

## NEW INEQUALITIES FOR THE VOLUME OF THE UNIT BALL IN $\mathbb{R}^n$

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(Communicated by G. Nemes)

*Abstract.* Many interesting monotonicity properties and inequalities for the volume of the unit ball in  $\mathbb{R}^n$  have been established. The main object of this paper is to establish new inequalities for the volume of the unit ball in  $\mathbb{R}^n$ .

### 1. Introduction

In the recent past, several researchers established interesting monotonicity properties and inequalities of the volume of the unit ball in  $\mathbb{R}^n$ ,

$$\Omega_n = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2} + 1)}, \quad n \in \mathbb{N} := \{1, 2, 3, \dots\},$$

where  $\Gamma$  denotes the gamma function.

Böhm and Hertel [7, p. 264] pointed out that the sequence  $(\Omega_n)_{n \geq 1}$  is not monotonic for  $n \geq 1$ . Indeed, we have

$$\Omega_n < \Omega_{n+1} \quad \text{if } 1 \leq n \leq 4 \quad \text{and} \quad \Omega_n > \Omega_{n+1} \quad \text{if } n \geq 5.$$

Anderson et al. [5] showed that  $(\Omega_n^{1/n})_{n \geq 1}$  is monotonically decreasing to zero. Anderson and Qiu [6] proved that the sequence  $(\Omega_n^{1/(n \ln n)})_{n \geq 2}$  decreases to  $e^{-1/2}$ . Guo and Qi [12] proved that the sequence  $(\Omega_n^{1/(n \ln n)})_{n \geq 2}$  is logarithmically convex. Klain and Rota [13] proved that the sequence  $(n\Omega_n/\Omega_{n-1})_{n \geq 1}$  is increasing.

Many interesting inequalities for the volume of the unit ball in  $\mathbb{R}^n$  have been established [3, 4, 8, 11, 16, 17, 18, 19]. For example, Alzer [3] proved that for all integers  $n \geq 1$ ,

$$a_1 \Omega_{n+1}^{n/(n+1)} \leq \Omega_n < b_1 \Omega_{n+1}^{n/(n+1)}, \quad (1)$$

$$\sqrt{\frac{n+a_2}{2\pi}} < \frac{\Omega_{n-1}}{\Omega_n} \leq \sqrt{\frac{n+b_2}{2\pi}} \quad (2)$$

*Mathematics subject classification* (2010): 33B15, 26D07.

*Keywords and phrases:* Volume of the unit  $n$ -dimensional ball, gamma function, inequality.

and

$$\left(1 + \frac{1}{n}\right)^{a_3} \leq \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < \left(1 + \frac{1}{n}\right)^{b_3} \tag{3}$$

with the best possible constants

$$a_1 = \frac{2}{\sqrt{\pi}} = 1.1283\dots, \quad b_1 = \sqrt{e} = 1.6487\dots,$$

$$a_2 = \frac{1}{2}, \quad b_2 = \frac{\pi}{2} - 1 = 0.5707\dots$$

and

$$a_3 = 2 - \frac{\ln \pi}{\ln 2} = 0.3485\dots, \quad b_3 = \frac{1}{2}.$$

Double inequality (2) was rediscovered by Qiu and Vuorinen [19]. Double inequality (2) refines a result due to Borgwardt [8, p. 253] who proved (2) with  $a_2 = 0$  and  $b_2 = 1$ . Merkle [16] improved the left-hand side of (3) and obtained the following result:

$$\left(1 + \frac{1}{n+1}\right)^{1/2} \leq \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}, \quad n \geq 1. \tag{4}$$

Recently, Mortici [17] improved the inequalities (1) to (3) and obtained the following results:

$$\frac{64 \cdot 720^{11/12} \cdot 2^{1/22}}{10395 \cdot \pi^{5/11} \sqrt[2n]{2\pi}} \leq \frac{\Omega_n}{\Omega_{n+1}^{n/(n+1)}} < \frac{\sqrt{e}}{\sqrt[2n]{2\pi}}, \quad n \geq 4, \tag{5}$$

$$\sqrt{\frac{n + \frac{1}{2}}{2\pi}} < \frac{\Omega_{n-1}}{\Omega_n} < \sqrt{\frac{n + \frac{1}{2}}{2\pi} + \frac{1}{16\pi n}}, \quad n \geq 1 \tag{6}$$

and

$$\left(1 + \frac{1}{n}\right)^{\frac{1}{2} - \frac{1}{4n}} \leq \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < \left(1 + \frac{1}{n}\right)^{\frac{1}{2}}, \quad n \geq 4. \tag{7}$$

Very recently, Chen and Lin [11] presented sharp inequalities for the volume of the unit ball in  $\mathbb{R}^n$ . More precisely, the authors proved that, for all integers  $n \geq 1$ ,

$$\left(1 + \frac{1}{n+1}\right)^\alpha < \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} \leq \left(1 + \frac{1}{n+1}\right)^\beta, \tag{8}$$

$$\left(1 + \frac{1}{2n} - \frac{3}{8n^2}\right)^\lambda < \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} \leq \left(1 + \frac{1}{2n} - \frac{3}{8n^2}\right)^\mu, \tag{9}$$

$$\sqrt{\frac{2\pi}{n+a}} \leq \frac{\Omega_n}{\Omega_{n-1} + \Omega_{n+1}} < \sqrt{\frac{2\pi}{n+b}} \tag{10}$$

and

$$\frac{1}{\sqrt{\pi(n+\theta)}} \left(\frac{2\pi e}{n}\right)^{n/2} \leq \Omega_n < \frac{1}{\sqrt{\pi(n+\vartheta)}} \left(\frac{2\pi e}{n}\right)^{n/2} \tag{11}$$

with best possible constants

$$\alpha = \frac{1}{2}, \quad \beta = \frac{2\ln 2 - \ln \pi}{\ln 3 - \ln 2} = 0.5957713\dots,$$

$$\lambda = 1, \quad \mu = \frac{2\ln 2 - \ln \pi}{2\ln 3 - 3\ln 2} = 2.0509275\dots,$$

$$a = \frac{\pi(1+\pi)^2}{2} - 1 = 25.94353\dots, \quad b = \frac{1}{2} + 4\pi = 13.06637\dots$$

and

$$\theta = \frac{e}{2} - 1 = 0.3591409\dots, \quad \vartheta = \frac{1}{3}.$$

The main object of this paper is to establish new inequalities for the volume of the unit ball in  $\mathbb{R}^n$ .

