{p}}\left[\sum_{k=1}^{n+1}b_k^{qt}(1-e_{n+1}+e_k)\right]^{\frac{1}{qt}-\frac{1}{pt}}\times a_{n+1}\left[\sum_{k=1}^{n+1}b_k^{qt}(1-e_{n+1}+e_k)\right]^{\frac{1}{pt}}$$
$$\times b_{n+1}^{\frac{q}{p}}\left[\sum_{k=1}^{n+1}a_k^{pt}(1-e_{n+1}+e_k)\right]^{\frac{1}{pt}}$$
$$+\sum_{r=1}^{n}\left\{\left[b_r^{qt}b_{n+1}^{qt}(1-e_r+e_{n+1})\right]^{\frac{1}{qt}-\frac{1}{pt}}\times\left[a_r^{pt}b_{n+1}^{qt}(1-e_r+e_{n+1})\right]^{\frac{1}{pt}}\right.$$
$$\left.\times\left[b_r^{qt}a_{n+1}^{pt}(1-e_r+e_{n+1})\right]^{\frac{1}{pt}}\right\}$$
$$\leqslant b_{n+1}^{1-\frac{q}{p}}\left[\sum_{k=1}^{n+1}b_k^{qt}(1-e_{n+1}+e_k)\right]^{\frac{1}{qt}-\frac{1}{pt}}\times a_{n+1}\left[\sum_{k=1}^{n+1}b_k^{qt}(1-e_{n+1}+e_k)\right]^{\frac{1}{pt}}$$
$$\times b_{n+1}^{\frac{q}{p}}\left[\sum_{k=1}^{n+1}a_k^{pt}(1-e_{n+1}+e_k)\right]^{\frac{1}{pt}}$$
$$+\left[\sum_{r=1}^{n}b_r^{qt}b_{n+1}^{qt}(1-e_r+e_{n+1})\right]^{\frac{1}{qt}-\frac{1}{pt}}\times\left[\sum_{r=1}^{n}a_r^{pt}b_{n+1}^{qt}(1-e_r+e_{n+1})\right]^{\frac{1}{pt}}$$
$$\times\left[\sum_{r=1}^{n}b_r^{qt}a_{n+1}^{pt}(1-e_r+e_{n+1})\right]^{\frac{1}{pt}}$$
$$\leqslant b_{n+1}^{1-\frac{q}{p}}\left[\sum_{k=1}^{n+1}(1-e_{n+1}+e_k)\right]^{(1-t)(\frac{1}{qt}-\frac{1}{pt})}\times\left[\sum_{k=1}^{n+1}b_k^{q}(1-e_{n+1}+e_k)\right]^{\frac{1}{q}-\frac{1}{p}}$$
$$\times a_{n+1}\left[\sum_{k=1}^{n+1}(1-e_{n+1}+e_k)\right]^{(1-t)\frac{1}{pt}}\times\left[\sum_{k=1}^{n+1}b_k^{q}(1-e_{n+1}+e_k)\right]^{\frac{1}{p}}$$

