Journal of
Mathematical
Inequalities
Volume 11, Number 2 (2017), 565–576 doi:10.7153/jmi-11-46

EXISTENCE AND UNIQUENESS SOLUTIONS FOR A CLASS OF HEMIVARIATIONAL INEQUALITIES

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(*Communicated by J. Pečarić*)

Abstract. This paper deals with the existence and uniqueness of results for a class of hemivariational inequality problem.

$$\beta_1(x,y)+\beta_2(x,y)+J^0(x;y-x)\geqslant 0.$$

Moreover, we enhance the main results an application to the existence of solution for a differential inclusion.

1. Introduction

The theory of hemivariational inequalities was introduced by P. D. Panagiotopoulos at the beginning of the 1980s (see [21]). Within a very short period of time, this theory witnessed a remarkable development in both pure and applied mathematics. It has been proved very efficient to describe a variety of mechanical problems and engineering sciences, economics, differential inclusion and optimal control (see [3], [8], [13], [18–20], [24–26]). In these papers, based on Clarke's generalized directional derivative and Clarke's generalized gradient for locally Lipchitz functions, the researchers study the existence and uniqueness of solutions by using such as fixed point Theorems, KKM Theorems, critical point Theory, surjectivity Theorems for pseudomonotone and coercive operators (see [1–2], [28]).

Recently, a number of authors have proposed many essential generalizations of monotonicity, such as α-monotonicity, relaxed monotonicity, relaxed $\Psi-\alpha$ monotonicity and quasimononicity (see [17], [23], [27], [30–31]).

The main purpose of this work is to give a new contribution in this area. In particular, we establish the existence and uniqueness of solutions for new type of hemivariational inequalities. It is worth mentioning that we do not deal with a classical technique to proof our results. Thus, several difficulties occur in finding an application to the main results, because the classical methods fail to be applied directly.

In order to achieve the aim, the study is divided into the following sections. In Section 2, we refer to some definitions and results that will assist us in the study. In Section 3, we prove the existence and uniqueness of solutions for the problem. The

Mathematics subject classification (2010): 47H04, 47H05, 47J20, 49J53, 26D15.
Keywords and phrases: Hemivariatinal inequality, monotone bifunction, fixed point theorem, differential inclusion.

Paper JMI-11-46

proof of the first result is based on arguments of α-monotone operators and a version of the well known KKM Principle due to Ky Fan [11]. However, the second result of this section relies essentially on the Schauder's fixed point Theorem. In the last section of this paper, we illustrate the applicability of our approach by a differential inclusion in the special case of our main results. We point out the fact that the results of this work can be viewed as generalization of many known results (see [9], [14], [29]).

2. Preliminaries

In the sequel unless stated otherwise, authors always assume that E is Banach space and E^* is a topological dual space of E, while $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote the duality pairing between E and E^* and norm in E^*, respectively.

For the convenience of the reader, we recall some definitions and results that need to be imposed in order to prove our main results.

We say that a functional $J : E \to \mathbb{R}$ is called *locally Lipschitz* if for every $u \in X$ there exists a neighborhood U of u and a constant $L_u > 0$ such that

$$|J(w) - J(v)| \leqslant L_u \|w - v\|_X, \quad \text{for all } v, w \in U.$$

DEFINITION 2.1. Let $J : E \to \mathbb{R}$ be a *locally Lipschitz functional*. The generalized derivative of J at $u \in E$ in the direction $v \in X$, denoted $J^0(u;v)$, is defined by

$$J^0(u;v) = \limsup_{\substack{w\to u\\ \lambda\downarrow 0}} \frac{J(w+\lambda v) - J(w)}{\lambda}.$$

The *generalized gradient* of J at $u \in E$ is defined by

$$\partial J(u) = \left\{\xi \in E^* : \langle \xi, z\rangle \leqslant J^0(u;z), \forall z \in E\right\}.$$

We point out the fact that for each $u \in E$ we have $\partial J(u) \neq \emptyset$ (see e.g., [6]).

LEMMA 2.2. *Let $J : E \to \mathbb{R}$ be locally Lipschitz of rank L_u near the point $u \in X$. Then*

(i) The function $J^0(u;\cdot)$ is finite, positively homogeneous, subadditive and satisfies

$$|J^0(u;v)| \leqslant L_u \|v\|_X,$$

(ii) $J^0(u;v)$ is upper semicontinuous as a function of (u,v),

(iii) $J^0(u;-v) = (-J)^0(u;v)$,

(iv) $J^0(u;v) = \max\{\langle \zeta, v\rangle_X, \zeta \in \partial J(u)\}$.

One can found it's proof in [8].

