

## WEIGHTED INTEGRAL INEQUALITY FOR THE SECOND DERIVATIVE OF 4-CONVEX FUNCTIONS

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*Abstract.* In this paper, we establish an energy estimate for the second derivative of 4-convex functions. Such kinds of estimates for the first derivative of 2-convex (convex) functions were obtained by Hussain, Pečarić and Shashivili [5].

### 1. Introduction

Let  $f$  be a real valued function defined on a closed and bounded interval  $[a, b]$ , and let  $x_0 < x_1 < \dots < x_n$  be points in  $[a, b]$ . A  $n$ th order divided difference of  $f$  at these points is defined recursively as

$$[x_i]f = f(x_i) \quad (0 \leq i \leq n)$$

and

$$[x_0, x_1, \dots, x_n]f = \frac{[x_1, x_2, \dots, x_n]f - [x_0, x_1, \dots, x_{n-1}]f}{x_n - x_0}.$$

The function  $f$  is called  $n$ -convex in  $[a, b]$  if we have

$$[x_0, x_1, \dots, x_n]f \geq 0$$

for all  $a \leq x_0 < x_1 < \dots < x_n \leq b$ . We say  $f$  is  $n$ -concave if  $-f$  is  $n$ -convex. We recall some facts from the theory of  $n$ -convex functions (for the proofs see, for example, [15, 14]). 1-convex function is simply non-decreasing and a 2-convex function is convex in the usual sense. If  $f^{(n)}$  exists, then  $f$  is  $n$ -convex if and only if  $f^{(n)} \geq 0$ . For a  $n$ -convex function,  $f^{(k)}$  exists and is  $(n-k)$ -convex for  $2 \leq k \leq n$ .

The weighted integral inequalities for convex functions have applications in financial mathematics (see [6]). Therefore, it will be of interest to study these inequalities for functions having higher order convexity. In this paper, we derive a weighted integral inequality for the difference of two arbitrary 4-convex functions, not necessarily smooth. Similar weighted inequalities for convex functions have been obtained in [5, 13]. See

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also [7, 10, 11] For some related refinements and generalizations of convexity with applications in inequalities, the reader is referred to recent papers [1, 2, 3, 8, 9, 12] and references therein.

The paper is organized as follows. In the next section, we derive the inequality in a simpler case when functions are smooth, and then making use of approximation of general 4-functions by smooth 4-convex functions, we prove the inequality for general 4-convex functions in the second section.

## 2. The case of smooth 4-convex functions

**THEOREM 2.1.** *Let  $f, F \in C^4[a, b]$  be such that the following condition*

$$\left. \begin{aligned} |f''(x)| &\leq F''(x) & \forall x \in (a, b) \\ |f^{(iv)}(x)| &\leq F^{(iv)}(x) & \forall x \in (a, b) \end{aligned} \right\} \quad (2.1)$$

*is fulfilled. Let  $h$  be a non-negative 2-concave function in  $C^4[a, b]$  satisfying*

$$h(x) = h'(x) = h''(x) = h'''(x) = 0, \quad x \in \{a, b\}. \quad (2.2)$$

*Then the following energy estimate is valid:*

$$\int_a^b |f''(x)|^2 h(x) dx \leq \int_a^b \left( \frac{(f(x))^2}{2} - \|f\|_{L^\infty} F(x) \right) h^{(iv)}(x) dx. \quad (2.3)$$

*Proof.* Put

$$I = \int_a^b |f''(x)|^2 h(x) dx.$$

Then write

$$I = \int_a^b f''(x)(f''(x)h(x)) dx,$$

and integrate by parts to get

$$I = - \int_a^b f'(x)f'''(x)h(x) dx - \int_a^b f'(x)f''(x)h'(x) dx.$$

Note here that we have made use of the condition (2.2) on the weight function  $h$ . Again using integration by parts formula on first integral of right hand side, we obtain that

$$\begin{aligned} I &= \int_a^b f(x)f^{(iv)}(x)h(x) dx + \int_a^b f(x)f'''(x)h'(x) dx \\ &\quad - \int_a^b f'(x)f''(x)h'(x) dx, \end{aligned}$$

