Journal of Mathematical Inequalities
Volume 11, Number 2 (2017), 577–594 doi:10.7153/jmi-11-47

SOME NEW ITERATED HARDY–TYPE INEQUALITIES AND APPLICATIONS

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(*Communicated by A. Kalybay*)

Abstract. We characterize some iterated Hardy inequalities previously studied by R. Oinarov and A. Kalybay and, as an application, we give new simpler proofs of the characterizations of the weighted inequalities for the bilinear Hardy operator.

1. Introduction and results

Let u, v and w weights, i.e., positive and locally integrable functions on (a,b) with $-\infty \leqslant a < b \leqslant +\infty$. In [8], R. Oinarov and A. Kalybay characterize the weights u, v and w such that the inequality

$$\left(\int_a^b \left(\int_x^b \left(\int_x^t f\right)^q w(t)\,dt\right)^{r/q} u(x)\,dx\right)^{\frac{1}{r}} \leqslant C\left(\int_a^b f^p v\right)^{\frac{1}{p}} \tag{1.1}$$

holds for all nonnegative functions f on (a,b) in the case $0<q<\infty$ and $1 \leqslant p \leqslant r < \infty$. For $p \in (0,\infty]$, we write

$$||f||_{p,w,(a,b)} = \begin{cases} \left(\int_a^b |f(x)|^p w(x)\,dx\right)^{1/p} & \text{if } 0<p<\infty, \\ ess\sup_{a<x<b} |f(x)|w(x) & \text{if } p=\infty. \end{cases}$$

When $w \equiv 1$, we simply write $||f||_{p,(a,b)}$. With this notation we can write the above inequality as

$$\left|\left|\left|\left|\int_x^t f(s)\,ds\right|\right|_{q,w,(x,b)}\right|\right|_{r,u,(a,b)} \leqslant C||f||_{p,v,(a,b)}. \tag{1.2}$$

For $(\alpha,\beta) \subset (a,b)$ we write

$$\underline{v}(\alpha,\beta) = ess \inf_{\alpha<t<\beta} v(t) \qquad \text{and} \qquad U(\alpha,\beta) = \left(\int_\alpha^\beta u(s)\,ds\right)^{1/r}.$$

Mathematics subject classification (2010): 26D10, 26D15, 42B99.
Keywords and phrases: Bilinear operators, Hardy type operators, weighted inequalities, weights.
This research has been partially supported by Spanish Government, Ministerio de Economía y Competitividad (grants no. MTM2011-28149-C02-02 and MTM2015-66157-C2-2-P) and Junta de Andalucía (grant FQM354). First author was partially supported by CAI+D+UNL, CONICET (Argentina).

If $p' = \frac{p}{p-1}$ when $1 < p < \infty$ and $p' = \infty$ when $p = 1$, we define

$$A^{p,q}_{(v,w)}(\alpha,\beta) = \sup_{\alpha<x<\beta} \left(\int_x^\beta w(s)\,ds\right)^{1/q} \left(\int_\alpha^x v^{1-p'}(s)\,ds\right)^{1/p'}.$$

Finally, if $q < p$ and $\frac{1}{\theta} = \frac{1}{q} - \frac{1}{p}$, we define

$$B^{p,q}_{(v,w)}(\alpha,\beta) = \left\{\int_\alpha^\beta \left(\int_x^\beta w(s)\,ds\right)^{\theta/p} \left(\int_\alpha^x v^{1-p'}(s)\,ds\right)^{\theta/p'} w(x)\,dx\right\}^{1/\theta}$$

and

$$B^{1,q}_{(v,w)}(\alpha,\beta) = \left\{\int_\alpha^\beta \left(\int_x^\beta w(s)\,ds\right)^{q/(1-q)} [\underline{v}(\alpha,x)]^{q/(q-1)} w(x)\,dx\right\}^{(1-q)/q}.$$

The following theorem collects the results obtained in [8].

THEOREM 1.1. ([8]) *The inequality* (1.1) *holds if and only if*

(i) $F = \sup_{a<z<b} U(a,z) A^{p,q}_{(v,w)}(z,b) < \infty$ *for* $1 \leqslant p \leqslant \min\{q,r\} < \infty$*;*

(ii) $F = \sup_{a<z<b} U(a,z) B^{p,q}_{(v,w)}(z,b) < \infty$ *for* $0 < q < p$ *and* $1 < p \leqslant r < \infty$*.*

(iii) $F = \sup_{a<z<b} U(a,z) B^{1,q}_{(v,w)}(z,b) < \infty$ *for* $0 < q < 1 = p \leqslant r < \infty$ *and* $\underline{v}(\alpha,\beta) > 0$ *for any* α,β *such that* $a < \alpha < \beta < b$*.*

Moreover, $F \approx C$*, where* C *is the best constant in* (1.1)*.*

Let us observe that reversing the orientation in the real line it is easy to obtain from the above theorem the corresponding characterization for the inequality

$$\left\| \left\| \int_t^x f(s)\,ds \right\|_{q,w,(a,x)} \right\|_{r,u,(a,b)} \leqslant C\|f\|_{p,v,(a,b)}.$$

It is well known that condition $A^{p,q}_{(v,u)}(a,b) < \infty$ characterizes the weighted Hardy inequality

$$\left(\int_a^b \left(\int_a^x f\right)^q u(x)dx\right)^{\frac{1}{q}} \leqslant C\left(\int_a^b f^p v\right)^{\frac{1}{p}} \tag{1.3}$$

in the case $1 < p \leqslant q < \infty$ (see [7] and [2]) and $B^{p,q}_{(v,u)}(a,b) < \infty$ characterizes this inequality in the case $0 < q < p$ and $1 < p < \infty$ (see [6]).

Let us observe that if $r = +\infty$ to prove inequality (1.2) is equivalent to prove that for almost every $x \in (a,b)$,

$$\left(\int_x^b \left(\int_x^t f(s)\,ds\right)^q \tilde{w}_x(t)\,dt\right)^{\frac{1}{q}} \leqslant C\|f\|_{p,v},$$

with a constant C independent of f and x, where $\tilde{w}_x(t) = w(t)u(x)^q$. But this is equivalent to prove that the pair of weights $(v, \tilde{w}_x)$ verifies the necessary and sufficient condition for the operator $H_x f(t) = \int_x^t f$ to be bounded from $L^p((x,b),v)$ to $L^q((x,b),\tilde{w}_x)$, i.e. $A^{p,q}_{(v,\tilde{w}_x)}(x,b) < \infty$ if $p \leqslant q$ or $B^{p,q}_{(v,\tilde{w}_x)}(x,b) < \infty$ if $q < p$. Then, we get the following result.