In 2016 in [14] introduced a new type of of monotone bifunction. They called it α-monotone bifunction, as follows:

DEFINITION 2.3. A bifunction $\beta : K \times K \to \mathbb{R}$ is called α*-monotone* if

$$\beta(x,y) + \beta(y,x) + \alpha(x,y) \leqslant 0 \quad (\forall x,y \in K). \tag{2.1}$$

REMARK 2.4. If $\alpha \equiv 0$ then from 2.1, is it follows that β is monotone; that is,

$$\beta(x,y) + \beta(y,x) \leqslant 0 \quad (\forall x,y \in K).$$

EXAMPLE 2.5. Let $E = \mathbb{R}$, $K = \mathbb{R}$ and let $\beta : K \times K \to \mathbb{R}$ be bifunction defined by

$$\beta(u,v) = \cos(u-v)^2 + (u-v)^2,$$

for all $u,v \in K$. Then

$$\beta(u,v) + \beta(v,u) = 2\cos(u-v)^2 + 2(u-v)^2 \nleqslant 0,$$

where $u \neq v$. Therefore β is not monotone bifunction.

But, it easy to see that β is α-monotone bifunction with $\alpha(u,v) = -5(u-v)^2$. In fact,

$$\begin{aligned} \beta(u,v) + \beta(v,u) &= 2\cos(u-v)^2 + 2(u-v)^2 \\ &\leqslant 5(u+v)^2 \\ &= -\alpha(u,v). \end{aligned}$$

The following notions of a KKM mapping and Schauder's fixed point theorem play an important role in the proof of main results.

DEFINITION 2.6. [16] Assume that K is a nonempty subset of a Hausdorff topological vector space E. A mapping $G : K \multimap E$ is said to be a KKM mapping for any finite subset $\{u_1, u_2, \cdots, u_n\}$ of K, we have $co\{u_1, u_2, \cdots, u_n\} \subset \bigcup_{i=1}^{n} G(u_i)$, where $co\{u_1, u_2, \cdots, u_n\}$ denotes the convex hull of $\{u_1, \cdots, u_n\}$.

LEMMA 2.7. [10] Assume that K is a nonempty subset of a Hausdorff topological vector space E and let $G : K \multimap E$ be a KKM mapping. If $G(x)$ is closed in E for every $x \in K$ and compact for some $u_0 \in K$, then $\bigcap_{u \in K} G(u) \neq \phi$.

THEOREM 2.8. [4] *Assume that K is a convex compact set in a Banach space E and that $G : K \to K$ is a continuous mapping. Then G has a fixed point in the set K.*

DEFINITION 2.9. [7] A real-valued function, defined on a convex subset K of E, is said to be *hemicontinuous*, if

$$\lim_{t \to 0^+} \Omega(tx + (1-t)y) = \Omega(y) \quad (\forall x,y \in K).$$

DEFINITION 2.10. Let X be a Banach space. A mapping $\Lambda : X \to \mathbb{R}$ is said to be

(i) *lower semicontinuous* (for short, (l.s.c)) at $x_0 \in X$, if

$$\Lambda(x_0) \leqslant \liminf_n \Lambda(x_n)$$

(ii) *upper semicontinuous* (for short, (u.s.c)) at $x_0 \in X$, if

$$\Lambda(x_0) \geqslant \limsup_n \Lambda(x_n)$$

for any sequence x_n of X such that $x_n \to x_0$.

DEFINITION 2.11. [14] Assume that E is a Banach space, and $\zeta : E \to \mathbb{R} \cup \{+\infty\}$ a proper function. One can say that $x^* \in E^*$ is a α-subdifferential of ζ in $x \in dom\zeta = \{x : \zeta(x) < \infty\}$, if

$$\partial_\alpha \zeta(x) = \Big\{x^* \in X^* : \zeta(y) - \frac{\alpha(y,x)}{2} \geqslant \zeta(x) + \langle x^*, y-x\rangle \, (\forall y \in X)\Big\}.$$

Now, we consider the following problem.

Find $x \in K$ such that

$$\beta_1(x,y) + \beta_2(x,y) + J^0(x; y-x) \geqslant 0 \ \ (\forall y \in K). \tag{2.2}$$

where $\beta_1, \beta_2 : K \times K \to \mathbb{R}$ are two real-valued bifunctions, K is a nonempty subset of a Banach space E.

In order to highlight the generality of a problem 2.2, we recall some special cases, as below:

(i) $\beta_1(x,y) = \langle Ax, y-x\rangle$ and $J \equiv \beta_2 \equiv 0$ then problem 2.2 is reduces to the standard variational inequality (see [12]).

(ii) $\beta_1(x,y) = \langle Ax, y-x\rangle$ and $\beta_2 \equiv 0$ then problem 2.2 is reduces to the hemivariational inequality (see [22]).