next using integration by parts formula on the middle integral yields

$$\begin{aligned} I &= \int_a^b f(x)f^{(iv)}(x)h(x)dx - 2 \int_a^b f'(x)f''(x)h'(x)dx \\ &\quad - \int_a^b f(x)f''(x)h''(x)dx \\ &= \int_a^b f(x)f^{(iv)}(x)h(x)dx - \int_a^b [(f'(x))^2]'h'(x)dx \\ &\quad - \int_a^b f(x)f''(x)h''(x)dx, \end{aligned}$$

once again using integration by parts formula on the middle integral, we achieve

$$\begin{aligned} I &= \int_a^b f(x)f^{(iv)}(x)h(x)dx + \int_a^b (f'(x))^2h''(x)dx \\ &\quad - \int_a^b f(x)f''(x)h''(x)dx \\ &= \int_a^b f(x)f^{(iv)}(x)h(x)dx + \int_a^b f'(x)(f'(x)h''(x))dx \\ &\quad - \int_a^b f(x)f''(x)h''(x)dx, \end{aligned}$$

at this point applying the integration by parts formula on the middle integral will lead us to

$$\begin{aligned} I &= \int_a^b f(x)f^{(iv)}(x)h(x)dx - 2 \int_a^b f(x)f''(x)h''(x)dx \\ &\quad - \frac{1}{2} \int_a^b [f^2(x)]'h'''(x)dx, \end{aligned}$$

next we use integration by parts formula on the last integral to obtain

$$\begin{aligned} I &= \int_a^b f(x)f^{(iv)}(x)h(x)dx - 2 \int_a^b f(x)f''(x)h''(x)dx \\ &\quad + \frac{1}{2} \int_a^b f^2(x)h^{(iv)}(x)dx, \end{aligned}$$

whence we get

$$\begin{aligned} I &\leq \sup_{a \leq x \leq b} |f(x)| \int_a^b |f^{(iv)}(x)|h(x)dx \\ &\quad + 2 \sup_{a \leq x \leq b} |f(x)| \int_a^b |f''(x)||h''(x)|dx \\ &\quad + \frac{1}{2} \int_a^b f^2(x)h^{(iv)}(x)dx, \end{aligned}$$

which further turns into

$$\begin{aligned} I &\leq \sup_{a \leq x \leq b} |f(x)| \int_a^b F^{(iv)}(x)h(x)dx \\ &\quad - 2 \sup_{a \leq x \leq b} |f(x)| \int_a^b F''(x)h''(x)dx \\ &\quad + \frac{1}{2} \int_a^b f^2(x)h^{(iv)}(x)dx, \end{aligned}$$

in view of (2.1) and the fact that  $h$  is 2-concave. Finally, using integration by parts formula four times on the first integral and twice on the second integral, we get

$$\begin{aligned} I &\leq \sup_{a \leq x \leq b} |f(x)| \int_a^b F(x)h^{(iv)}(x)dx \\ &\quad - 2 \sup_{a \leq x \leq b} |f(x)| \int_a^b F(x)h^{(iv)}(x)dx \\ &\quad + \frac{1}{2} \int_a^b f^2(x)h^{(iv)}(x)dx, \end{aligned}$$

from which follows the desired inequality (2.3).  $\square$

The following weighted energy inequality for the smooth 4-convex functions simply follows from the previous theorem by taking  $F = f$ .

**COROLLARY 2.2.** *Let  $f \in C^4[a, b]$  be 4-convex as well as 2-convex, and let  $h$  be the same as in the previous theorem. Then the following energy estimate is valid:*

$$\int_a^b |f''(x)|^2 h(x) dx \leq \int_a^b \left( \frac{(f(x))^2}{2} - \|f\|_{L^\infty} f(x) \right) h^{(iv)}(x) dx. \quad (2.4)$$

The next result describes the energy estimates for the difference of two 4-convex functions.