$$
\begin{aligned}
&\times b_{n+1}^{\frac{q}{p}}\left[\sum_{k=1}^{n+1}(1-e_{n+1}+e_k)\right]^{(1-t)\frac{1}{pt}} \times \left[\sum_{k=1}^{n+1} a_k^p(1-e_{n+1}+e_k)\right]^{\frac{1}{p}} \\
&+\left[\sum_{r=1}^{n}(1-e_r+e_{n+1})\right]^{(1-t)(\frac{1}{qt}-\frac{1}{pt})}\left[\sum_{r=1}^{n} b_r^q b_{n+1}^q(1-e_r+e_{n+1})\right]^{\frac{1}{q}-\frac{1}{p}} \\
&\times\left[\sum_{r=1}^{n}(1-e_r+e_{n+1})\right]^{(1-t)\frac{1}{pt}}\left[\sum_{r=1}^{n} a_r^p b_{n+1}^q(1-e_r+e_{n+1})\right]^{\frac{1}{p}} \\
&\times\left[\sum_{r=1}^{n}(1-e_r+e_{n+1})\right]^{(1-t)\frac{1}{pt}}\left[\sum_{r=1}^{n} b_r^q a_{n+1}^p(1-e_r+e_{n+1})\right]^{\frac{1}{p}} \\
=&\left[\sum_{k=1}^{n+1}(1-e_{n+1}+e_k)\right]^{(1-t)} \times b_{n+1}^{1-\frac{q}{p}}\left[\sum_{k=1}^{n+1} b_k^q(1-e_{n+1}+e_k)\right]^{\frac{1}{q}-\frac{1}{p}} \\
&\times a_{n+1}\left[\sum_{k=1}^{n+1} b_k^q(1-e_{n+1}+e_k)\right]^{\frac{1}{p}} \times b_{n+1}^{\frac{q}{p}}\left[\sum_{k=1}^{n+1} a_k^p(1-e_{n+1}+e_k)\right]^{\frac{1}{p}} \\
&+\left[\sum_{r=1}^{n}(1-e_r+e_{n+1})\right]^{(1-t)}\left[\sum_{r=1}^{n} b_r^q b_{n+1}^q(1-e_r+e_{n+1})\right]^{\frac{1}{q}-\frac{1}{p}} \\
&\times\left[\sum_{r=1}^{n} a_r^p b_{n+1}^q(1-e_r+e_{n+1})\right]^{\frac{1}{p}}\left[\sum_{r=1}^{n} b_r^q a_{n+1}^p(1-e_r+e_{n+1})\right]^{\frac{1}{p}} \\
\leqslant&\left[\sum_{k=1}^{n+1}\sum_{r=1}^{n+1}(1-e_r+e_k)\right]^{(1-t)} \times\Bigg\{b_{n+1}^{1-\frac{q}{p}}\left[\sum_{k=1}^{n+1} b_k^q(1-e_{n+1}+e_k)\right]^{\frac{1}{q}-\frac{1}{p}} \\
&\times a_{n+1}\left[\sum_{k=1}^{n+1} b_k^q(1-e_{n+1}+e_k)\right]^{\frac{1}{p}} \times b_{n+1}^{\frac{q}{p}}\left[\sum_{k=1}^{n+1} a_k^p(1-e_{n+1}+e_k)\right]^{\frac{1}{p}} \\
&+\left[\sum_{r=1}^{n} b_r^q b_{n+1}^q(1-e_r+e_{n+1})\right]^{\frac{1}{q}-\frac{1}{p}}\left[\sum_{r=1}^{n} a_r^p b_{n+1}^q(1-e_r+e_{n+1})\right]^{\frac{1}{p}} \\
&\times\left[\sum_{r=1}^{n} b_r^q a_{n+1}^p(1-e_r+e_{n+1})\right]^{\frac{1}{p}}\Bigg\} \\
=&\,(n+1)^{2(1-t)}\Bigg\{b_{n+1}^{1-\frac{q}{p}}\left[\sum_{k=1}^{n+1} b_k^q(1-e_{n+1}+e_k)\right]^{\frac{1}{q}-\frac{1}{p}} \\
&\times a_{n+1}\left[\sum_{k=1}^{n+1} b_k^q(1-e_{n+1}+e_k)\right]^{\frac{1}{p}} \times b_{n+1}^{\frac{q}{p}}\left[\sum_{k=1}^{n+1} a_k^p(1-e_{n+1}+e_k)\right]^{\frac{1}{p}} \\
&+\left[\sum_{r=1}^{n} b_r^q b_{n+1}^q(1-e_r+e_{n+1})\right]^{\frac{1}{q}-\frac{1}{p}}\left[\sum_{r=1}^{n} a_r^p b_{n+1}^q(1-e_r+e_{n+1})\right]^{\frac{1}{p}} \\
&\times\left[\sum_{r=1}^{n} b_r^q a_{n+1}^p(1-e_r+e_{n+1})\right]^{\frac{1}{p}}\Bigg\}. \qquad (29)
\end{aligned}
$$