(iii) If $\beta_2 \equiv J \equiv 0$ then problem 2.2 is reduces to the classical equilibrium problem (for short, (EP)), which is to find $x \in K$ such that $\beta_1(x,y) \geqslant 0 \ \ (\forall y \in K)$ (see [5]).

(iv) If $J \equiv 0$ and $\beta_2(x,y) = \beta_2(y) - \beta_2(x) \ \ \forall y \in K$ then problem 2.2 is reduces to the mixed equilibrium problem (for short, (MEP)) (see [17]).

(v) If $\beta_2(x,y) \equiv 0 \ \ \forall y \in K$ then problem 2.2 is reduces to the generalized equilibrium problem (for short, (GEP)) (see [15]).

(vi) If $J \equiv 0$ then problem 2.2 is reduces to the new type of generalized equilibrium problem (for short, (EP_Ψ)) (see [14]).

Throughout this work, let us assume that $\alpha : K \times K \to \mathbb{R}$ in which

$$\lim_{\varepsilon \to 0} \frac{\alpha(x, x_\varepsilon)}{\varepsilon} = 0, \tag{2.3}$$

$$\alpha(x,y) \leqslant \lim_{\varepsilon \to 0} \frac{\varepsilon - 1}{\varepsilon}\big[\beta_2(x,x) + \alpha(x,x)\big], \tag{2.4}$$

$\forall \varepsilon \in [0,1]$.

3. Main results

In this section we establish existence and uniqueness of results for a class of hemivariational inequalities. It is worth mentioning that through the results of this section, we prove the existence of a solution of the problem 2.2 without any monotonicity assumption on β_1, nor we assume E to be a reflexive space.

LEMMA 3.1. *Let K be a nonempty subset of a real reflexive Banach space E, and $J : X \to \mathbb{R}$ be a locally Lipschitz functional. Assume that*

(i) $\beta_1 : K \times K \to \mathbb{R}$ is α-monotone bifunction, hemicontinuous in first argument, and convex in second argument, where $\beta_1(u,u) = 0$ for all $u \in K$,

(ii) $\beta_2, \alpha : K \times K \to \mathbb{R}$ is convex in second argument.

Then problem 2.2 is equivalent to the following problem:
Find a $x \in K$ such that

$$\beta_1(y,x) + \alpha(x,y) \leqslant \beta_2(x,y) + J^0(x; y-x) \quad (\forall y \in K). \tag{3.1}$$

Proof. Suppose that x is a solution of 2.2, and by definition of α-monotone bifunction,

$$\beta_1(y,x) + \beta_1(x,y) + \alpha(x,y) \leqslant 0 \quad (\forall x, y \in K), \tag{3.2}$$

Therefore, by 2.2 and 3.2 we have a solution of problem 3.1.
Conversely, assume that $x \in K$ is a solution of problem 3.1 and fix $y \in K$.
Letting $x_\lambda = x - \lambda(x-y)$, $\lambda \in]0,1[$. Then $x_\lambda \in K$, since K is a convex, so,

$$\begin{aligned} \beta_1(x_\lambda, x) + \alpha(x, x_\lambda) - \beta_2(x, x_\lambda) &\leqslant J^0(x; x_\lambda - x) \\ &= \lambda J^0(x; y-x) \end{aligned} \tag{3.3}$$

Since β_1 is convex in the second argument

$$0 = \beta_1(x_\lambda, x_\lambda) \leqslant \beta_1(x_\lambda, x) - \lambda\left[\beta_1(x_\lambda, x) - \beta_1(x_\lambda, y)\right]$$

so,

$$\lambda\left[\beta_1(x_\lambda, x) - \beta_1(x_\lambda, y)\right] \leqslant \beta_1(x_\lambda, x) \tag{3.4}$$

By the convexity of $\beta_2(x,\cdot)$ and $\alpha(x,\cdot)$

$$\alpha(x, x_\lambda) \leqslant \alpha(x,x) - \lambda\left[\alpha(x,x) - \alpha(x,y)\right] \tag{3.5}$$

$$\beta_2(x, x_\lambda) \leqslant \beta_2(x,x) - \lambda\left[\beta_2(x,x) - \beta_2(x,y)\right] \tag{3.6}$$

Then, from (3.3), (3.4), (3.5) and (3.6),

$$\begin{aligned} &\lambda\left[\beta_1(x_\lambda, x) - \beta_1(x_\lambda, y) + \alpha(x,x) - \alpha(x,y) + \beta_2(x,x) - \beta_2(x,y)\right] \\ &\leqslant \beta_1(x_\lambda, x) + \alpha(x,x) - \alpha(x, x_\lambda) + \beta_2(x,x) - \beta_2(x, x_\lambda) \\ &\leqslant \lambda J^0(x; y-x) - 2\alpha(x, x_\lambda) + \alpha(x,x) + \beta_2(x,x). \end{aligned}$$