THEOREM 1.2. *Let $r = +\infty$ and $1 < p,q < \infty$. The inequality (1.2) holds if and only if*

(i) $F = \sup_{a<z<b} u(z) A^{p,q}_{(v,w)}(z,b) < \infty$ for $p \leqslant q$;

(ii) $F = \sup_{a<z<b} u(z) B^{p,q}_{(v,w)}(z,b) < \infty$ for $q < p$.

Moreover, $F \approx C$, where C is the best constant in (1.2).

Motivated by the close connection between the inequality (1.2) and the weighted bilinear Hardy inequality (see (1.4) below), we worked in the case $r < p$ of (1.2) obtaining the desired result.

To state the result we shall need the following definition. We say that $\{x_k\}$ is a covering sequence if $\{x_k\}$ is a decreasing sequence in (a,b) such that $\lim_{k\to\infty} x_k = a$ and $\lim_{k\to-\infty} x_k = b$. We also admit decreasing sequences $\{x_k\}_{k=K}^{k=J}$, where either $J \in \mathbb{Z}$ and $x_J = a$, or $K \in \mathbb{Z}$ and $x_K = b$, or both. Our result is the following one.

THEOREM 1.3. *Let $0 < q,r < \infty$ and $1 < p < \infty$. Let us consider the following cases: (a) $0 < r < p$ and $1 < p \leqslant q$ and (b) $\max\{r,q\} < p$. Let $\frac{1}{\eta} = \frac{1}{r} - \frac{1}{p}$. The following statements are equivalent.*

(i) The inequality (1.1) holds for all $f \geqslant 0$.

(ii)

$$F_1 = \sup_{\{x_k\}} \left\{ \sum_k [D^{p,q}_{(v,w)}(x_k, x_{k-1})]^\eta \left(\int_{x_{k+1}}^{x_k} u \right)^{\eta/r} \right\}^{1/\eta} < \infty$$

and

$$F_2 = \sup_{\{x_k\}} \left\{ \sum_k \left(\int_{x_{k-1}}^b w \right)^{\eta/q} \left(\int_{x_k}^{x_{k-1}} v^{1-p'} \right)^{\eta/p'} \left(\int_{x_{k+1}}^{x_k} u \right)^{\eta/r} \right\}^{1/\eta} < \infty,$$

where the suprema are taken over all covering sequences $\{x_k\}$.

(iii)

$$F = \left\{ \int_a^b [D^{p,q}_{(v,w)}(x,b)]^\eta [U(a,x)]^{\eta r/p} u(x)\, dx \right\}^{1/\eta} < \infty,$$

where $D^{p,q}_{(v,w)} = A^{p,q}_{(v,w)}$ in case (a) and $D^{p,q}_{(v,w)} = B^{p,q}_{(v,w)}$ in case (b).

Moreover, $F \approx C$, where C is the best constant in (1.1).

Recently, we learned of a new article of R. Oinarov and A. Kalybay [9] where the authors studied the case $r < p$ of the inequality (1.2). They prove $(iii) \Rightarrow (i)$ in Theorem 1.3 and $(i) \Rightarrow (iii)$ only for the case (b) $\max\{r,q\} < p$. The proof of the implication $(iii) \Rightarrow (i)$ is the same in this paper and in [9]. However the implication $(i) \Rightarrow (iii)$ is different even in the case (b). The introduction of an intermediate condition defined in terms of covering sequences, allows us to obtain the complete result.

In [1], M. I. Aguilar Cañestro, P. Ortega Salvador and C. Ramírez Torreblanca have studied a bilinear version of the Hardy operator defined, for pairs of nonnegative measurable functions (f,g) on (a,b), $-\infty \leqslant a < b \leqslant \infty$, by

$$\mathscr{H}(f,g)(x) = \left(\int_a^x f(t)\,dt\right)\left(\int_a^x g(t)dt\right).$$

The purpose of the paper [1] was to characterize the positive measurable functions w, w_1, w_2 such that the weighted bilinear Hardy inequality

$$\left(\int_a^b (\mathscr{H}(f,g)(x))^q w(x)dx\right)^{\frac{1}{q}} \leqslant C\left(\int_a^b f^{p_1} w_1\right)^{\frac{1}{p_1}}\left(\int_a^b g^{p_2} w_2\right)^{\frac{1}{p_2}} \tag{1.4}$$

holds for all pairs of nonnegative measurable functions (f,g) with a positive constant C independent of f and g, where $q, p_1, p_2 > 1$.

The proofs of the necessity of the characterizing conditions in [1] were based on reducing the bilinear problem to a linear one. However, this was not the philosophy in the proofs of the sufficiency of the conditions, specially in the cases $q < p_1$, $q < p_2$, where splitting and discretization techniques were applied. Our second purpose in this article is to provide new simpler proofs of the sufficiency of the conditions by means of reducing the problem to prove a weighted Hardy inequality with variable weight and then applying the weighted results for the classical Hardy operator. This process leads us to apply the weighted iterated Hardy inequality characterized in Theorems 1.1, 1.2 and 1.3. In passing, we reduce the number of characterizing conditions and we obtain the results also for $0 < q < 1$. We think that to give simpler proofs of the weighted bilinear Hardy inequalities is of interest, because the techniques applied in [1] are difficult to handle in order to get new results on bilinear Hardy operators, such as bilinear weak type inequalities or bilinear Hardy-type inequalities with Oinarov kernels.

We will use the following notation:

$$Hf(x) = \int_a^x f(t)\,dt \qquad \text{and} \qquad \tilde{H}f(x) = \int_x^b f(t)\,dt.$$

For $i = 1,2$, let $\sigma_i = w_i^{1-p_i'}$ and, if $q < p_i$, let r_i such that $\dfrac{1}{r_i} = \dfrac{1}{q} - \dfrac{1}{p_i}$. If $q < p_1, p_2$, let s such that $\dfrac{1}{s} = \dfrac{1}{r_1} - \dfrac{1}{p_2} = \dfrac{1}{q} - \dfrac{1}{p_1} - \dfrac{1}{p_2}$. With the following notation we describe the conditions on the weights for the different cases of relationships between the parameters:

$$\mathscr{A} = \sup_{a<x<b} \left(\tilde{H}w(x)\right)^{\frac{1}{q}} (H\sigma_1(x))^{\frac{1}{p_1'}} (H\sigma_2(x))^{\frac{1}{p_2'}},$$

$$\mathscr{A}\mathscr{B}^{p_1;p_2,q}_{w_1;(w_2,w)} = \sup_{a<x<b} \left(H\sigma_1(x)\right)^{\frac{1}{p_1'}} B^{p_2,q}_{w_2,w}(x,b),$$

and

$$\mathscr{B}\mathscr{B}^{p_1;p_2,q}_{w_1;(w_2,w)} = \left\{ \int_a^b \left[B^{p_2,q}_{(w_2,w)}(x,b)\right]^s \left(H\sigma_1(x)\right)^{\frac{s}{r_2'}} \sigma_1(x)dx \right\}^{\frac{1}{s}}.$$

With $\mathscr{A}\mathscr{B}^{p_2;p_1,q}_{w_2;(w_1,w)}$ we shall denote the term $\mathscr{A}\mathscr{B}^{p_1;p_2,q}_{w_1;(w_2,w)}$ but interchanging the roles of w_1 and w_2 and the roles of p_1 and p_2. The same with $\mathscr{B}\mathscr{B}^{p_2;p_1,q}_{w_2;(w_1,w)}$ with respect to $\mathscr{B}\mathscr{B}^{p_1;p_2,q}_{w_1;(w_2,w)}$.