Hence, from (29), (28) and (25), we have

$$
\begin{aligned}
& y_{n+1}-y_n+x_n \\
& \leqslant (n+1)^{2(1-t)}\Bigg\{\Bigg[b_{n+1}^q\Bigg(\sum_{k=1}^{n+1} b_k^q(1-e_{n+1}+e_k)\Bigg)\Bigg]^{\frac{1}{q}-\frac{1}{p}} \\
& \quad\times\Bigg[a_{n+1}^p\Bigg(\sum_{k=1}^{n+1} b_k^q(1-e_{n+1}+e_k)\Bigg)\Bigg]^{\frac{1}{p}}\Bigg[b_{n+1}^q\Bigg(\sum_{k=1}^{n+1} a_k^p(1-e_{n+1}+e_k)\Bigg)\Bigg]^{\frac{1}{p}} \\
& \quad+\Bigg[\sum_{k=1}^{n} b_k^q b_{n+1}^q(1-e_k+e_{n+1})\Bigg]^{\frac{1}{q}-\frac{1}{p}}\Bigg[\sum_{k=1}^{n} a_k^p b_{n+1}^q(1-e_k+e_{n+1})\Bigg]^{\frac{1}{p}} \\
& \quad\times\Bigg[\sum_{k=1}^{n} b_k^q a_{n+1}^p(1-e_k+e_{n+1})\Bigg]^{\frac{1}{p}}+\Bigg[\sum_{k=1}^{n} b_k^q\sum_{r=1}^{n} b_r^q(1-e_k+e_r)\Bigg]^{\frac{1}{q}-\frac{1}{p}} \\
& \quad\times\Bigg[\sum_{k=1}^{n}\sum_{r=1}^{n} a_k^p b_r^q(1-e_k+e_r)\Bigg]^{\frac{1}{p}}\Bigg[\sum_{k=1}^{n}\sum_{r=1}^{n} b_k^q a_r^p(1-e_k+e_r)\Bigg]^{\frac{1}{p}}\Bigg\} \\
& \leqslant (n+1)^{2(1-t)}\Bigg\{\Bigg[b_{n+1}^q\Bigg(\sum_{k=1}^{n+1} b_k^q(1-e_{n+1}+e_k)\Bigg)\Bigg] \\
& \quad+\sum_{k=1}^{n}\Bigg[b_k^q b_{n+1}^q(1-e_k+e_{n+1})+b_k^q\sum_{r=1}^{n} b_r^q(1-e_k+e_r)\Bigg]\Bigg\}^{\frac{1}{q}-\frac{1}{p}} \\
& \quad\times\Bigg\{a_{n+1}^p\Bigg(\sum_{k=1}^{n+1} b_k^q(1-e_{n+1}+e_k)\Bigg) \\
& \quad+\sum_{k=1}^{n}\Bigg[a_k^p b_{n+1}^q(1-e_k+e_{n+1})+a_k^p\sum_{r=1}^{n} b_r^q(1-e_k+e_r)\Bigg]\Bigg\}^{\frac{1}{p}} \\
& \quad\times\Bigg\{b_{n+1}^q\Bigg(\sum_{k=1}^{n+1} a_k^p(1-e_{n+1}+e_k)\Bigg) \\
& \quad+\sum_{k=1}^{n}\Bigg[b_k^q a_{n+1}^p(1-e_k+e_{n+1})+b_k^q\sum_{r=1}^{n} a_r^p(1-e_k+e_r)\Bigg]\Bigg\}^{\frac{1}{p}} \\
& = (n+1)^{2(1-t)}\Bigg\{b_{n+1}^q\Bigg[\sum_{k=1}^{n+1} b_k^q(1-e_{n+1}+e_k) \\
& \quad+\sum_{k=1}^{n} b_k^q(1-e_k+e_{n+1})\Bigg]+\Bigg(\sum_{k=1}^{n} b_k^q\Bigg)^2\Bigg\}^{\frac{1}{q}-\frac{1}{p}} \\
& \quad\times\Bigg\{a_{n+1}^p\Bigg(\sum_{k=1}^{n+1} b_k^q(1-e_{n+1}+e_k)\Bigg)+\sum_{k=1}^{n} a_k^p\Bigg[\sum_{r=1}^{n+1} b_r^q(1-e_k+e_r)\Bigg]\Bigg\}^{\frac{1}{p}}
\end{aligned}
$$