Since $\beta_1(\cdot,y)$ is hemicontinuous,

$$\lambda\left[-\beta_1(x,y)-\beta_2(x,y)-J^0(x;y-x)\right] \leqslant -2\alpha(x,x_\lambda)+\lambda\alpha(x,y) \\ +(1-\lambda)\left[\beta_2(x,x)+\alpha(x,x)\right],$$

so

$$\beta_1(x,y)+\beta_2(x,y)+J^0(x;y-x) \geqslant \frac{2\alpha(x,x_\lambda)}{\lambda}-\alpha(x,y)+\frac{(\lambda-1)}{\lambda}\left[\beta_2(x,x)+\alpha(x,x)\right].$$

From (2.3) and (2.4),

$$\beta_1(x,y)+\beta_2(x,y)+J^0(x;y-x) \geqslant 0 \ \ (\forall y \in K).$$

Therefore, 2.2 admits at least one solution. □

The first main result of this work is given by the following theorem.

THEOREM 3.2. *Assume that K is a nonempty closed bounded convex subset of a real reflexive Banach space E. Let β_1, β_2, $\alpha : K \times K \to \mathbb{R}$ be three bifunctions, where $\beta_n(x,x)=0 \ \forall x \in K$, $n \in \{1,2\}$. If the following conditions hold*

(i) β_1 is α-monotone bifunction, hemicontinuous in first argument, and l.s.c , convex in second argument,

(ii) β_2 is convex in second argument, u.s.c in first argument,

(iii) α is convex in second argument and l.s.c in first argument,

then problem 2.2 admits at least one solution.

Proof. Consider two set valued mappings $\Psi,\Gamma : K \multimap K$ defined by

$$\Psi(y)=\left\{x\in K : \beta_1(x,y)+\beta_2(x,y)+J^0(x;y-x)\geqslant 0 \ \ (\forall y\in K)\right\}$$

$$\Gamma(y)=\{x\in K : \beta_1(y,x)+\alpha(x,y)\leqslant \beta_2(x,y)+J^0(x;y-x) \ \ (\forall y\in K)\}.$$

Then, $\forall x \in K$ the problem 2.2 has a solution iff $\bigcap_{y\in K}\Psi(y)\neq\emptyset$. Now, we show that Ψ is a KKM-mapping. On the contrary, Ψ is not a KKM-mapping. Then, there exists a finite subset $\{x_1,x_2,\cdots,x_n\}$ of K and $\lambda_i \geqslant 0$ for every $i=\overline{1,n}$ with $\sum_{i=1}^{n}\lambda_i=1$ such that $x_0=\sum_{i=1}^{n}\lambda_i x_i \notin \bigcup_{i=1}^{n}\Psi(x_i)$. Then

$$\beta_1(x_0,x_i)+\beta_2(x_0,x_i)+J^0(x_0;x_i-x_0)<0$$

for every $i=\overline{1,n}$.

By convexity of β_n $\forall n \in \{1,2\}$, and the fact that $J^0(x_0;\cdot)$ is asubadditive,

$$\begin{aligned}
0 &= \beta_1(x_0,x_0)+\beta_2(x_0,x_0)+J^0(x_0;x_0-x_0)\\
&= \beta_1(x_0,\sum_{i=1}^{n}\lambda_i x_i)+\beta_2(x_0,\sum_{i=1}^{n}\lambda_i x_i)+J^0(x_0;\sum_{i=1}^{n}\lambda_i x_i-x_0)\\
&\leqslant \sum_{i=1}^{n}\lambda_i\left[\beta_1(x_0,x_i)+\beta_2(x_0,x_i)+J^0(x_0;x_i-x_0)\right]\\
&< 0,
\end{aligned}$$

for every $i=\overline{1,n}$.

This is a contradiction. Therefore Ψ is a KKM-mapping. Next, From Lemma 3.1. $\Psi(y)\subset\Gamma(y)$ $(\forall y\in K)$. Therefore, $\Gamma(y)$ is a KKM-mapping. Since $\alpha(.,y)$, $\beta_1(y,.)$ are l.s.c , $\beta_2(.,y)$ is u.s.c, and Lemma 2.2 (ii), then

$$\begin{aligned}
\beta_1(y,x)+\alpha(x,y) &\leqslant \liminf_n\left[\beta_1(y,x_n)+\alpha(x_n,y)\right]\\
&\leqslant \limsup_n\left[\beta_2(x_n,y)+J^0(x_n;y-x_n)\right]\\
&\leqslant \beta_2(x,y)+J^0(x;y-x).
\end{aligned}$$

Therefore, $\Gamma(y)$ is a weakly closed $\forall y\in K$. Since K is nonempty, bounded, closed and convex and X is real reflexive, then K is weakly compact. Hence, $\Gamma(y)$ is weakly compact $\forall y\in K$. From Lemma 2.7 and Lemma 3.1.

$$\bigcap_{y\in K}\Psi(y)=\bigcap_{y\in K}\Gamma(y)\neq\phi.$$

So, any element of this intersection is a solution. Therefore, the problem 2.2 has a solution. □

For uniqueness of solutions we present the next result.