### 2. Lemmas and preliminaries

Euler’s gamma function:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \quad x > 0$$

is one of the most important functions in mathematical analysis and its applications in various diverse areas. The logarithmic derivative of the gamma function  $\psi(x) = \Gamma'(x)/\Gamma(x)$  is known as the psi (or digamma) function. The derivatives of the psi function  $\psi^{(n)}(x)$ :

$$\psi^{(n)}(x) := \frac{d^n}{dx^n} \{ \psi(x) \}, \quad n \in \mathbb{N}$$

are called the polygamma functions.

The following asymptotic formulas are well known [1, p. 257] that, as  $x \rightarrow \infty$ ,

$$\ln \Gamma(x) \sim \left(x - \frac{1}{2}\right) \ln x - x + \ln \sqrt{2\pi} + \frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} - \frac{1}{1680x^7} + \dots \tag{12}$$

The following asymptotic expansion can be found (see [15, p. 33]):

$$\frac{\Gamma(x+t)}{\Gamma(x+s)} \sim x^{t-s} \sum_{n=0}^\infty (-1)^n \frac{(s-t)_n}{n!} B_n^{(t-s+1)}(t) \frac{1}{x^n}, \quad x \rightarrow \infty, \tag{13}$$

where the symbol  $B_n^{(a)}(t)$  stands for the generalized Bernoulli polynomials, defined by the following generating function:

$$\frac{x^a e^{tx}}{(e^x - 1)^a} = \sum_{n=0}^\infty B_n^{(a)}(t) \frac{x^n}{n!}, \tag{14}$$

and  $(t)_n$  is Pochhammer's symbol defined as

$$(t)_n = t(t+1)\cdots(t+n-1) = \frac{\Gamma(t+n)}{\Gamma(t)}, \quad (t)_0 = 1.$$

Formula (13) yields, as  $x \rightarrow \infty$ ,

$$\left[ \frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})} \right]^2 \sim x + \frac{1}{4} + \frac{1}{32x} - \frac{1}{128x^2} - \frac{5}{2048x^3} + \frac{23}{8192x^4} + \frac{53}{65536x^5} - \cdots \quad (15)$$

and

$$\left[ \frac{\Gamma(x+\frac{3}{2})}{\Gamma(x+1)} \right]^2 \sim x + \frac{3}{4} + \frac{1}{32x} - \frac{3}{128x^2} + \frac{27}{2048x^3} - \frac{27}{8192x^4} - \frac{171}{65536x^5} + \cdots \quad (16)$$

The following lemmas are required in our present investigation.

LEMMA 2.1. ([9, Corollary 1]) *Let  $m, n \in \mathbb{N}$ . Then for  $x > 0$ ,*

$$\begin{aligned} & \sum_{j=1}^{2m} \left( 1 - \frac{1}{2^{2j}} \right) \frac{2B_{2j}}{(2j)!} \frac{(2j+n-2)!}{x^{2j+n-1}} \\ & < (-1)^n \left( \psi^{(n-1)}(x+1) - \psi^{(n-1)}\left(x + \frac{1}{2}\right) \right) + \frac{(n-1)!}{2x^n} \\ & < \sum_{j=1}^{2m-1} \left( 1 - \frac{1}{2^{2j}} \right) \frac{2B_{2j}}{(2j)!} \frac{(2j+n-2)!}{x^{2j+n-1}}, \end{aligned} \quad (17)$$

where  $B_n$  are the Bernoulli numbers.

It follows from (17) that, for  $x > 0$ ,

$$\begin{aligned} \frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{64x^4} - \frac{1}{128x^6} & < \psi(x+1) - \psi\left(x + \frac{1}{2}\right) \\ & < \frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{64x^4} - \frac{1}{128x^6} + \frac{17}{2048x^8}. \end{aligned} \quad (18)$$

It was proved in [11] that for  $x > 0$ ,

$$\frac{1}{2x} - \frac{3}{8x^2} + \frac{1}{4x^3} - \frac{9}{64x^4} < \psi\left(x + \frac{3}{2}\right) - \psi(x+1) < \frac{1}{2x} - \frac{3}{8x^2} + \frac{1}{4x^3} - \frac{9}{64x^4} + \frac{1}{16x^5} \quad (19)$$

and

$$\psi\left(x + \frac{3}{2}\right) - \psi(x+1) > \frac{1}{2x} - \frac{3}{8x^2} + \frac{1}{4x^3} - \frac{9}{64x^4} + \frac{1}{16x^5} - \frac{3}{128x^6}. \quad (20)$$

LEMMA 2.2. *For  $x \geq 2$ ,*

$$x + \frac{1}{4} + \frac{1}{32x} - \frac{1}{128x^2} - \frac{5}{2048x^3} < \left[ \frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})} \right]^2. \quad (21)$$

*Proof.* Consider the function  $P(x)$  defined for  $x > 0$  by

$$P(x) = 2 \ln \Gamma(x + 1) - 2 \ln \Gamma\left(x + \frac{1}{2}\right) - \ln\left(x + \frac{1}{4} + \frac{1}{32x} - \frac{1}{128x^2} - \frac{5}{2048x^3}\right).$$