$$\times\left\{b_{n+1}^q\left(\sum_{k=1}^{n+1}a_k^p(1-e_{n+1}+e_k)\right)+\sum_{k=1}^{n}b_k^q\left[\sum_{r=1}^{n+1}a_r^p(1-e_k+e_r)\right]\right\}^{\frac{1}{p}}$$
$$=(n+1)^{2(1-t)}\left[\left(b_{n+1}^q\right)^2+2b_{n+1}^q\sum_{k=1}^{n}b_k^q+\left(\sum_{k=1}^{n}b_k^q\right)^2\right]^{\frac{1}{q}-\frac{1}{p}}$$
$$\times\left[\sum_{k=1}^{n+1}\sum_{r=1}^{n+1}a_k^pb_r^q(1-e_k+e_r)\right]^{\frac{1}{p}}\left[\sum_{k=1}^{n+1}\sum_{r=1}^{n+1}b_k^qa_r^p(1-e_k+e_r)\right]^{\frac{1}{p}}$$
$$=(n+1)^{2(1-t)}\left(\sum_{k=1}^{n+1}b_k^q\right)^{\frac{2}{q}-\frac{2}{p}}\left\{\left[\left(\sum_{k=1}^{n+1}b_k^q\right)\left(\sum_{k=1}^{n+1}a_k^p\right)\right]^2\right.$$
$$\left.-\left[\left(\sum_{k=1}^{n+1}b_k^qe_k\right)\left(\sum_{k=1}^{n+1}a_k^p\right)-\left(\sum_{k=1}^{n+1}b_k^q\right)\left(\sum_{k=1}^{n+1}a_k^pe_k\right)\right]^2\right\}^{\frac{1}{p}}$$
$$=x_{n+1}. \tag{30}$$

So
$$F_1(n)\leqslant F_1(n+1).$$

From $y_{n+1}-y_n\leqslant x_{n+1}-x_n$, it is easy to see that the equality
$$y_{n+1}^s-y_n^s\leqslant x_{n+1}^s-x_n^s$$
holds with $s=2,3,\cdots$. That is, $F_s(n)\leqslant F_s(n+1)$, $s=2,3,\cdots$. The proof of Theorem 2.2 is completed. □

Proof of Theorem 2.3. Since $a<t_1<t_2<b$, there exists $\beta\in(0,1)$ such that
$$\frac{t_1-a}{t_2-a}=\beta \text{ or } t_1=(1-\beta)a+\beta t_2.$$

Case 1. Let β is a rational number. Then $\beta=\frac{m_1}{m_2}$, where $m_1,m_2\in\mathbb{N}$ with $m_1<m_2$. Let $n_1=m_1n$ and $n_2=m_2n$. Then we have an equidistant partition of $[a,t_1]$:
$$a=x_0<x_1<x_2<\cdots<x_i<\cdots<x_{n_1-1}<x_{n_1}=t_1,$$
where
$$x_i=a+\frac{t_1-a}{n_1}i \text{ and } \Delta x_i=\frac{t_1-a}{n_1},\quad i=1,2,\ldots,n_1.$$

Therefore we have
$$\left(\frac{t_1-a}{n_1}\right)^{2s}F_s(n_1)$$
$$\triangleq\left(\frac{t_1-a}{n_1}\right)^{2s}n_1^{2s-\frac{2s}{p}-\frac{2s}{q}}\left(\sum_{i=1}^{n_1}g^q(x_i)\right)^{\frac{2s}{q}-\frac{2s}{p}}\left[\left(\sum_{i=1}^{n_1}g^q(x_i)\sum_{i=1}^{n_1}f^p(x_i)\right)^2\right.$$