**COROLLARY 2.3.** *Let  $f_0, f_1 \in C^4[a, b]$  be both 4-convex as well as 2-convex functions. Let the weight function  $h$  satisfies the conditions of Theorem 2.1. Then we have*

$$\int_a^b |f_2''(x) - f_1''(x)|^2 h(x) dx \leq \left( \frac{1}{2} \|f_2 - f_1\|_{L^\infty}^2 + \|f_2 - f_1\|_{L^\infty} (\|f_1\|_{L^\infty} + \|f_2\|_{L^\infty}) \right) \|h^{(iv)}\|_{L^1}.$$

*Proof.* Apply Theorem 2.1 to  $f = f_2 - f_1$  and  $F = f_1 + f_2$  to get

$$\begin{aligned} \int_a^b |f_2''(x) - f_1''(x)|^2 h(x) dx &\leq \int_a^b \left( \frac{(f_2(x) - f_1(x))^2}{2} \right. \\ &\quad \left. - \|f_2 - f_1\|_{L^\infty} (f_2(x) + f_1(x)) \right) h^{(iv)}(x) dx, \end{aligned}$$

whence we get the desired inequality.  $\square$

We conclude this section with the following remark.

REMARK 2.4. Let  $f_0, f_1$  and  $h$  be the same as in the previous corollary. Then using Hölder inequality, we obtain that

$$\int_a^b |f_2''(x) - f_1''(x)|^2 h(x) dx \leq \|\tilde{f}\|_{L^p} \|h^{(iv)}\|_{L^q},$$

where  $\frac{1}{p} + \frac{1}{q} = 1$  and

$$\tilde{f}(x) = \frac{(f_2(x) - f_1(x))^2}{2} - \|f_2 - f_1\|_{L^\infty} (f_1(x) + f_2(x)).$$

### 3. The case of arbitrary 4-convex functions

First we define the mollification of arbitrary 4-convex function in  $[a, b]$ . The mollification of arbitrary function is very well explained in the book by L. C. Evans [4]. Let  $f$  be an arbitrary 4-convex function and also 2-convex function. Then by the property of the differentiability of the 4-convex function,  $f \in C^2[a, b]$ . Let  $\theta_\varepsilon \in C^\infty(\mathbb{R})$  have support on interval  $I_\varepsilon = I(x_0, r_\varepsilon)$ . The  $\theta$  is called approximation identity or mollifier. Take

$$\theta_\varepsilon(x) = \begin{cases} C \exp\frac{1}{x^2-1}, & x \leq 0, \\ 0, & x > 0, \end{cases}$$

where  $C$  is a constant such that

$$\int_{\mathbb{R}} \theta_\varepsilon(x) dx = 1.$$

Now using  $\theta_\varepsilon$  as a kernel, we define the convolution of  $f$  as

$$f_\varepsilon(x) = \int_{\mathbb{R}} f(x-y)\theta_\varepsilon(y)dy = \int_{\mathbb{R}} f(y)\theta_\varepsilon(x-y)dy$$

Since  $\theta_\varepsilon \in C^\infty(\mathbb{R})$ , so  $f_\varepsilon \in C^\infty(\mathbb{R})$ .

If  $f$  is continuous, then  $f_\varepsilon$  converges uniformly to  $f$  in any compact subset  $K \subseteq I$ .

$$|f_\varepsilon - f| \xrightarrow{\varepsilon \rightarrow 0} 0$$

if  $\varepsilon = \frac{1}{m}$  then  $|f_m - f| \xrightarrow{m \rightarrow \infty} 0$ .

Since

$$\begin{aligned}
 f_{\varepsilon}(\lambda x_1 + (1-\lambda)x_2) &= \int_{\mathbb{R}} f(\lambda x_1 + (1-\lambda)x_2 - y) \theta_{\varepsilon}(y) dy \\
 &= \int_{\mathbb{R}} f(\lambda(x_1 - y) + (1-\lambda)(x_2 - y)) \theta_{\varepsilon} y dy \\
 &\leq \int_{\mathbb{R}} [\lambda f(x_1 - y) + (1-\lambda)f(x_2 - y)] \theta_{\varepsilon} y dy \\
 &= \int_{\mathbb{R}} \lambda f(x_1 - y) \theta_{\varepsilon} y dy + \int_{\mathbb{R}} (1-\lambda) f(x_2 - y) \theta_{\varepsilon} y dy \\
 &= \lambda f_{\varepsilon}(x_1) + (1-\lambda) f_{\varepsilon}(x_2),
 \end{aligned}$$

so  $f_{\varepsilon}$  is 2-convex. Similarly, the 2-convexity of  $f^{(2)}$  yields the 2-convexity of  $f_{\varepsilon}^{(2)}$ . Therefore,  $f_{\varepsilon}$  is 4-convex.