We conclude from the asymptotic formula (15) that

$$\lim_{x \rightarrow \infty} P(x) = 0.$$

Differentiating and applying the second inequality in (18), we find that for  $x \geq 2$ ,

$$\begin{aligned} P'(x) &= 2 \left( \psi(x + 1) - \psi\left(x + \frac{1}{2}\right) \right) - \frac{2048x^4 - 64x^2 + 32x + 15}{x(2048x^4 + 512x^3 + 64x^2 - 16x - 5)} \\ &< 2 \left( \frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{64x^4} - \frac{1}{128x^6} + \frac{17}{2048x^8} \right) \\ &\quad - \frac{2048x^4 - 64x^2 + 32x + 15}{x(2048x^4 + 512x^3 + 64x^2 - 16x - 5)} \\ &= - \left( 1548853 + 5160656(x-2) + 6899824(x-2)^2 + 4780544(x-2)^3 \right. \\ &\quad \left. + 1819808(x-2)^4 + 361984(x-2)^5 + 29440(x-2)^6 \right) \\ &\quad / \left( 1024x^8(2048x^4 + 512x^3 + 64x^2 - 16x - 5) \right) \\ &< 0. \end{aligned}$$

Hence, we have for  $x \geq 2$ ,

$$P(x) > \lim_{x \rightarrow \infty} P(x) = 0.$$

By rearranging the terms in the last expression, inequality (21) follows.  $\square$

LEMMA 2.3. For  $x \geq 3$ ,

$$x + \frac{3}{4} + \frac{1}{32x} - \frac{3}{128x^2} < \left[ \frac{\Gamma(x + \frac{3}{2})}{\Gamma(x + 1)} \right]^2 < x + \frac{3}{4} + \frac{1}{32x}. \tag{22}$$

*Proof.* In [11, Lemma 2.2], it was proved that the first inequality in (22) is valid for all  $x \geq 2$ . We now prove the second inequality in (22) for  $x \geq 3$ . Consider the function  $V(x)$  defined for  $x > 0$  by

$$V(x) = 2 \ln \Gamma\left(x + \frac{3}{2}\right) - 2 \ln \Gamma(x + 1) - \ln\left(x + \frac{3}{4} + \frac{1}{32x}\right).$$

We conclude from the asymptotic formula (16) that

$$\lim_{x \rightarrow \infty} V(x) = 0.$$

Differentiating and applying the first inequality in (19), we find that for  $x \geq 3$ ,

$$\begin{aligned} V'(x) &= 2 \left( \psi \left( x + \frac{3}{2} \right) - \psi(x+1) \right) - \frac{(32x^2 - 1)}{x(32x^2 + 24x + 1)} \\ &> 2 \left( \frac{1}{2x} - \frac{3}{8x^2} + \frac{1}{4x^3} - \frac{9}{64x^4} \right) - \frac{(32x^2 - 1)}{x(32x^2 + 24x + 1)} \\ &= \frac{39 + 232(x - 3) + 72(x - 3)^2}{32x^4(32x^2 + 24x + 1)} > 0. \end{aligned}$$

Hence, we have for  $x \geq 3$ ,

$$V(x) < \lim_{x \rightarrow \infty} V(x) = 0.$$

By rearranging the terms in the last expression, the second inequality in (22) follows.  $\square$

From (19), we obtain that for  $x \geq 1$ ,

$$\begin{aligned} (2x + 1) \left( -\psi \left( x + \frac{3}{2} \right) + \psi(x+1) \right) + 1 &> (2x + 1) \left( -\frac{1}{2x} + \frac{3}{8x^2} - \frac{1}{4x^3} + \frac{9}{64x^4} - \frac{1}{16x^5} \right) + 1 \\ &= \frac{1}{4x} - \frac{1}{8x^2} + \frac{1}{32x^3} + \frac{1}{64x^4} - \frac{1}{16x^5} \\ &= \frac{7 + 45(x - 1) + 74(x - 1)^2 + 56(x - 1)^3 + 16(x - 1)^4}{64x^4} > 0 \end{aligned}$$

and

$$\begin{aligned} (2x + 1) \left( -\psi \left( x + \frac{3}{2} \right) + \psi(x+1) \right) + 1 &< (2x + 1) \left( -\frac{1}{2x} + \frac{3}{8x^2} - \frac{1}{4x^3} + \frac{9}{64x^4} \right) + 1 \\ &= \frac{1}{4x} - \frac{1}{8x^2} + \frac{1}{32x^3} + \frac{9}{64x^4}, \end{aligned}$$

we then obtain

$$0 < (2x + 1) \left( -\psi \left( x + \frac{3}{2} \right) + \psi(x+1) \right) + 1 < \frac{1}{4x} - \frac{1}{8x^2} + \frac{1}{32x^3} + \frac{9}{64x^4}, \quad x \geq 1. \tag{23}$$

The proof of Theorem 3.2 makes use of the inequalities (22) and (23).

### 3. Main results

In view of the second inequality in (6), it is natural to ask: what is the smallest number  $\vartheta_1$  and what is the largest number  $\vartheta_2$  such that the inequality

$$\sqrt{\frac{n + \frac{1}{2}}{2\pi} + \frac{1}{16\pi(n + \vartheta_1)}} \leq \frac{\Omega_{n-1}}{\Omega_n} \leq \sqrt{\frac{n + \frac{1}{2}}{2\pi} + \frac{1}{16\pi(n + \vartheta_2)}}$$

holds for all integers  $n \geq 1$ ? Theorem 3.1 answers this question.