The result for the bilinear Hardy operator is the following.

THEOREM 1.4. ([1]) *Let $q>0$ and $p_1,p_2>1$. Let w,w_1,w_2 be positive measurable functions defined on (a,b) and let $\sigma_i = w_i^{1-p_i'}$, $i=1,2$. Then there exists a positive constant C such that the inequality (1.4) holds for all nonnegative measurable functions f and g if and only if*

(i) $\mathscr{A}<\infty$, *for* $q \geqslant p_1$ *and* $q \geqslant p_2$;

(ii) $\mathscr{A}\mathscr{B}^{p_1;p_2,q}_{w_1;(w_2,w)} < \infty$, *for* $q \geqslant p_1$ *and* $q < p_2$;

(iii) $\mathscr{A}\mathscr{B}^{p_1;p_2,q}_{w_1;(w_2,w)}, \mathscr{A}\mathscr{B}^{p_2;p_1,q}_{w_2;(w_1,w)} < \infty$, *for* $q<p_1$, $q<p_2$ *and* $\dfrac{1}{p_1}+\dfrac{1}{p_2} \geqslant \dfrac{1}{q}$;

(iv) $\mathscr{B}\mathscr{B}^{p_1;p_2,q}_{w_1;(w_2,w)}, \mathscr{B}\mathscr{B}^{p_2;p_1,q}_{w_2;(w_1,w)} < \infty$, *for* $q<p_1$, $q<p_2$ *and* $\dfrac{1}{p_1}+\dfrac{1}{p_2} < \dfrac{1}{q}$.

In the next section we shall prove Theorem 1.3. Although the implication $(iii) \Rightarrow (i)$ is very similar to that given in [9], we include the complete proof of Theorem 1.3 to make the paper self-contained. In Section 3 we will show how to apply the iterated Hardy inequalities in order to prove Theorem 1.4.

2. Proof of Theorem 1.3

$(iii) \Rightarrow (i)$. Working as in the proof of Theorem 2.1 in [8] we define

$$x_k = \inf\left\{ x \in (a,b) : \int_x^b \left(\int_x^t f(s)\,ds\right)^q w(t)\,dt \leqslant 2^{qk} \right\}.$$

The sequence $\{x_k\}$ decreases and verifies $(a,b) = \cup_k (x_{k+1}, x_k]$. Now, if $1 \leqslant q < \infty$,

$$\begin{aligned} 2^{k-1} &= \left(\int_{x_k}^b \left(\int_{x_k}^t f\right)^q w(t)\,dt\right)^{1/q} - \left(\int_{x_{k-1}}^b \left(\int_{x_{k-1}}^t f\right)^q w(t)\,dt\right)^{1/q} \\ &\leqslant \left(\int_{x_k}^{x_{k-1}} \left(\int_{x_k}^t f\right)^q w(t)\,dt\right)^{1/q} + \left(\int_{x_{k-1}}^b w\right)^{1/q} \left(\int_{x_k}^{x_{k-1}} f\right). \end{aligned}$$

In the case $0<q<1$ we get that

$$\begin{aligned} 2^{q(k-1)} &= \frac{1}{2^q-1}\left(2^{qk}-2^{q(k-1)}\right) \\ &= \frac{1}{2^q-1}\left\{\int_{x_k}^b\left(\int_{x_k}^t f\right)^q w(t)\,dt - \int_{x_{k-1}}^b\left(\int_{x_{k-1}}^t f\right)^q w(t)\,dt\right\} \\ &\leqslant \frac{1}{2^q-1}\left\{\int_{x_k}^{x_{k-1}}\left(\int_{x_k}^t f\right)^q w(t)\,dt + \left(\int_{x_{k-1}}^b w\right)\left(\int_{x_k}^{x_{k-1}} f\right)^q\right\}. \end{aligned}$$

Then,

$$\begin{aligned} T &= \int_a^b\left(\int_x^b\left(\int_x^t f(s)\,ds\right)^q w(t)\,dt\right)^{r/q} u(x)\,dx \\ &= \sum_k \int_{x_{k+1}}^{x_k}\left(\int_x^b\left(\int_x^t f(s)\,ds\right)^q w(t)\,dt\right)^{r/q} u(x)\,dx \\ &\leqslant \sum_k 2^{r(k+1)}\int_{x_{k+1}}^{x_k} u(x)\,dx \\ &= 2^{2r}\sum_k 2^{r(k-1)}\int_{x_{k+1}}^{x_k} u(x)\,dx. \end{aligned}$$

If $q \geqslant 1$,

$$\begin{aligned} T &\leqslant C\sum_k\left(\int_{x_k}^{x_{k-1}}\left(\int_{x_k}^t f\right)^q w(t)\,dt\right)^{r/q}\int_{x_{k+1}}^{x_k} u(x)\,dx \\ &\quad + C\sum_k\left(\int_{x_{k-1}}^b w\right)^{r/q}\left(\int_{x_k}^{x_{k-1}} f\right)^r\int_{x_{k+1}}^{x_k} u(x)\,dx = T_1+T_2. \end{aligned}$$

If $0<q<1$,

$$\begin{aligned} T &\leqslant C\sum_k\left(2^{q(k-1)}\right)^{r/q}\int_{x_{k+1}}^{x_k} u(x)\,dx \\ &\leqslant C\sum_k\left\{\int_{x_k}^{x_{k-1}}\left(\int_{x_k}^t f\right)^q w(t)\,dt + \left(\int_{x_{k-1}}^b w\right)\left(\int_{x_k}^{x_{k-1}} f\right)^q\right\}^{r/q}\int_{x_{k+1}}^{x_k} u(x)\,dx \\ &\leqslant C\sum_k\left(\int_{x_k}^{x_{k-1}}\left(\int_{x_k}^t f\right)^q w(t)\,dt\right)^{r/q}\int_{x_{k+1}}^{x_k} u(x)\,dx \\ &\quad + C\sum_k\left(\int_{x_{k-1}}^b w\right)^{r/q}\left(\int_{x_k}^{x_{k-1}} f\right)^r\int_{x_{k+1}}^{x_k} u(x)\,dx = T_1+T_2. \end{aligned}$$