THEOREM 3.3. *In addition to the assumptions $(i-iii)$ in Theorem 3.2 hold. We assume that the following hypotheses are fulfilled:*

$\boldsymbol{H}_1$: *there exists $M_n>0$ such that $\beta_n(u,v)+\beta_n(v,u)+M_n\|v_2-v_1\|^2\leqslant 0$ for all $u,v\in K$, $n\in\{1,2\}$.*

$\boldsymbol{H}_2$: *there exists a positive constant $S\leqslant M_n$ such that $|J^0(u;v)|\leqslant S\|v\|^2$, $\forall n\in\{1,2\}$.*

Then problem 2.1 has a unique solution.

Proof. Towards to a contradiction, let us assume that $x_1,x_2\in K$ be two solutions to problem 2.1. So, if write in the problem 2.1 for x_2 with $x=x_1$, we have

$$\beta_1(x_1,x_2)+\beta_2(x_1,x_2)+J^0(x_1;x_2-x_1)\geqslant 0. \tag{3.7}$$

and then for x_1 with $x = x_2$,

$$\beta_1(x_2,x_1)+\beta_2(x_2,x_1)+J^0(x_2;x_1-x_2)\geqslant 0. \tag{3.8}$$

Taking $M=\min\{M_1,M_2\}$ and multiplying each of the equations 3.7 and 3.8 by -1 and summing together,

$$\begin{aligned}0 &\geqslant -\beta_1(x_1,x_2)-\beta_1(x_2,x_1)-\beta_2(x_1,x_2)-\beta_2(x_2,x_1)-J^0(x_1;x_2-x_1)-J^0(x_2;x_1-x_2)\\ &\geqslant M_1\|x_2-x_1\|^2+M_2\|x_2-x_1\|^2-|J^0(x_1;x_2-x_1)|-|J^0(x_2;x_1-x_2)|\\ &\geqslant (2M-2S)\|x_2-x_1\|^2.\end{aligned}$$

which shows that $\|x_2-x_1\|^2\leqslant 0$ since $M-S\geqslant 0$. Consequently, $x_1=x_2\in K$. $\square$

In the next result, we prove the problem (2.1) admits at least one solution in the case K is a compact convex subset of E without using any monotonicity conditions on β_1 in a Banach space E.

THEOREM 3.4. *Assume that K is a nonempty compact convex subset of the Banach space E. If bifunctions $\beta_1,\beta_2\colon K\times K\to\mathbb{R}$ are convex in second argument and u.s.c in first argument, then the problem* (2.1) *admits at least one solution.*

Proof. Towards to a contradiction, we assume that problem (2.1) has no solution. Then, for each $x\in K$, there exists $y\in K$ such that

$$\beta_1(x,y)+\beta_2(x,y)+J^0(x,y-x)<0. \tag{3.9}$$

Let us define the set-valued mapping $\eta : K \multimap K$ as follows:

$$\eta(y):=\big\{x\in K:\beta_1(x,y)+\beta_2(x,y)+J^0(x,y-x)\geqslant 0\big\}.$$

Claim 1. $\eta(y)$ is a nonempty and closed for each $y\in K$.

Clearly, $\eta(y)$ is nonempty since $y\in\eta(y)$ for each $y\in K$ according to definition of set η. Assume that $\{x_n\}_{n\geqslant 1}\subset\eta(y)$ is a sequence which converges weakly to x. We must prove that $x\in\eta(y)$. For each $n\geqslant 1$,

$$\beta_1(x_n,y)+\beta_2(x_n,y)+J^0(x_n,y-x_n)\geqslant 0.$$

Since $\beta_i(\cdot,y)$ is u.s.c $\forall i\in\{1,2\}$, and take into account Lemma 2.2 (ii), then $x\in\eta(y)$ for each $n\geqslant 1$. To do this, passing to $\limsup$ as $n\to\infty$ in 2.18 we have

$$\beta_1(x,y)+\beta_2(x,y)+J^0(x,y-x)\geqslant 0.$$

Therefore, $x\in\eta(y)$, and $\eta(y)$ is a weakly closed subset of K.