THEOREM 3.1. For  $n \geq 1$ ,

$$\sqrt{\frac{n+\frac{1}{2}}{2\pi} + \frac{1}{16\pi(n+\vartheta_1)}} \leq \frac{\Omega_{n-1}}{\Omega_n} < \sqrt{\frac{n+\frac{1}{2}}{2\pi} + \frac{1}{16\pi(n+\vartheta_2)}} \tag{24}$$

with best possible constants

$$\vartheta_1 = \frac{13-4\pi}{4\pi-12} = 0.7656283\dots \quad \text{and} \quad \vartheta_2 = \frac{1}{2}.$$

*Proof.* Inequality (24) can be written as

$$\vartheta_1 \geq \frac{1}{16\pi \left(\frac{\Omega_{n-1}}{\Omega_n}\right)^2 - 8n - 4} - n > \vartheta_2.$$

We now show that the sequence

$$y_n = \frac{1}{16\pi \left(\frac{\Omega_{n-1}}{\Omega_n}\right)^2 - 8n - 4} - n = \frac{1}{16 \left(\frac{\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n}{2}+\frac{1}{2})}\right)^2 - 8n - 4} - n$$

is strictly decreasing for  $n \geq 1$ . To this end, we consider the function  $g(x)$  defined by

$$g(x) = \frac{1}{16 \left(\frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})}\right)^2 - 16x - 4} - 2x.$$

Elementary calculations show that

$$\begin{aligned} & -\frac{1}{2} \left( \left( \frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})} \right)^2 - x - \frac{1}{4} \right) g'(x) \\ &= \left( \left( \frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})} \right)^2 - x - \frac{1}{4} \right) + \frac{1}{16} \left( \psi(x+1) - \psi\left(x+\frac{1}{2}\right) \right) \left( \frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})} \right)^2 - \frac{1}{32}. \end{aligned}$$

By using inequalities (21) and the first inequality in (18), we obtain that for  $x \geq 4$ ,

$$\begin{aligned} & -\frac{1}{2} \left( \left( \frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})} \right)^2 - x - \frac{1}{4} \right) g'(x) \\ &> \left( \frac{1}{32x} - \frac{1}{128x^2} - \frac{5}{2048x^3} \right)^2 \\ &\quad + \frac{1}{16} \left( \frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{64x^4} - \frac{1}{128x^6} \right) \left( x + \frac{1}{4} + \frac{1}{32x} - \frac{1}{128x^2} - \frac{5}{2048x^3} \right) - \frac{1}{32} \\ &= \frac{125413+281712(x-4)+201058(x-4)^2+64761(x-4)^3+9840(x-4)^4+576(x-4)^5}{4194304x^9} \\ &> 0. \end{aligned}$$

Hence, the function  $g(x)$  is strictly decreasing on  $[4, \infty)$  and the sequence  $(x_n)$  is strictly decreasing for  $n \geq 8$ .

Direct computation would yield

$$\begin{aligned} y_1 &= g\left(\frac{1}{2}\right) = 0.76562832\dots, & y_2 &= g(1) = 0.68938142\dots, \\ y_3 &= g\left(\frac{3}{2}\right) = 0.64519319\dots, & y_4 &= g(2) = 0.61698646\dots, \\ y_5 &= g\left(\frac{5}{2}\right) = 0.59764076\dots, & y_6 &= g(3) = 0.58363594\dots, \\ y_7 &= g\left(\frac{7}{2}\right) = 0.57306769\dots, & y_8 &= g(4) = 0.56482818\dots \end{aligned}$$

Consequently, the sequence  $(y_n)_{n \geq 1}$  is strictly decreasing. This leads to

$$\lim_{n \rightarrow \infty} y_n < y_n \leq y_1 = \frac{13 - 4\pi}{4\pi - 12} \quad \text{for all } n \geq 1.$$

It remains to prove that

$$\lim_{n \rightarrow \infty} y_n = \frac{1}{2}. \tag{25}$$

We conclude from the asymptotic formula (15) that

$$g(x) = \frac{1}{2} + O\left(\frac{1}{x}\right) \quad \text{as } x \rightarrow \infty,$$

which implies (25). This completes the proof of Theorem 3.1.  $\square$

As numerical computations confirm, the ratio  $\Omega_n^2/(\Omega_{n-1}\Omega_{n+1})$  in (8) becomes closer to  $(1 + 1/(n + 1))^{1/2}$ , as  $n$  approaches infinity. This fact motivated us to pose the following question: what is the smallest number  $\theta_1$  and what is the largest number  $\theta_2$  such that the inequality

$$\left(1 + \frac{1}{n + \theta_1}\right)^{1/2} \leq \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} \leq \left(1 + \frac{1}{n + \theta_2}\right)^{1/2}$$

holds for all integers  $n \geq 1$ ? Theorem 3.2 answers this question.

**THEOREM 3.2.** For  $n \geq 1$ ,

$$\left(1 + \frac{1}{n + \theta_1}\right)^{1/2} \leq \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < \left(1 + \frac{1}{n + \theta_2}\right)^{1/2} \tag{26}$$

with best possible constants

$$\theta_1 = \frac{2\pi^2 - 16}{16 - \pi^2} = 0.60994576\dots \quad \text{and} \quad \theta_2 = \frac{1}{2}.$$

*Proof.* Inequality (26) can be written as

$$\theta_1 \geq \frac{1}{\left(\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}\right)^2 - 1} - n > \theta_2.$$



We now show that the sequence

$$x_n = \frac{1}{\left(\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}\right)^2 - 1} - n = \frac{1}{\frac{4}{(n+1)^2} \left(\frac{\Gamma(\frac{n}{2} + \frac{3}{2})}{\Gamma(\frac{n}{2} + 1)}\right)^4 - 1} - n$$

is strictly decreasing for  $n \geq 1$ . To this end, we consider the function  $f(x)$  defined by

$$f(x) = \frac{1}{\frac{4}{(2x+1)^2} \left(\frac{\Gamma(x+\frac{3}{2})}{\Gamma(x+1)}\right)^4 - 1} - 2x.$$