Condition $F<\infty$ implies that $D^{p,q}_{(v,w)}(x_k,x_{k-1})<\infty$ for all k and then, the Hardy operator $H_kf(t)=\int_{x_k}^t f$ is bounded from $L^p((x_k,x_{k-1}),v)$ to $L^q((x_k,x_{k-1}),w)$, i.e., there is a constant C_k such that

$$\left(\int_{x_k}^{x_{k-1}}\left(\int_{x_k}^{t} f\right)^q w(t)\,dt\right)^{1/q} \leqslant C_k\left(\int_{x_k}^{x_{k-1}} f^p v\right)^{1/p}$$

and $C_k \approx C_{p,q} D^{p,q}_{(v,w)}(x_k,x_{k-1})$, where $C_{p,q}$ is a constant independent of k and only dependent on q and p. Then we have

$$\begin{aligned}
T_1 &= C\sum_k \frac{\left(\int_{x_k}^{x_{k-1}}\left(\int_{x_k}^{t} f\right)^q w(t)\,dt\right)^{r/q}}{\left(\int_{x_k}^{x_{k-1}} f^p v\right)^{r/p}}\left(\int_{x_k}^{x_{k-1}} f^p v\right)^{r/p}\int_{x_{k+1}}^{x_k} u(x)\,dx\\
&\leqslant C\sum_k [D^{p,q}_{(v,w)}(x_k,x_{k-1})]^r\left(\int_{x_k}^{x_{k-1}} f^p v\right)^{r/p}\int_{x_{k+1}}^{x_k} u(x)\,dx\\
&\leqslant C\sum_k [U(x_{k+1},x_k)D^{p,q}_{(v,w)}(x_k,b)]^r\left(\int_{x_k}^{x_{k-1}} f^p v\right)^{r/p}.
\end{aligned}$$

On the other hand, by Hölder inequality,

$$T_2 \leqslant C\sum_k\left(\int_{x_{k-1}}^{b} w\right)^{r/q}\left(\int_{x_k}^{x_{k-1}} f^p v\right)^{r/p}\left(\int_{x_k}^{x_{k-1}} v^{1-p'}\right)^{r/p'}\int_{x_{k+1}}^{x_k} u(x)\,dx.$$

For the case $p \leqslant q$, clearly

$$T_2 \leqslant C\sum_k [U(x_{k+1},x_k)A^{p,q}_{(v,w)}(x_k,b)]^r\left(\int_{x_k}^{x_{k-1}} f^p v\right)^{r/p}.$$

If $q < p$, observe that since $\frac{1}{\theta} = \frac{1}{q} - \frac{1}{p}$,

$$\left(\int_{x_{k-1}}^{b} w\right)^{1/q} = C\left(\int_{x_{k-1}}^{b}\left(\int_x^b w\right)^{\frac{\theta}{p}} w(x)\,dx\right)^{\frac{1}{\theta}}.$$

Then

$$\begin{aligned}
&\left(\int_{x_{k-1}}^{b} w\right)^{1/q}\left(\int_{x_k}^{x_{k-1}} v^{1-p'}\right)^{1/p'}\\
&= C\left(\int_{x_{k-1}}^{b}\left(\int_x^b w\right)^{\frac{\theta}{p}}\left(\int_{x_k}^{x_{k-1}} v^{1-p'}\right)^{\frac{\theta}{p'}} w(x)\,dx\right)^{\frac{1}{\theta}}\\
&\leqslant C\left(\int_{x_{k-1}}^{b}\left(\int_x^b w\right)^{\frac{\theta}{p}}\left(\int_{x_k}^{x} v^{1-p'}\right)^{\frac{\theta}{p'}} w(x)\,dx\right)^{\frac{1}{\theta}}\\
&\leqslant C\left(\int_{x_k}^{b}\left(\int_x^b w\right)^{\frac{\theta}{p}}\left(\int_{x_k}^{x} v^{1-p'}\right)^{\frac{\theta}{p'}} w(x)\,dx\right)^{\frac{1}{\theta}} = CB^{p,q}_{(v,w)}(x_k,b).
\end{aligned}$$

Therefore,

$$T_2 \leqslant C\sum_k [U(x_{k+1},x_k)B^{p,q}_{(v,w)}(x_k,b)]^r \left(\int_{x_k}^{x_{k-1}} f^p v\right)^{r/p}.$$

Putting together the estimates of T_1 and T_2 and applying Hölder inequality with exponents $p/r>1$ and η/r we have that

$$T \leqslant C\left(\int_a^b f^p v\right)^{r/p}\left\{\sum_k [U(x_{k+1},x_k)D^{p,q}_{(v,w)}(x_k,b)]^\eta\right\}^{\frac{r}{\eta}}.$$

Observe that

$$\begin{aligned}
[U(x_{k+1},x_k)D^{p,q}_{(v,w)}(x_k,b)]^\eta &= \left(\int_{x_{k+1}}^{x_k} u\right)^{\frac{\eta}{r}} [D^{p,q}_{(v,w)}(x_k,b)]^\eta \\
&= C[D^{p,q}_{(v,w)}(x_k,b)]^\eta \int_{x_{k+1}}^{x_k}\left(\int_{x_{k+1}}^{x} u\right)^{\frac{\eta}{r}-1} u(x)\,dx \\
&= C\int_{x_{k+1}}^{x_k} [D^{p,q}_{(v,w)}(x_k,b)]^\eta \left(\int_{x_{k+1}}^{x} u\right)^{\frac{\eta}{p}} u(x)\,dx \\
&\leqslant C\int_{x_{k+1}}^{x_k} [D^{p,q}_{(v,w)}(x,b)]^\eta \left(\int_{a}^{x} u\right)^{\frac{\eta}{p}} u(x)\,dx.
\end{aligned}$$

Then, summing up in k, we obtain

$$T \leqslant C\left(\int_a^b f^p v\right)^{r/p}\left\{\int_a^b [D^{p,q}_{(v,w)}(x,b)]^\eta U(a,x)^{\frac{r\eta}{p}} u(x)\,dx\right\}^{\frac{r}{\eta}} = CF^r\left(\int_a^b f^p v\right)^{r/p}.$$

$(i)\Rightarrow(ii)$. Let $\{x_k\}$ be a covering sequence of (a,b). Clearly,

$$\begin{aligned}
T &= \int_a^b\left(\int_x^b\left(\int_x^t f(s)\,ds\right)^q w(t)\,dt\right)^{r/q} u(x)\,dx \\
&= \sum_k \int_{x_{k+1}}^{x_k}\left(\int_x^b\left(\int_x^t f(s)\,ds\right)^q w(t)\,dt\right)^{r/q} u(x)\,dx \\
&\geqslant \sum_k \left(\int_{x_k}^{x_{k-1}}\left(\int_{x_k}^t f\right)^q w(t)\,dt\right)^{r/q}\int_{x_{k+1}}^{x_k} u(x)\,dx.
\end{aligned}$$