According to 3.9 for each $x\in K$, there exists $y\in K$ such that $x\in[\eta(y)]^c=X-\eta(y)$. Therefore, the family $\{[\eta(y)]^c\}$ is an open covering of the compact set K, for each $y\in K$. This means that there exists a finite subset $\{y_1,y_2,\cdots,y_N\}$ of K such that $\{[\eta(y_r)]^c\}$ is a finite subcover of K for every $r=\overline{1,N}$.

Assume that $\Theta_r(x) := \mathrm{dis}(x;\eta(y_r))$ (i.e., the distance between x and the set $\eta(y_r)$) for every $r = \overline{1,N}$ and let $S_r : K \to \mathbb{R}$ be a function defined as follows:

$$S_r(x) := \frac{\Theta_r(x)}{\sum\limits_{i=1}^{N} \Theta_i(x)}.$$

Notice that S_r is a Lipschitz continuous function for every $r = \overline{1,N}$, $S_r(x) \in [0,1]$, for all $x \in K$ and $\sum\limits_{r=1}^{N} S_r(x) = 1$. Let $\Upsilon : K \to K$ be a mapping defined by:

$$\Upsilon(x) := \sum_{r=1}^{N} S_r(x) y_r.$$

Claim 2: The mapping Υ is continuous.

To do this, we can obtain for any $x_1, x_2 \in K$ the following estimation:

$$\begin{aligned}
\|\Upsilon(x_1) - \Upsilon(x_2)\| &= \| \sum_{r=1}^{N} (S_r(x_1) - S_r(x_2))\, y_r\| \\
&\leqslant \sum_{r=1}^{N} \|y_r\| \|S_r(x_1) - S_r(x_2)\| \\
&\leqslant J_r \sum_{r=1}^{N} \|y_r\| \|x_1 - x_2\| \\
&\leqslant J\|x_1 - x_2\|.
\end{aligned}$$

This shows that Υ is continuous map. Taking into account Theorem 2.8, there exists $x_0 \in K$ such that $\Upsilon(x_0) = x_0$.

Let us consider the functional $\chi : K \to \mathbb{R}$ as follows:

$$\chi(x) := \beta_1(x,\Upsilon(x)) + \beta_2(x,\Upsilon(x)) + J^0(x,\Upsilon(x) - x).$$

Applying Lemma 2.2 , $\beta_n(x,\cdot)$ is convex $\forall n \in \{1,2\}$,

$$\begin{aligned}
\chi(x) &= \beta_1(x, \sum_{r=1}^{N} S_r(x) y_r) + \beta_2(x, \sum_{r=1}^{N} S_r(x) y_r) + J^0(x, \sum_{r=1}^{N} S_r(x)(y_r - x)) \\
&\leqslant \sum_{r=1}^{N} S_r(x) \big[\beta_1(x,y_r) + \beta_2(x,y_r) + J^0(x,y_r - x)\big].
\end{aligned}$$

On the other hand, since $K \subset \bigcup\limits_{r=1}^{N} \left[\eta(y_r)\right]^c$ for every $r = \overline{1,N}$, there exists at least one index $r_0 = \overline{1,N}$ such that $x \in \left[\eta(y_{r_0})\right]^c$. This shows that $\chi(u) < 0$ for all $y \in K$ which contradicts the fact that $\chi(u_0) = 0$. $\square$

REMARK 3.5. Notice that the solutions of hemiequilibrium inequality on unbounded domains exist if we expand the conditions for the bounded domains with a coercivity condition. As, if we put some coercivity conditions, it will ensure that Theorem 3.2 or Theorem 3.4 will also satisfy when the set K is unbounded (for more details, see [9] and [31]).

4. Application

It is important to say that there has been an increased interest in differential problems governed by higher order operators. In this section, we apply our main result, expressed in Theorem 3.2, to some partial differential inclusion problems. Let us consider the usual Sobolev space as $W^{1,p}(\Omega)$ and the Banach $W^{-1,p'}(\Omega)$ is dual space of $W^{1,p}(\Omega)$, where $\frac{1}{p}+\frac{1}{p'}=1$. $p>1$ is a real constant, and Ω is a bounded domain of $\mathbb{R}^N$, $N \geqslant 1$ with smooth boundary $\partial\Omega$.

In what follows, let us consider the partial differential inclusion problem

$$\begin{cases} -w-g(x) \in \partial_{-2\alpha} f(u), & x \in \Omega \\ u=0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

such that $w \in \partial J(u)$, $g:\Omega \to \mathbb{R}$ is continuous with compact support. $f:K \to \mathbb{R}$ is a continuous concave function and K is a bounded convex subset of Sobolev space $W^{1,p}(\Omega)$. For technical reasons, let us define $\beta_2 : W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega) \to \mathbb{R}$ as follows:

$$\beta_2(\xi,\rho) := \int_{\Omega} g(x)(\xi-\rho)(x)dx.$$

We suppose that K is a nonempty, closed, bounded and convex subset of Sobolev space $W_0^{1,p}(\Omega)$.