Elementary calculations show that

$$\begin{aligned} & -\frac{1}{2} \left( \frac{4}{(2x+1)^2} \left( \frac{\Gamma(x+\frac{3}{2})}{\Gamma(x+1)} \right)^4 - 1 \right)^2 f'(x) \\ &= \left( \frac{4}{(2x+1)^2} \left( \frac{\Gamma(x+\frac{3}{2})}{\Gamma(x+1)} \right)^4 - 1 \right)^2 \\ & \quad - \frac{8 \left( (2x+1) \left( -\psi(x+\frac{3}{2}) + \psi(x+\frac{1}{2}) \right) + 1 \right)}{(2x+1)^3} \left( \frac{\Gamma(x+\frac{3}{2})}{\Gamma(x+1)} \right)^4. \end{aligned}$$

By using inequalities (22) and (23), we obtain that for  $x \geq 12$ ,

$$\begin{aligned} & -\frac{1}{2} \left( \frac{4}{(2x+1)^2} \left( \frac{\Gamma(x+\frac{3}{2})}{\Gamma(x+1)} \right)^4 - 1 \right)^2 f'(x) \\ & > \left( \frac{4}{(2x+1)^2} \left( x + \frac{3}{4} + \frac{1}{32x} - \frac{3}{128x^2} \right)^2 - 1 \right)^2 \\ & \quad - \frac{8 \left( \frac{1}{4x} - \frac{1}{8x^2} + \frac{1}{32x^3} + \frac{9}{64x^4} \right)}{(2x+1)^3} \left( x + \frac{3}{4} + \frac{1}{32x} \right)^2 \\ & = \frac{\lambda(x-12)}{16777216x^8(2x+1)^4}, \end{aligned}$$

with

$$\begin{aligned} \lambda(x) &= 45704971569681 + 200145001247568x + 105831697102560x^2 \\ & \quad + 25217680734464x^3 + 3382791477504x^4 + 273737302016x^5 \\ & \quad + 13326352384x^6 + 360972288x^7 + 4194304x^8. \end{aligned}$$

Hence,  $f(x)$  is strictly decreasing on  $[12, \infty)$  and the sequence  $(x_n)$  is strictly decreasing for  $n \geq 24$ .

Direct computation would yield

$$\begin{aligned}
 x_1 &= f\left(\frac{1}{2}\right) = 0.609945759\dots, & x_2 &= f(1) = 0.577896831\dots, \\
 x_3 &= f\left(\frac{3}{2}\right) = 0.559987003\dots, & x_4 &= f(2) = 0.548648634\dots, \\
 x_5 &= f\left(\frac{5}{2}\right) = 0.540861512\dots, & x_6 &= f(3) = 0.535197735\dots, \\
 x_7 &= f\left(\frac{7}{2}\right) = 0.530899584\dots, & x_8 &= f(4) = 0.527529486\dots, \\
 x_9 &= f\left(\frac{9}{2}\right) = 0.524817824\dots, & x_{10} &= f(5) = 0.522589737\dots, \\
 x_{11} &= f\left(\frac{11}{2}\right) = 0.520727001\dots, & x_{12} &= f(6) = 0.519146891\dots, \\
 x_{13} &= f\left(\frac{13}{2}\right) = 0.517789828\dots, & x_{14} &= f(7) = 0.516611833\dots, \\
 x_{15} &= f\left(\frac{15}{2}\right) = 0.515579748\dots, & x_{16} &= f(8) = 0.514668106\dots, \\
 x_{17} &= f\left(\frac{17}{2}\right) = 0.513857031\dots, & x_{18} &= f(9) = 0.513130781\dots, \\
 x_{19} &= f\left(\frac{19}{2}\right) = 0.512476735\dots, & x_{20} &= f(10) = 0.511884651\dots, \\
 x_{21} &= f\left(\frac{21}{2}\right) = 0.511346135\dots, & x_{22} &= f(11) = 0.510854241\dots, \\
 x_{23} &= f\left(\frac{23}{2}\right) = 0.510403173\dots, & x_{24} &= f(12) = 0.509988057\dots
 \end{aligned}$$

Consequently, the sequence  $(x_n)_{n \geq 1}$  is strictly decreasing. This leads to

$$\lim_{n \rightarrow \infty} x_n < x_n \leq x_1 = \frac{2\pi^2 - 16}{16 - \pi^2} \quad \text{for all } n \geq 1.$$

It remains to prove that

$$\lim_{n \rightarrow \infty} x_n = \frac{1}{2}. \tag{27}$$

We conclude from the asymptotic formula (16) that

$$f(x) = \frac{1}{2} + O\left(\frac{1}{x}\right) \quad \text{as } x \rightarrow \infty,$$

which implies (27). This completes the proof of Theorem 3.2.  $\square$

REMARK 3.1. The lower bound in (26) is sharper than one in (8) for  $n \geq 1$ ; the upper bound in (26) is sharper than one in (8) for  $n \geq 2$ .

Theorem 3.3 improves the inequality (7).

THEOREM 3.3. For  $n \geq 1$ ,

$$\left(1 + \frac{1}{n+1}\right)^{\frac{1}{2} + \frac{1}{4n} - \frac{3}{8n^2}} < \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < \left(1 + \frac{1}{n+1}\right)^{\frac{1}{2} + \frac{1}{4n}}. \tag{28}$$

*Proof.* In order to prove the first inequality in (28), it suffices to show that the sequence

$$y_n = \ln 2 + 2 \ln \Gamma\left(\frac{n+3}{2}\right) - 2 \ln \Gamma\left(\frac{n+2}{2}\right) - \ln(n+1) - \left(\frac{1}{2} + \frac{1}{4n} - \frac{3}{8n^2}\right) \ln\left(\frac{n+2}{n+1}\right)$$

is positive for  $n \geq 1$ .