$$
\begin{aligned}
&-\left(\sum_{i=1}^{n_1} g^q(x_i)\, e(x_i) \sum_{i=1}^{n_1} f^p(x_i) - \sum_{i=1}^{n_1} g^q(x_i) \sum_{i=1}^{n_1} f^p(x_i)\, e(x_i)\right)^2\Bigg]^{\frac{s}{p}} \\
&-\left(\frac{t_1-a}{n_1}\right)^{2s}\left(\sum_{i=1}^{n_1} f(x_i)\, g(x_i)\right)^{2s} \\
={}& (t_1-a)^{2s-\frac{2s}{p}-\frac{2s}{q}}\left(\sum_{i=1}^{n_1} g^q(x_i)\,\frac{t_1-a}{n_1}\right)^{\frac{2s}{q}-\frac{2s}{p}} \\
&\times\Bigg[\left(\sum_{i=1}^{n_1} g^q(x_i)\,\frac{t_1-a}{n_1} \sum_{i=1}^{n_1} f^p(x_i)\,\frac{t_1-a}{n_1}\right)^2 \\
&\quad -\Bigg(\sum_{i=1}^{n_1} g^q(x_i)\, e(x_i)\frac{t_1-a}{n_1} \sum_{i=1}^{n_1} f^p(x_i)\,\frac{t_1-a}{n_1} \\
&\qquad -\sum_{i=1}^{n_1} g^q(x_i)\,\frac{t_1-a}{n_1} \sum_{i=1}^{n_1} f^p(x_i)\, e(x_i)\frac{t_1-a}{n_1}\Bigg)^2\Bigg]^{\frac{s}{p}} \\
&\quad -\left(\sum_{i=1}^{n_1} f(x_i)\, g(x_i)\,\frac{t_1-a}{n_1}\right)^{2s}.
\end{aligned}
$$

Similarly, for the equidistant partition of $[a,t_2]$:

$$a = x'_0 < x'_1 < x'_2 < \cdots < x'_i < \cdots < x'_{n_2-1} < x'_{n_2} = t_2,$$

where

$$x'_i = a + \frac{t_2-a}{n_2} i \quad \text{and} \quad \Delta x'_i = \frac{t_2-a}{n_2}, \quad i = 1,2,\ldots,n_2,$$

we have

$$
\begin{aligned}
&\left(\frac{t_2-a}{n_2}\right)^{2s} F_s(n_2)' \\
\triangleq{}& \left(\frac{t_2-a}{n_2}\right)^{2s}\Bigg\{ n_2^{2s-\frac{2s}{p}-\frac{2s}{q}}\left(\sum_{i=1}^{n_2} g^q(x'_i)\right)^{\frac{2s}{q}-\frac{2s}{p}}\Bigg[\left(\sum_{i=1}^{n_2} g^q(x'_i) \sum_{i=1}^{n_2} f^p(x'_i)\right)^2 \\
&-\left(\sum_{i=1}^{n_2} g^q(x'_i)\, e(x'_i) \sum_{i=1}^{n_2} f^p(x'_i) - \sum_{i=1}^{n_2} g^q(x'_i) \sum_{i=1}^{n_2} f^p(x'_i)\, e(x'_i)\right)^2\Bigg]^{\frac{s}{p}} \\
&-\left(\sum_{i=1}^{n_2} f(x'_i)\, g(x'_i)\right)^{2s}\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
= &(t_2-a)^{2s-\frac{2s}{p}-\frac{2s}{q}}\left(\sum_{i=1}^{n_2} g^q\left(x_i'\right)\frac{t_2-a}{n_2}\right)^{\frac{2s}{q}-\frac{2s}{p}} \\
&\times\left[\left(\sum_{i=1}^{n_2} g^q\left(x_i'\right)\frac{t_2-a}{n_2}\sum_{i=1}^{n_2} f^p\left(x_i'\right)\frac{t_2-a}{n_2}\right)^2\right. \\
&\quad-\left(\sum_{i=1}^{n_2} g^q\left(x_i'\right)e(x_i')\frac{t_2-a}{n_2}\sum_{i=1}^{n_2} f^p\left(x_i'\right)\frac{t_2-a}{n_2}\right. \\
&\quad\left.\left.-\sum_{i=1}^{n_2} g^q\left(x_i'\right)\frac{t_2-a}{n_2}\sum_{i=1}^{n_2} f^p\left(x_i'\right)e(x_i')\frac{t_2-a}{n_2}\right)^2\right]^{\frac{s}{p}} \\
&\quad-\left(\sum_{i=1}^{n_2} f\left(x_i'\right)g\left(x_i'\right)\frac{t_2-a}{n_2}\right)^{2s}.
\end{aligned}
$$