Applying (i) we get

$$\sum_k \frac{\left(\int_{x_k}^{x_{k-1}}\left(\int_{x_k}^t f\right)^q w(t)\,dt\right)^{r/q}}{\left(\int_{x_k}^{x_{k-1}} f^p v\right)^{r/p}}\left(\int_{x_k}^{x_{k-1}} f^p v\right)^{r/p}\left(\int_{x_{k+1}}^{x_k} u\right) \leqslant C\left(\int_a^b f^p v\right)^{r/p}. \quad (2.1)$$

Let $\gamma > 1$. Using the characterization of the boundedness of the Hardy operator $H_k f(t) = \int_{x_k}^t f$ from $L^p((x_k, x_{k-1}), v)$ to $L^q((x_k, x_{k-1}), w)$, for each k there is a non negative function f_k defined on (x_k, x_{k-1}) such that

$$D^{p,q}_{(v,w)}(x_k, x_{k-1}) < \gamma \frac{\left(\int_{x_k}^{x_{k-1}} \left(\int_{x_k}^t f_k\right)^q w(t)\,dt\right)^{1/q}}{\left(\int_{x_k}^{x_{k-1}} f_k^p v\right)^{1/p}}.$$

Let $\{a_k\}$ be a sequence of non negative numbers and let f be the function defined by $f = \sum_k \chi_{(x_k, x_{k-1})} a_k f_k$. With this function f in (2.1) we obtain

$$\frac{1}{\gamma^r} \sum_k [D^{p,q}_{(v,w)}(x_k, x_{k-1})]^r a_k^r \left(\int_{x_k}^{x_{k-1}} f_k^p v\right)^{r/p} \left(\int_{x_{k+1}}^{x_k} u\right) \leqslant C \left(\sum_k a_k^p \int_{x_k}^{x_{k-1}} f_k^p v\right)^{r/p}.$$

Using the notation $v_k = \int_{x_k}^{x_{k-1}} f_k^p v$ and $w_k = [D^{p,q}_{(v,w)}(x_k, x_{k-1})]^r \left(\int_{x_k}^{x_{k-1}} f_k^p v\right)^{r/p} \left(\int_{x_{k+1}}^{x_k} u\right)$, the above inequality can be written as

$$\left(\sum_k a_k^r w_k\right)^{1/r} \leqslant C\gamma \left(\sum_k a_k^p v_k\right)^{1/p}.$$

By Proposition 4.1 in [3] we get that $\left(\sum_k w_k^{\eta/r} v_k^{-\eta/p}\right)^{1/\eta} \leqslant C\gamma$, i.e.,

$$\left\{\sum_k [D^{p,q}_{(v,w)}(x_k, x_{k-1})]^\eta \left(\int_{x_{k+1}}^{x_k} u\right)^{\eta/r}\right\}^{1/\eta} \leqslant C\gamma.$$

Letting γ tend to 1 and having into account that the covering sequence $\{x_k\}$ is arbitrary, we get that $F_1 < +\infty$. On the other hand, for any covering sequence $\{x_k\}$ we have that

$$\begin{aligned} T &= \int_a^b \left(\int_x^b \left(\int_x^t f(s)\,ds\right)^q w(t)\,dt\right)^{r/q} u(x)\,dx \\ &= \sum_k \int_{x_{k+1}}^{x_k} \left(\int_x^b \left(\int_x^t f(s)\,ds\right)^q w(t)\,dt\right)^{r/q} u(x)\,dx \\ &\geqslant \sum_k \left(\int_{x_{k-1}}^b w\right)^{r/q} \left(\int_{x_k}^{x_{k-1}} f\right)^r \int_{x_{k+1}}^{x_k} u(x)\,dx. \end{aligned}$$

Let $\{a_k\}$ be a sequence of non negative numbers and let f be the function defined by $f = \sum_k \chi_{(x_k, x_{k-1})} a_k v^{1-p'}$. Applying (i) to this function f, we get

$$\sum_k a_k^r \left(\int_{x_{k-1}}^b w\right)^{r/q} \left(\int_{x_k}^{x_{k-1}} v^{1-p'}\right)^r \left(\int_{x_{k+1}}^{x_k} u\right) \leqslant C \left(\sum_k a_k^p \int_{x_k}^{x_{k-1}} v^{1-p'}\right)^{r/p}.$$

If $v_k = \int_{x_k}^{x_{k-1}} v^{1-p'}$ and

$$w_k = \left(\int_{x_{k-1}}^{b} w\right)^{r/q} \left(\int_{x_k}^{x_{k-1}} v^{1-p'}\right)^{r} \left(\int_{x_{k+1}}^{x_k} u\right)$$

we get that $\left(\sum_k a_k^r w_k\right)^{1/r} \leqslant C\left(\sum_k a_k^p v_k\right)^{1/p}$. Then, as above, $\left(\sum_k w_k^{\eta/r} v_k^{-\eta/p}\right)^{1/\eta} \leqslant C$ and since this inequality holds for all covering sequences $\{x_k\}$, we obtain $F_2 < +\infty$.

$(ii) \Rightarrow (iii)$. We define two sequences: $\{x_k\}$ and $\{y'_s\}$ where

$$\int_a^{x_k} u = 2^{-k} \quad \text{and} \quad [D_{(v,w)}^{p,q}(y'_s, b)]^{p'} = 2^s.$$

According to a principle similar to that which was introduced by Q. Lai in [5], we reduce the sequence $\{y'_s\}$ to a subsequence $\{y_n\}$ in the following way: if $(y'_{s+1}, y'_s] \cap \{x_k\} = \emptyset$ then we delete the term y'_s from the sequence $\{y'_s\}$. Then, if $y_{n+1} < x_{k+1} \leqslant y_n = y'_s$, we get that $y'_{s+1} \leqslant x_{k+1}$ and $y_{n+2} \leqslant x_{k+2}$. Now,

$$\begin{aligned} F^{\eta} &= \int_a^b \left[D_{(v,w)}^{p,q}(x,b)\right]^{\eta} \left(\int_a^x u\right)^{\eta/p} u(x)\,dx \\ &\leqslant \sum_k \left[D_{(v,w)}^{p,q}(x_{k+1},b)\right]^{\eta} \left(\int_{x_{k+1}}^{x_k} \left(\int_a^x u\right)^{\eta/p} u(x)\,dx\right). \end{aligned}$$

Then, for k such that $y_{n+1} < x_{k+1} \leqslant y_n = y'_s$ we have

$$\begin{aligned} [D_{(v,w)}^{p,q}(x_{k+1},b)]^{p'} &\leqslant [D_{(v,w)}^{p,q}(y'_{s+1},b)]^{p'} \\ &= 4[D_{(v,w)}^{p,q}(y'_s,b)]^{p'} - 4[D_{(v,w)}^{p,q}(y'_{s-1},b)]^{p'}. \end{aligned}$$