**THEOREM 3.1.** *Let  $f_0, f_1$  be both 4-convex as well as 2-convex functions. Let the weight function  $h$  satisfies the conditions of Theorem 2.1. Then we have*

$$\int_a^b |f_2''(x) - f_1''(x)|^2 h(x) dx \leq \left( \frac{1}{2} \|f_2 - f_1\|_{L^\infty}^2 + \|f_2 - f_1\|_{L^\infty} (\|f_1\|_{L^\infty} + \|f_2\|_{L^\infty}) \right) \|h^{(iv)}\|_{L^1}.$$

*Proof.* Let  $I = [a, b]$ , and choose an increasing sequence  $(I_k)$  of subintervals of  $I$  such that  $\cup I_k = I$ . Consider the smooth approximations  $f_{m,i}(x)$ ,  $i = 1, 2$ . Then, for the interval  $I_{k+l}$  there exist an integer  $m_{k+l}$  such that  $f_{m,i}(x)$  converges uniformly to  $f_i(x)$ ,  $i = 1, 2$ . and also  $f_{m,i}(x)$  is smooth for  $m \geq m_{k+l}$ .

Now using the Corollary 2.3 for the functions  $f_{m,1}$  and  $f_{m,2}$  over interval  $I_{k+l}$ , we get

$$\begin{aligned}
 &\int_{I_{k+l}} |f_{m,2}''(x) - f_{m,1}''(x)|^2 h_{k+l}(x) dx \\
 &\leq c_{k+l} \left[ \frac{1}{2} \|f_{m,2} - f_{m,1}\|_{L^\infty}^2 + \|f_{m,2} - f_{m,1}\|_{L^\infty} \left( \|f_{m,1}\|_{L^\infty} + \|f_{m,2}\|_{L^\infty} \right) \right],
 \end{aligned}$$

where  $c_{k+l} = \int_{I_{k+l}} |h_{k+l}^{(iv)}(x)| dx$ . Now taking limit  $m \rightarrow \infty$ , we obtain

$$\begin{aligned}
 &\int_{I_{k+l}} |f_2''(x) - f_1''(x)|^2 h_{k+l}(x) dx \\
 &\leq c_{k+l} \left[ \frac{1}{2} \|f_2 - f_1\|_{L^\infty(I_{k+l})}^2 + \|f_2 - f_1\|_{L^\infty(I_{k+l})} \times \left( \|f_1\|_{L^\infty(I_{k+l})} + \|f_2\|_{L^\infty(I_{k+l})} \right) \right].
 \end{aligned}$$

Now writing left hand integral for the smaller interval  $I_k \subset I_{k+l}$  and also taking limit  $l \rightarrow \infty$  we obtain

$$\begin{aligned} & \int_{I_k} |f_2''(x) - f_1''(x)|^2 h(x) dx \\ & \leq c_\infty \left[ \frac{1}{2} \|f_2 - f_1\|_{L^\infty(I)}^2 + \|f_2 - f_1\|_{L^\infty(I)} \left( \|f_1\|_{L^\infty(I)} + \|f_2\|_{L^\infty(I)} \right) \right]. \end{aligned}$$

Since we have

$$\int_I |f_i''(x)|^2 h(x) dx < \infty \quad i = 1, 2,$$

Taking limit as  $k \rightarrow \infty$  we obtain the required result.  $\square$

REMARK 3.2. The previous theorem implies that if two 4-convex functions are closed in  $L^\infty$ -norm then their second derivatives are also closed in weighted  $L^2$ -norm.

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