DEFINITION 4.1. We say that $u \in K$ has a K-weak subsolution of the problem (4.1) if,

$$\langle -w-g(x), u-v\rangle \leqslant f(u)-f(v)-\alpha(u,v) \quad (\forall v \in K). \tag{4.2}$$

Here, we show that the K-weak solvability of (4.2) By the integral form of $\langle\cdot,\cdot\rangle$, one can get that

$$\langle w, v-u\rangle - \int_{\Omega} g(x)(u-v)(x)dx + \alpha(u,v) \leqslant f(u)-f(v).$$

Set $\beta_1(v,u) := f(v)-f(u)$ and $\beta_2(u,v) := \int_{\Omega} g(x)(u-v)(x)dx$. Therefore, we obtain

$$\beta_1(v,u)+\alpha(u,v) \leqslant \beta_2(u,v)+J^0(u;v-u) \quad (\forall v \in K)$$

Consideration $\alpha(u,v) = -(u-v)^2$. Then β_1 is α-monotone bifunction. In Lemma 3.1, we proved that (3.5) and (2.1) are equivalent under some assumptions. Therefore, we must prove that β_1 and β_2 hold all assumptions of Theorem 3.2.

It is clear that the bifunction $\alpha(u,v) = -(u-v)^2$ satisfies all assumptions in Theorem 3.2.

Claim 1: β_2 is a convex in second argument.
Let $z = tv_1 + (1-t)v_2 \in K$, $t \in [0,1]$, so

$$\begin{aligned}
\beta_2(u,z) &= \beta_2(u, tv_1 + (1-t)v_2) \\
&= \int_\Omega g(x)(u - tv_1 - (1-t)v_2))(x)dx \\
&= \int_\Omega g(x)u(x)dx - \int_\Omega g(x)(tv_1 + (1-t)v_2)(x)dx \\
&= \int_\Omega g(x)(tu + (1-t)u)(x)dx - \int_\Omega g(x)tv_1(x)dx - \int_\Omega g(x)(1-t)v_2(x)dx \\
&= t\int_\Omega g(x)(u(x) - v_1(x))dx + (1-t)\int_\Omega g(x)(u(x) - v_2(x))dx \\
&= t\beta_2(u,v_1) + (1-t)\beta_2(u,v_2).
\end{aligned}$$

Moreover, if $u_n \to u \in W_0^{1,p}(\Omega)$. By Sobolev embedding we can assume that $u_n \to u \in L^p(\Omega)$

$$\begin{aligned}
\left|\beta_2(u_n,v) - \beta_2(u,v)\right| &= \Big| \int_\Omega g(x)(u_n(x) - u(x))dx \Big| \\
&\leqslant \Big(\int_\Omega |g(x)|^{p'}\Big)^{\frac{1}{p'}} . \Big(\int_\Omega |u_n(x) - u(x)|^p\Big)^{\frac{1}{p}} \\
&\leqslant \Upsilon \|u_n - u\|_{L^p} \leqslant \Upsilon \|u_n - u\| \rightharpoonup 0.
\end{aligned}$$

This shows that β_2 is continuous. So, it is *u.s.c* in the first argument.

Notice that β_1 is *hemicontinuous* in first argument, *l.s.c* and convex in second argument, because f is concave and continuous function. Therefore, all conditions of Theorem 3.2 satisfied.

REFERENCES

[1] B. ALLECHE, V. RĂDULESCU, M. SEBAOUI, *The Tikhonov regularization for equilibrium problems and applications to quasi-hemivariationalinequalities*, Optim., **9** (2015) 483–503.

[2] I. ANDREI AND N. COSTEA, *Nonlinear hemivariational inequalities and applications to nonsmooth mechanics*, Adv. Nonlinear Var. Inequal. **13** (2010) 1–17.

[3] J. P. AUBIN, AND F. H. CLARKE, *Shadow prices and duality for a class of optimal control problems*, SIAM J. Control Optim. **17** (1979) 567–586.

[4] M. BERGER, *Nonlinearity and Functional Analysis*, Academic Press, New York(1977).

[5] E. BLUM AND W. OETTLI, *From optimization and variational inequalities to equilibrium problems*, The Mathematics Student **63** (1994) 123–145.

[6] H. BREZIS, *Analyse Fonctionnelle: Théorie et Applications*, Masson, Paris (1992).

[7] F. E. BROWDER, *The solvability of non-linear functional equations*, Duke Math. J. **30** (1963) 557–566.

[8] F. H. CLARKE, *Optimization and Nonsmooth Analysis*, Wiley (1983).