If $p \leqslant q$ we get that

$$\begin{aligned} &[D_{(v,w)}^{p,q}(x_{k+1},b)]^{p'} = [A_{(v,w)}^{p,q}(x_{k+1},b)]^{p'} \\ &\leqslant 4 \sup_{y'_s<x<b} \left(\int_{y'_s}^x v^{1-p'}\right)\left(\int_x^b w\right)^{p'/q} - 4 \sup_{y'_{s-1}<x<b} \left(\int_{y'_{s-1}}^x v^{1-p'}\right)\left(\int_x^b w\right)^{p'/q} \\ &\leqslant 4 \sup_{y'_s<x<y'_{s-1}} \left(\int_{y'_s}^x v^{1-p'}\right)\left(\int_x^b w\right)^{p'/q} + 4\left(\int_{y'_s}^{y'_{s-1}} v^{1-p'}\right)\left(\int_{y'_{s-1}}^b w\right)^{p'/q} \\ &\leqslant C \sup_{y'_s<x<y'_{s-1}} \left(\int_{y'_s}^x v^{1-p'}\right)\left(\int_x^b w\right)^{p'/q} + C\left(\int_{y'_s}^{y'_{s-1}} v^{1-p'}\right)\left(\int_{y'_{s-1}}^{y_{n-2}} w\right)^{p'/q} \\ &\quad + C\left(\int_{y'_s}^{y'_{s-1}} v^{1-p'}\right)\left(\int_{y_{n-2}}^b w\right)^{p'/q} \\ &\leqslant C \sup_{y_n<x<y_{n-2}} \left(\int_{y_n}^x v^{1-p'}\right)\left(\int_x^{y_{n-2}} w\right)^{p'/q} + C\left(\int_{y_n}^{y_{n-2}} v^{1-p'}\right)\left(\int_{y_{n-2}}^b w\right)^{p'/q}. \end{aligned}$$

On the other hand, if $q<p$, using the notation $\bar{w}(t)=\left(\int_t^b w\right)^{\theta/p} w(t)$, we get that

$$
\begin{aligned}
&[D^{p,q}_{(v,w)}(x_{k+1},b)]^{p'}=[B^{p,q}_{(v,w)}(x_{k+1},b)]^{p'}\\
&\leqslant 4\left(\int_{y'_s}^b\left(\int_{y'_s}^t v^{1-p'}\right)^{\theta/p'}\bar{w}(t)\,dt\right)^{p'/\theta}-4\left(\int_{y'_{s-1}}^b\left(\int_{y'_{s-1}}^t v^{1-p'}\right)^{\theta/p'}\bar{w}(t)\,dt\right)^{p'/\theta}\\
&\leqslant 4\left(\int_{y'_s}^{y'_{s-1}}\left(\int_{y'_s}^t v^{1-p'}\right)^{\theta/p'}\bar{w}(t)\,dt\right)^{p'/\theta}+4\left(\int_{y'_{s-1}}^b\left(\int_{y'_s}^{y'_{s-1}} v^{1-p'}\right)^{\theta/p'}\bar{w}(t)\,dt\right)^{p'/\theta}\\
&\leqslant 4\left(\int_{y'_s}^{y'_{s-1}}\left(\int_{y'_s}^t v^{1-p'}\right)^{\theta/p'}\bar{w}(t)\,dt\right)^{p'/\theta}+4\left(\int_{y'_{s-1}}^{y_{n-2}}\left(\int_{y'_s}^{y'_{s-1}} v^{1-p'}\right)^{\theta/p'}\bar{w}(t)\,dt\right)^{p'/\theta}\\
&+4\left(\int_{y'_s}^{y'_{s-1}} v^{1-p'}\right)\left(\int_{y_{n-2}}^b \bar{w}\right)^{p'/\theta}\\
&\leqslant C\left(\int_{y_n}^{y_{n-2}}\left(\int_{y_n}^t v^{1-p'}\right)^{\theta/p'}\bar{w}(t)\,dt\right)^{p'/\theta}+C\left(\int_{y_n}^{y_{n-2}} v^{1-p'}\right)\left(\int_{y_{n-2}}^b \bar{w}\right)^{p'/\theta}.
\end{aligned}
$$

Then, since

$$\int_{x_{k+1}}^{x_k}\left(\int_a^x u\right)^{\eta/p} u(x)\,dx\leqslant C\left(\int_a^{x_k} u\right)^{\eta/r}\approx\left(\int_{x_{k+2}}^{x_{k+1}} u\right)^{\eta/r}$$

and

$$\sum_{y_{n+1}<x_{k+1}\leqslant y_n}\int_{x_{k+2}}^{x_{k+1}} u\leqslant\int_{y_{n+2}}^{y_n} u,$$

we get, in the case $p\leqslant q$, that

$$
\begin{aligned}
F^\eta\leqslant C\sum_n&\left(\sup_{y_n<x<y_{n-2}}\left(\int_{y_n}^x v^{1-p'}\right)^{1/p'}\left(\int_x^{y_{n-2}} w\right)^{1/q}\right)^\eta\left(\sum_{y_{n+1}<x_{k+1}\leqslant y_n}\int_{x_{k+2}}^{x_{k+1}} u\right)^{\eta/r}\\
&+C\sum_n\left(\int_{y_n}^{y_{n-2}} v^{1-p'}\right)^{\eta/p'}\left(\int_{y_{n-2}}^b w\right)^{\eta/q}\left(\sum_{y_{n+1}<x_{k+1}\leqslant y_n}\int_{x_{k+2}}^{x_{k+1}} u\right)^{\eta/r}\\
\leqslant{}& C\sum_n\left[A^{p,q}_{(v,w)}(y_n,y_{n-2})\right]^\eta\left(\int_{y_{n+2}}^{y_n} u\right)^{\eta/r}\\
&+C\sum_n\left(\int_{y_n}^{y_{n-2}} v^{1-p'}\right)^{\eta/p'}\left(\int_{y_{n-2}}^b w\right)^{\eta/q}\left(\int_{y_{n+2}}^{y_n} u\right)^{\eta/r}=I_a+II_a.
\end{aligned}
$$

Then, clearly $I_a\leqslant C\sum_{n=2m}\cdots+C\sum_{n=2m+1}\cdots\leqslant CF_1^\eta$ and $II_a\leqslant C\sum_{n=2m}\cdots+C\sum_{n=2m+1}\cdots\leqslant CF_2^\eta$.