Due to

$$
\frac{t_1-a}{t_2-a}=\frac{m_1 n}{m_2 n}=\frac{n_1}{n_2},
$$

we see that $\Delta x_i=\Delta x_i'$ and $x_i=x_i'$ for $i=1,2,\ldots,n_1$. By Theorem 2.2, we deduce that

$$
\left(\frac{t_1-a}{n_1}\right)^{2s}F_s\left(n_1\right)<\left(\frac{t_2-a}{n_2}\right)^{2s}F_s\left(n_2\right)',
$$

which, by letting $n\to\infty$, implies that $G_s\left(t_1\right)\leqslant G_s\left(t_2\right)$ for $a\leqslant t_1\leqslant t_2\leqslant b$.

Case 2. Let β is an irrational number. Then there exists an rational sequence $\{\beta_j\}$ such that $\lim_{j\to\infty}\beta_j=\beta$. From Case 1 we have

$$
G_s\left(\left(1-\beta_j\right)a+\beta_j t_2\right)\leqslant G_s\left(t_2\right),
$$

then letting $j\to\infty$ gives

$$
G_s\left(\left(1-\beta\right)a+\beta t_2\right)=G_s\left(t_1\right)\leqslant G_s\left(t_2\right),
$$

which completes the proof. □

By the same way as in Theorem 2.2 and Theorem 2.3, we can obtain the following properties of (8) and (9).

THEOREM 2.4. *Let $a_k \geqslant 0,\ b_k > 0\ (k=1,2,\cdots),\ 1-e_i+e_j \geqslant 0\ (i,j=1,2,\cdots)$, let $p>0,\ q<0,\ \vartheta = \max\{\frac{1}{p}+\frac{1}{q},1\}$, and let*

$$F_s(n) = n^{2s(1-\vartheta)}\left(\sum_{k=1}^n a_k^p\right)^{\frac{2s}{p}-\frac{2s}{q}}\left\{\left[\left(\sum_{k=1}^n a_k^p\right)\left(\sum_{k=1}^n b_k^q\right)\right]^2 - \left[\left(\sum_{k=1}^n a_k^p e_k\right)\left(\sum_{k=1}^n b_k^q\right) - \left(\sum_{k=1}^n a_k^p\right)\left(\sum_{k=1}^n b_k^q e_k\right)\right]^2\right\}^{\frac{s}{q}} - \left(\sum_{k=1}^n a_k b_k\right)^{2s}, \quad s=1,2,\cdots. \tag{31}$$

Then

$$F_s(n) \geqslant F_s(n+1). \tag{32}$$

The integral form is as follows:

THEOREM 2.5. *Let $f(x),\ g(x),\ e(x)$ be integrable functions defined on $[a,b]$ and $f(x) \geqslant 0,\ g(x)>0,\ 1-e(x)+e(y) \geqslant 0$ for all $x,y \in [a,b]$, let $p>0,\ q<0,\ \frac{1}{p}+\frac{1}{q} \geqslant 1$, and let*

$$G_s(t) = (t-a)^{2s-\frac{2s}{p}-\frac{2s}{q}}\left(\int_a^t f^p(x)dx\right)^{\frac{2s}{p}-\frac{2s}{q}}\left[\left(\int_a^t f^p(x)dx\int_a^t g^q(x)dx\right)^2 - \left(\int_a^t f^p(x)e(x)dx\int_a^t g^q(x)dx - \int_a^t f^p(x)dx\int_a^t g^q(x)e(x)dx\right)^2\right]^{\frac{s}{q}} - \left(\int_a^t f(x)g(x)dx\right)^{2s}, \quad s=1,2,\cdots. \tag{33}$$

Then we have

$$G_s(t_1) \geqslant G_s(t_2), \quad a \leqslant t_1 \leqslant t_2 \leqslant b. \tag{34}$$

Acknowledgements. The authors would like to express hearty thanks to the anonymous referees for their making great efforts to improve this paper.

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(Received December 3, 2015)

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