We consider the function  $F(x)$  defined by

$$F(x) = \ln 2 + 2 \ln \Gamma\left(x + \frac{3}{2}\right) - 2 \ln \Gamma(x + 1) - \ln(2x + 1) - \left(\frac{1}{2} + \frac{1}{8x} - \frac{3}{32x^2}\right) \ln\left(\frac{2x + 2}{2x + 1}\right).$$

We conclude from the asymptotic formula (16) that

$$\lim_{x \rightarrow \infty} F(x) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} y_n = 0.$$

It is not difficult to show that

$$\frac{1}{2x} - \frac{3}{8x^2} < \ln\left(\frac{2x + 2}{2x + 1}\right) < \frac{1}{2x} - \frac{3}{8x^2} + \frac{7}{24x^3}, \quad x > 0. \tag{29}$$

Differentiating  $F(x)$  and applying inequalities (19) and (29), we obtain that for  $x \geq 3/2$ ,

$$\begin{aligned} F'(x) &= 2 \left( \psi\left(x + \frac{3}{2}\right) - \psi(x + 1) \right) + \left(\frac{2x - 3}{16x^3}\right) \ln\left(\frac{2x + 2}{2x + 1}\right) - \frac{64x^3 + 48x^2 - 4x + 3}{32x^2(2x + 1)(x + 1)} \\ &< 2 \left( \frac{1}{2x} - \frac{3}{8x^2} + \frac{1}{4x^3} - \frac{9}{64x^4} + \frac{1}{16x^5} \right) + \left(\frac{2x - 3}{16x^3}\right) \left( \frac{1}{2x} - \frac{3}{8x^2} + \frac{7}{24x^3} \right) \\ &\quad - \frac{64x^3 + 48x^2 - 4x + 3}{32x^2(2x + 1)(x + 1)} \\ &= - \frac{24 + 124(x - 1) + 213(x - 1)^2 + 92(x - 1)^3}{384x^6(2x + 1)(x + 1)} < 0. \end{aligned}$$

Hence,  $F(x)$  is strictly decreasing on  $[3/2, \infty)$ , and therefore, the sequence  $(y_n)$  is strictly decreasing for  $n \geq 3$ .

Direct computation would yield

$$\begin{aligned} y_1 &= F\left(\frac{1}{2}\right) = -\frac{3}{8} \ln 3 + \frac{19}{8} \ln 2 - \ln \pi = 0.089515\dots, \\ y_2 &= F(1) = -\frac{65}{16} \ln 2 + \frac{49}{32} \ln 3 + \ln \pi = 0.0110695\dots, \\ y_3 &= F\left(\frac{3}{2}\right) = \frac{73}{12} \ln 2 - \frac{13}{24} \ln 5 - 2 \ln 3 - \ln \pi = 0.002912\dots \end{aligned}$$

Consequently, the sequence  $(y_n)_{n \geq 1}$  is strictly decreasing. This leads to

$$y_n > \lim_{n \rightarrow \infty} y_n = 0, \quad n \geq 1.$$

In order to prove the second inequality in (28), it suffices to show that the sequence

$$z_n = \ln 2 + 2 \ln \Gamma\left(\frac{n + 3}{2}\right) - 2 \ln \Gamma\left(\frac{n + 2}{2}\right) - \ln(n + 1) - \left(\frac{1}{2} + \frac{1}{4n}\right) \ln\left(\frac{n + 2}{n + 1}\right)$$

is negative for  $n \geq 1$ .

We consider the function  $G(x)$  defined by

$$G(x) = \ln 2 + 2 \ln \Gamma \left( x + \frac{3}{2} \right) - 2 \ln \Gamma(x + 1) - \ln(2x + 1) - \left( \frac{1}{2} + \frac{1}{8x} \right) \ln \left( \frac{2x + 2}{2x + 1} \right).$$

We conclude from the asymptotic formula (16) that

$$\lim_{x \rightarrow \infty} G(x) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} z_n = 0.$$

Differentiating  $G(x)$  and applying inequalities (19) and (29), we obtain that for  $x \geq 3$ ,

$$\begin{aligned} G'(x) &= 2 \left( \psi \left( x + \frac{3}{2} \right) - \psi(x + 1) \right) + \frac{1}{8x^2} \ln \left( \frac{2x + 2}{2x + 1} \right) - \frac{16x^2 + 12x - 1}{8x(2x + 1)(x + 1)} \\ &> 2 \left( \frac{1}{2x} - \frac{3}{8x^2} + \frac{1}{4x^3} - \frac{9}{64x^4} \right) + \frac{1}{8x^2} \left( \frac{1}{2x} - \frac{3}{8x^2} \right) - \frac{16x^2 + 12x - 1}{8x(2x + 1)(x + 1)} \\ &= \frac{3(20 + 27(x - 3) + 6(x - 3)^2)}{64x^4(2x + 1)(x + 1)} > 0. \end{aligned}$$

Hence,  $G(x)$  is strictly increasing on  $[3, \infty)$ , and therefore, the sequence  $(z_n)$  is strictly increasing for  $n \geq 6$ .

Direct computation would yield

$$\begin{aligned} z_1 &= G\left(\frac{1}{2}\right) = -0.0625343\dots, & z_2 &= G(1) = -0.0159006\dots, \\ z_3 &= G\left(\frac{3}{2}\right) = -0.0063856\dots, & z_4 &= G(2) = -0.0031937\dots, \\ z_5 &= G\left(\frac{5}{2}\right) = -0.0018234\dots, & z_6 &= G(3) = -0.0011382\dots \end{aligned}$$

Consequently, the sequence  $(z_n)_{n \geq 1}$  is strictly increasing. This leads to

$$z_n < \lim_{n \rightarrow \infty} z_n = 0, \quad n \geq 1.$$

The proof of Theorem 3.3 is thus completed.  $\square$

REMARK 3.2. The lower bound in (28) is sharper than one in (7) for  $n \geq 2$ , and the upper bound in (28) is better than one in (7) for all  $n \geq 1$ .

Theorem 3.4 provides new bounds for  $\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}$ .