[9] N. COSTEA AND V. RĂDULESCU, *Existence results for hemivariational inequalities involving relaxed $\eta-\alpha$ monotone mappings*, Commun. Appl. Anal. **13** (2009) 293–304.

[10] K. FAN, *A generalization of Tychonoffs fixed point theorem*, Math. Ann. **142** (1961) 305–310.

[11] Y. P. FANG AND N. J. HUANG, *Variational-like inequalities with generalized monotone mappings in Banach spaces*, J. Optim. Theory Appl. **118** (2003) 327–338.

[12] P. HARTMAN AND G. STAMPACCHIA, *On some nonlinear elliptic differential functional equations*, Acta Math. **115** (1966) 271–310.

[13] A. E. HASHOOSH AND M. ALIMOHAMMADY, *On Well-Posed of Generalized Equilibrium Problems Involving α-Monotone Bifunction*, Journal of Hyperstructures **5** (2016), 151–168.

[14] A. E. HASHOOSH, M. ALIMOHAMMADY AND M. K. KALLEJI, *Existence Results for Some Equilibrium Problems involving α-Monotone Bifunction*, International Journal of Mathematics and Mathematical Sciences, **2016** (2016) 1–5.

[15] U. KAMRAKSA AND R. WANGKEEREE, *Generalized equilibrium problems and fixed point problems for nonexpansive semigroups in Hilbert spaces*, Journal of Global Optimization, **51** (2011) 689–714.

[16] B. KNASTER, K. KURATOWSKI AND S. MAZURKIEWICZ, *Ein Beweis des Fixpunktsatzes fur n-dimensionale Simplexe*, Fund. Math. **14** (1929) 132–137.

[17] N. K. MAHATO AND C. NAHAK, *Mixed equilibrium problems with relaxed α-monotone mapping in Banach spaces*, Rendiconti del Circolo Matematico di Palermo, (2013).

[18] D. MOTREANU, P. D. PANAGIOTOPOULOS, *Minimax theorems and qualitative properties of the solutions of hemivariational inequalities*, Nonconvex Optimization and its Applications **29**, Kluwer Academic Publishers, Dordrecht, 1999.

[19] D. MOTREANU, V. RĂDULESCU, *Variational and non-variational methods in nonlinear analysis and boundary value problems*, Nonconvex Optimization and its Applications **67**, Kluwer Academic Publishers, Dordrecht, 2003.

[20] Z. NANIEWICZ, P. D. PANAGIOTOPOULOS, *Mathematical theory of hemivariational inequalities and applications*, Monographs and Textbooks in Pure and Applied Mathematics **188**, Marcel Dekker, Inc., New York, 1995.

[21] P. D. PANAGIOTOPOULOS, *Nonconvex energy functions. Hemivariational inequalities and substationarity principles*, Acta Mech. **42** (1983) 160–183.

[22] P. D. PANAGIOTOPOULOS, M. R. FUNDO AND V. RĂDULESCU, *Existence theorems of Hartman-Stampacchia type for hemivariational inequalities and applications*, J. Global Optim. **15** (1999) 41–54.

[23] J. W. PENG AND J. YAO, *A viscosity approximation scheme for system of equilibrium problems,nonexpansive mappings and monotone mappings*, Nonlinear Anal. **71** (2009) 6001–6010.

[24] V. RĂDULESCU, D. REPOVŠ, *Existence results for variational-hemivariational problems with lack of convexity*, Nonlinear Anal. **73** (2010) 99–104.

[25] V. RĂDULESCU, D. REPOVS, *Partial Differential Equations with Variable Exponents: Variational Methods and Qualitative Analysis*, CRC Press, Taylor Francis Group, Boca Raton FL (2015).

[26] D. REPOVŠ AND C. VARGA, *A Nash type solutions for hemivariational inequality systems*, Nonlinear Analysis **74** (2011), 5585–5590.

[27] A. TADA AND W. TAKAHASHI, *Weak and strong convergence theorems for a nonexpansive mapping and equilibrium problem*, J. Optim. Theory Appl. **133** (2007) 359–370.

[28] E. TARAFDAR, *A fixed point theorem equivalent to the Fan-Knaster-Kuratowski-Mazurkiewicz Theorem*, J. Math. Anal. Appl. **2** (1987) 475–479.

[29] R. U. VERMA, *A-monotonicity and its role in nonlinear variational inclusions*, J. Optim. Theory Appl. **129** (2006) 457–467.

[30] R. U. VERMA, *On generalized variational inequalities involving relaxed Lipschitz and relaxed monotone operators*, Journal of Mathematical Analysis and Applications, **213** (1997) 387–392.

[31] R. U. VERMA, *On monotone nonlinear variational inequality problems*, Commentationes Mathematicae Universitatis Carolinae, **39** (1998) 91–98.

(Received August 8, 2016)

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