On the other hand, for the case $q < p$,

$$F^\eta \leqslant C\sum_n \left(\int_{y_n}^{y_{n-2}} \left(\int_{y_n}^{t} v^{1-p'} \right)^{\theta/p'} \bar{w}(t)\,dt \right)^{\eta/\theta} \left(\int_{y_{n+2}}^{y_n} u \right)^{\eta/r}$$
$$+C\sum_n \left(\int_{y_n}^{y_{n-2}} v^{1-p'} \right)^{\eta/p'} \left(\int_{y_{n-2}}^{b} \bar{w} \right)^{\eta/\theta} \left(\int_{y_{n+2}}^{y_n} u \right)^{\eta/r} = I_b + II_b.$$

Notice that, since $\bar{w}(t) = \left(\int_t^b w \right)^{\theta/p} w(t)$, integrating by parts we obtain

$$\begin{aligned}\int_{y_n}^{y_{n-2}} \left(\int_{y_n}^{t} v^{1-p'} \right)^{\theta/p'} \bar{w}(t)\,dt &\leqslant \int_{y_n}^{y_{n-2}} \left(\int_{y_n}^{t} v^{1-p'} \right)^{\theta/q'} \left(\int_{t}^{b} w \right)^{\theta/q} v^{1-p'}(t)\,dt \\ &\leqslant C\int_{y_n}^{y_{n-2}} \left(\int_{y_n}^{t} v^{1-p'} \right)^{\theta/q'} \left(\int_{t}^{y_{n-2}} w \right)^{\theta/q} v^{1-p'}(t)\,dt \\ &\quad +C\left(\int_{y_{n-2}}^{b} w \right)^{\theta/q} \int_{y_n}^{y_{n-2}} \left(\int_{y_n}^{t} v^{1-p'} \right)^{\theta/q'} v^{1-p'}(t)\,dt \\ &\approx \int_{y_n}^{y_{n-2}} \left(\int_{y_n}^{t} v^{1-p'} \right)^{\theta/q'} \left(\int_{t}^{y_{n-2}} w \right)^{\theta/q} v^{1-p'}(t)\,dt \\ &\quad +\left(\int_{y_{n-2}}^{b} w \right)^{\theta/q} \left(\int_{y_n}^{y_{n-2}} v^{1-p'} \right)^{\theta/p'}.\end{aligned}$$

We also get that

$$\int_{y_{n-2}}^{b} \bar{w} \approx \left(\int_{y_{n-2}}^{b} w \right)^{\theta/q}.$$

Therefore,

$$I_b \leqslant C\sum_n \left(\int_{y_n}^{y_{n-2}} \left(\int_{y_n}^{t} v^{1-p'} \right)^{\theta/q'} \left(\int_{t}^{y_{n-2}} w \right)^{\theta/q} v^{1-p'}(t)\,dt \right)^{\eta/\theta} \left(\int_{y_{n+2}}^{y_n} u \right)^{\eta/r}$$
$$+C\sum_n \left(\int_{y_{n-2}}^{b} w \right)^{\eta/q} \left(\int_{y_n}^{y_{n-2}} v^{1-p'} \right)^{\eta/p'} \left(\int_{y_{n+2}}^{y_n} u \right)^{\eta/r} = I_b^1 + I_b^2,$$

and

$$II_b \leqslant C\sum_n \left(\int_{y_{n-2}}^{b} w \right)^{\eta/q} \left(\int_{y_n}^{y_{n-2}} v^{1-p'} \right)^{\eta/p'} \left(\int_{y_{n+2}}^{y_n} u \right)^{\eta/r}.$$

Finally, let us notice that

$$I_b^1 = C\sum_n [B_{(v,w)}^{p,q}(y_n, y_{n-2})]^\eta \left(\int_{y_{n+2}}^{y_n} u \right)^{\eta/r} = C\sum_{n=2m} \cdots + C\sum_{n=2m+1} \cdots \leqslant CF_1^\eta$$

and $II_b \leqslant I_b^2 \leqslant C\sum_{n=2m} \cdots + C\sum_{n=2m+1} \cdots \leqslant CF_2^\eta$.

3. Proof of Theorem 1.4

(i). The necessity of condition $\mathscr{A}$ follows as in [1]. Let us see now that condition $\mathscr{A}$ is sufficient. Clearly, to prove (1.4) is equivalent to prove

$$\left(\int_a^b (Hf)(x))^q w_g(x)dx\right)^{\frac{1}{q}} \leqslant C\left(\int_a^b f^{p_1} w_1\right)^{\frac{1}{p_1}}, \tag{3.1}$$

where

$$w_g(x) = (H(g/||g||_{p_2,w_2})(x))^q w(x)$$

and the constant C does not depend on g.

We know that, since $p_1 \leqslant q$, (3.1) is equivalent to the next condition:

$$A^{p_1,q}_{(w_1,w_g)}(a,b) = \sup_{a<x<b} \left(\tilde{H}w_g(x)\right)^{\frac{1}{q}} (H\sigma_1(x))^{\frac{1}{p_1'}} < \infty, \tag{3.2}$$

uniformly on g.

Observe that if $a<x<b$ then

$$\begin{aligned}
\left(\tilde{H}w_g(x)\right)^{\frac{1}{q}} (H\sigma_1(x))^{\frac{1}{p_1'}} &= \left(\int_x^b w_g(t)\,dt\right)^{\frac{1}{q}} (H\sigma_1(x))^{\frac{1}{p_1'}} \\
&= \left(\int_x^b \left(\int_a^t g(s)/||g||_{p_2,w_2}\,ds\right)^q w(t)\,dt\right)^{\frac{1}{q}} (H\sigma_1(x))^{\frac{1}{p_1'}} \\
&\leqslant \left(\int_x^b \left(\int_a^x g(s)/||g||_{p_2,w_2}\,ds\right)^q w(t)\,dt\right)^{\frac{1}{q}} (H\sigma_1(x))^{\frac{1}{p_1'}} \\
&\quad + \left(\int_x^b \left(\int_x^t g(s)/||g||_{p_2,w_2}\,ds\right)^q w(t)\,dt\right)^{\frac{1}{q}} (H\sigma_1(x))^{\frac{1}{p_1'}} \\
&= I + II.
\end{aligned}$$

Applying Hölder inequality,

$$\begin{aligned}
I &= \left(\int_x^b w(t)\,dt\right)^{\frac{1}{q}} \left(\int_a^x g(s)/||g||_{p_2,w_2}\,ds\right) (H\sigma_1(x))^{\frac{1}{p_1'}} \\
&\leqslant \left(\int_x^b w(t)\,dt\right)^{\frac{1}{q}} (H\sigma_2(x))^{\frac{1}{p_2'}} (H\sigma_1(x))^{\frac{1}{p_1'}} \leqslant \mathscr{A}.
\end{aligned}$$