THEOREM 3.4. For  $n \geq 1$ ,

$$1 + \frac{1}{2n} - \frac{3}{8n^2} < \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < 1 + \frac{1}{2n} - \frac{3}{8n^2} + \frac{3}{16n^3} + \frac{3}{128n^4}. \tag{30}$$

*Proof.* Inequality (30) can be written as

$$1 + \frac{1}{2n} - \frac{3}{8n^2} < \frac{2 \left[ \Gamma \left( \frac{n+3}{2} \right) \right]^2}{(n+1) \left[ \Gamma \left( \frac{n}{2} + 1 \right) \right]^2} < 1 + \frac{1}{2n} - \frac{3}{8n^2} + \frac{3}{16n^3} + \frac{3}{128n^4}. \tag{31}$$

The lower bound in (31) is obtained by considering the function  $J(x)$  defined for  $x > 0$  by

$$J(x) = \ln 2 + 2 \ln \Gamma \left( x + \frac{3}{2} \right) - \ln(2x + 1) - 2 \ln \Gamma(x + 1) - \ln \left( 1 + \frac{1}{4x} - \frac{3}{8(2x)^2} \right).$$

We conclude from the asymptotic formula (16) that

$$\lim_{x \rightarrow \infty} J(x) = 0.$$

Differentiating and using the second inequality in (19), we find that for  $x \geq 1$ ,

$$\begin{aligned} J'(x) &= 2 \left( \psi \left( x + \frac{3}{2} \right) - \psi(x + 1) \right) - \frac{2(32x^3 - x + 3)}{x(2x + 1)(32x^2 + 8x - 3)} \\ &< 2 \left( \frac{1}{2x} - \frac{3}{8x^2} + \frac{1}{4x^3} - \frac{9}{64x^4} + \frac{1}{16x^5} \right) - \frac{2(32x^3 - x + 3)}{x(2x + 1)(32x^2 + 8x - 3)} \\ &= - \frac{67 + 505(x - 1) + 954(x - 1)^2 + 648(x - 1)^3 + 144(x - 1)^4}{32x^5(2x + 1)(32x^2 + 8x - 3)} < 0. \end{aligned}$$

Hence,  $J(x)$  is strictly decreasing on  $[1, \infty)$ , and therefore, the sequence  $\{J(\frac{n}{2})\}_{n=2}^\infty$  is strictly decreasing.

Direct computation would yield

$$\begin{aligned} J\left(\frac{1}{2}\right) &= 5 \ln 2 - \ln \pi - 2 \ln 3 = 0.1237814\dots, \\ J(1) &= 2 \ln 2 + \ln 3 + \ln \pi - \ln 37 = 0.0187186\dots \end{aligned}$$

Consequently, the sequence  $\{J(\frac{n}{2})\}_{n=1}^\infty$  is strictly decreasing. This leads to

$$J\left(\frac{n}{2}\right) = \ln \left( \frac{2[\Gamma(\frac{n+3}{2})]^2}{(n+1)[\Gamma(\frac{n}{2}+1)]^2} \right) - \ln \left( 1 + \frac{1}{2n} - \frac{3}{8n^2} \right) > \lim_{n \rightarrow \infty} J\left(\frac{n}{2}\right) = 0$$

for  $n \geq 1$ . By rearranging the terms in the last expression, the first inequality in (31) follows.

The upper bound in (31) is obtained by considering the function  $K(x)$  defined for  $x > 0$  by

$$\begin{aligned} K(x) &= \ln 2 + 2 \ln \Gamma \left( x + \frac{3}{2} \right) - \ln(2x + 1) - 2 \ln \Gamma(x + 1) \\ &\quad - \ln \left( 1 + \frac{1}{4x} - \frac{3}{8(2x)^2} + \frac{3}{16(2x)^3} + \frac{3}{128(2x)^4} \right). \end{aligned}$$

We conclude from the asymptotic formula (16) that

$$\lim_{x \rightarrow \infty} K(x) = 0.$$

Differentiating and using (20), we find that for  $x \geq \frac{3}{2}$ ,

$$\begin{aligned} K'(x) &= 2 \left( \psi \left( x + \frac{3}{2} \right) - \psi(x+1) \right) - \frac{2(2048x^5 - 64x^3 + 96x^2 - 81x - 6)}{x(2x+1)(2048x^4 + 512x^3 - 192x^2 + 48x + 3)} \\ &> 2 \left( \frac{1}{2x} - \frac{3}{8x^2} + \frac{1}{4x^3} - \frac{9}{64x^4} + \frac{1}{16x^5} - \frac{3}{128x^6} \right) \\ &\quad - \frac{2(2048x^5 - 64x^3 + 96x^2 - 81x - 6)}{x(2x+1)(2048x^4 + 512x^3 - 192x^2 + 48x + 3)} \\ &= \frac{9828 + 55941(x - \frac{3}{2}) + 103860(x - \frac{3}{2})^2 + 87492(x - \frac{3}{2})^3 + 34720(x - \frac{3}{2})^4 + 5280(x - \frac{3}{2})^5}{64x^6(2x+1)(2048x^4 + 512x^3 - 192x^2 + 48x + 3)} \\ &> 0. \end{aligned}$$

Hence,  $K(x)$  is strictly increasing on  $[\frac{3}{2}, \infty)$ , and therefore, the sequence  $\{K(\frac{n}{2})\}_{n=3}^\infty$  is strictly increasing.

Direct computation would yield

$$\begin{aligned} K\left(\frac{1}{2}\right) &= 9 \ln 2 - \ln \pi - 2 \ln 3 - \ln 19 = -0.0480688 \dots, \\ K(1) &= 8 \ln 2 + \ln 3 + \ln \pi - \ln 41 - \ln 59 = -0.0025898 \dots, \\ K\left(\frac{3}{2}\right) &= 12 \ln 2 + \ln 3 - \ln \pi - \ln 7 - \ln 13 - \ln 43 = -0.0004110 \dots \end{aligned}$$

Consequently, the sequence  $\{K(\frac{n}{2})\}_{n=1}^\infty$  is strictly increasing. This leads to

$$\begin{aligned} K\left(\frac{n}{2}\right) &= \ln \left( \frac{2[\Gamma(\frac{n+3}{2})]^2}{(n+1)[\Gamma(\frac{n}{2}+1)]^2} \right) - \ln \left( 1 + \frac{1}{2n} - \frac{3}{8n^2} + \frac{3}{16n^3} + \frac{3}{128n^4} \right) \\ &< \lim_{n \rightarrow \infty} K\left(\frac{n}{2}\right) = 0, \quad n \geq 1. \end{aligned}$$

By rearranging the terms in the last expression, the second inequality in (31) follows. This completes the proof of Theorem 3.4.  $\square$

Theorem 3.5 provides new bounds for  $\Omega_n$ .

**THEOREM 3.5.** For  $n \geq 1$ ,

$$\sqrt{\frac{1}{\pi} \left( \frac{1}{n} - \frac{1}{3n^2} \right) \left( \frac{2\pi e}{n} \right)^{n/2}} < \Omega_n < \sqrt{\frac{1}{\pi} \left( \frac{1}{n} - \frac{1}{3n^2} + \frac{1}{18n^3} + \frac{31}{810n^4} \right) \left( \frac{2\pi e}{n} \right)^{n/2}}. \tag{32}$$

*Proof.* The lower bound is obtained by considering the function  $C(x)$  defined for  $x > 0$  by

$$C(x) = x \ln \pi - \ln \Gamma(x+1) - \left[ \frac{1}{2} \ln \left( \frac{1}{\pi} \right) + \frac{1}{2} \ln \left( \frac{1}{2x} - \frac{1}{3(2x)^2} \right) + x(\ln \pi + 1 - \ln x) \right].$$

We conclude from the asymptotic formula (12) that

$$\lim_{x \rightarrow \infty} C(x) = 0.$$

It follows from the known result (see [2, Theorem 8]) that, for  $x > 0$ ,

$$\ln x - \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} < \psi(x) < \ln x - \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4}. \tag{33}$$

Differentiating and applying the first inequality in (33), we find that for  $x \geq \frac{1}{2}$ ,

$$\begin{aligned} C'(x) &= \ln x - \psi(x) - \frac{1}{x} + \frac{3x-1}{x(6x-1)} \\ &< -\frac{1}{2x} + \frac{1}{12x^2} - \frac{1}{120x^4} + \frac{1}{252x^6} + \frac{3x-1}{x(6x-1)} \\ &= -\frac{\frac{29}{8} + \frac{237}{2}(x-\frac{1}{2}) + 483(x-\frac{1}{2})^2 + 546(x-\frac{1}{2})^3 + 210(x-\frac{1}{2})^4}{2520x^6(6x-1)} < 0. \end{aligned}$$

Hence,  $C(x)$  is strictly decreasing for  $x \geq \frac{1}{2}$ , and we have for  $n \geq 1$ ,

$$C\left(\frac{n}{2}\right) = \ln\left(\frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}\right) - \ln\left[\sqrt{\frac{1}{\pi}\left(\frac{1}{n} - \frac{1}{3n^2}\right)}\left(\frac{2\pi e}{n}\right)^{n/2}\right] > \lim_{n \rightarrow \infty} C\left(\frac{n}{2}\right) = 0.$$

By rearranging the terms in the last expression, the first inequality in (32) follows.

The upper bound is obtained by considering the function  $D(x)$  defined for  $x > 0$  by

$$\begin{aligned} D(x) &= x \ln \pi - \ln \Gamma(x+1) \\ &\quad - \left[ \frac{1}{2} \ln\left(\frac{1}{\pi}\right) + \frac{1}{2} \ln\left(\frac{1}{2x} - \frac{1}{3(2x)^2} + \frac{1}{18(2x)^3} + \frac{31}{810(2x)^4}\right) + x(\ln \pi + 1 - \ln x) \right]. \end{aligned}$$

We conclude from the asymptotic formula (12) that

$$\lim_{x \rightarrow \infty} D(x) = 0.$$

Differentiating and applying the second inequality in (33), we find that for  $x \geq \frac{1}{2}$ ,

$$\begin{aligned} D'(x) &= \ln x - \psi(x) - \frac{1}{x} + \frac{3240x^3 - 1080x^2 + 135x + 62}{x(6480x^3 - 1080x^2 + 90x + 31)} \\ &> -\frac{1}{2x} + \frac{1}{12x^2} - \frac{1}{120x^4} + \frac{3240x^3 - 1080x^2 + 135x + 62}{x(6480x^3 - 1080x^2 + 90x + 31)} \\ &= \frac{\frac{543}{2} + 1300(x-\frac{1}{2}) + 1390(x-\frac{1}{2})^2}{120x^4\left[616 + 3870(x-\frac{1}{2}) + 8640(x-\frac{1}{2})^2 + 6480(x-\frac{1}{2})^3\right]} > 0. \end{aligned}$$

Hence,  $D(x)$  is strictly increasing for  $x \geq \frac{1}{2}$ , and we have for  $n \geq 1$ ,

$$\begin{aligned} D\left(\frac{n}{2}\right) &= \ln\left(\frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}\right) - \ln\left[\sqrt{\frac{1}{\pi}\left(\frac{1}{n} - \frac{1}{3n^2} + \frac{1}{18n^3} + \frac{31}{810n^4}\right)}\left(\frac{2\pi e}{n}\right)^{n/2}\right] \\ &< \lim_{n \rightarrow \infty} D\left(\frac{n}{2}\right) = 0. \end{aligned}$$

By rearranging the terms in the last expression, the second inequality in (32) follows.  $\square$

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(Received February 26, 2016)

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