On the other hand, if

$$w_x(t) = (H\sigma_1(x))^{\frac{q}{p_1'}} w(t),$$

we have to prove that

$$II = \left(\int_x^b \left(\int_x^t g(s)/||g||_{p_2,w_2}\,ds\right)^q w_x(t)\,dt\right)^{\frac{1}{q}} \leqslant C,$$

or equivalently $\left(\int_x^b \left(\int_x^t g(s)\,ds\right)^q w_x(t)\,dt\right)^{\frac{1}{q}} \leqslant C||g||_{p_2,w_2}$, with a constant C independent of g and x, which is (1.1) with $r=\infty$, $p=p_2$, $v=w_2$ and $u(x)=(H\sigma_1(x))^{\frac{q}{p_1'}}$. Then, the above inequality holds if and only if

$$\sup_{a<x<b} (H\sigma_1(x))^{\frac{q}{p_1'}} A^{p_2,q}_{(w_2,w)}(x,b) < \infty. \tag{3.3}$$

But since for all $t\in(x,b)$ we have that

$$\left(\int_t^b w(s)\,ds\right)^{\frac{1}{q}} \left(\int_a^x \sigma_1(s)\,ds\right)^{\frac{1}{p_1'}} \left(\int_x^t \sigma_2(s)\,ds\right)^{\frac{1}{p_2'}} \leqslant \mathscr{A},$$

we get that condition $\mathscr{A}<\infty$ implies condition (3.3).

(ii). Let us see first that condition $\mathscr{A}\mathscr{B}^{p_1;p_2,q}_{w_1;(w_2,w)}$ is sufficient for (1.4) to hold. Working as in the proof of the above case we have to see that $A^{p_1,q}_{(w_1,w_g)}(a,b)<\infty$. Clearly, as in the proof of (i),

$$\left(\tilde{H}w_g(x)\right)^{\frac{1}{q}} (H\sigma_1(x))^{\frac{1}{p_1'}} \leqslant I+II$$

and $I\leqslant\mathscr{A}$. Let us see now that $\mathscr{A}\leqslant\mathscr{A}\mathscr{B}^{p_1;p_2,q}_{w_1;(w_2,w)}$. Since $1/r_2=1/q-1/p_2$, if $a<x<b$, then

$$\begin{aligned}
\mathscr{A}(x) &= (H\sigma_1(x))^{\frac{1}{p_1'}} \left(\tilde{H}w(x)\right)^{\frac{1}{q}} (H\sigma_2(x))^{\frac{1}{p_2'}} \\
&= (H\sigma_1(x))^{\frac{1}{p_1'}} \left\{\left(\tilde{H}w(x)\right)^{\frac{r_2}{q}} (H\sigma_2(x))^{\frac{r_2}{p_2'}}\right\}^{1/r_2} \\
&= (H\sigma_1(x))^{\frac{1}{p_1'}} \left\{\left(\tilde{H}w(x)\right)^{\frac{r_2}{p_2}+1} (H\sigma_2(x))^{\frac{r_2}{p_2'}}\right\}^{1/r_2} \\
&= C\,(H\sigma_1(x))^{\frac{1}{p_1'}} \left\{\int_x^b \left(\tilde{H}w(t)\right)^{\frac{r_2}{p_2}} (H\sigma_2(x))^{\frac{r_2}{p_2'}} w(t)\,dt\right\}^{1/r_2} \\
&\leqslant C\,(H\sigma_1(x))^{\frac{1}{p_1'}} \left\{\int_x^b \left(\tilde{H}w(t)\right)^{\frac{r_2}{p_2}} (H\sigma_2(t))^{\frac{r_2}{p_2'}} w(t)\,dt\right\}^{1/r_2} \leqslant \mathscr{A}\mathscr{B}^{p_1;p_2,q}_{w_1;(w_2,w)}.
\end{aligned}$$

So that $I\leqslant\mathscr{A}\mathscr{B}^{p_1;p_2,q}_{w_1;(w_2,w)}$.

On the other hand, as mention in the above case, the inequality $II\leqslant C$, with a constant C independent of g and x is the inequality (1.1) with $r=\infty$, $p=p_2$, $v=w_2$ and $u(x)=(H\sigma_1(x))^{\frac{q}{p_1'}}$. Since $q<p_2$, the necessary and sufficient condition to obtain $II\leqslant C$ is the condition in Theorem 1.2 (ii) which is condition $\mathscr{A}\mathscr{B}^{p_1;p_2,q}_{w_1;(w_2,w)}<\infty$.

Let us see now that condition $\mathscr{A}\mathscr{B}^{p_1;p_2,q}_{w_1;(w_2,w)}<\infty$ is necessary. Assume that (1.4) holds. Then,

$$\left(\int_a^b (Hg)(x))^q w_f(x)dx\right)^{\frac{1}{q}} \leqslant C\left(\int_a^b g^{p_2}w_2\right)^{\frac{1}{p_2}}, \tag{3.4}$$

where

$$w_f(x) = (H(f/||f||_{p_1,w_1})(x))^q w(x).$$

Since $p_2 > q$, we know that (3.4) is equivalent to the following condition:

$$B^{p_2,q}_{(w_2,w_f)}(a,b) \approx \left\{ \int_a^b \left(\tilde{H}w_f(x)\right)^{\frac{r_2}{p_2}} \left(H\sigma_2(x)\right)^{\frac{r_2}{p_2'}} w_f(x)dx \right\}^{\frac{1}{r_2}} < \infty.$$

Observe that, if $a < x < t < b$, we have

$$w_f(t) = (H(f/||f||_{p_1,w_1})(t))^q w(t) \geqslant (H(f/||f||_{p_1,w_1})(x))^q w(t).$$

Then,

$$\left\{ \int_a^b [H(f/||f||_{p_1,w_1})(x)]^{q\left(\frac{r_2}{p_2}+1\right)} \left(\tilde{H}w(x)\right)^{\frac{r_2}{p_2}} \left(H\sigma_2(x)\right)^{\frac{r_2}{p_2'}} w(x)dx \right\}^{\frac{1}{r_2}} < \infty,$$

i. e.,

$$\left(\int_a^b (Hf(x))^{r_2} u(t)\, dt \right)^{1/r_2} \leqslant C||f||_{p_1,w_1},$$

where

$$u(x) = \left(\tilde{H}w(x)\right)^{r_2/p_2} (H\sigma_2(x))^{r_2/p_2'} w(x).$$

The above inequality means that the operator H is bounded from $L^{p_1}(w_1)$ to $L^{r_2}(u)$. Since $p_1 \leqslant r_2$, the pair of weights (w_1, u) must verify

$$A^{p_1,r_2}_{(w_1,u)}(a,b) = \sup_{a<x<b} \left(\int_x^b u(t)\, dt \right)^{1/r_2} [H\sigma_1(x)]^{1/p_1'} < \infty,$$

which is condition $\mathscr{A}\mathscr{B}^{p_1;p_2,q}_{w_1;(w_2,w)} < \infty$.

(iii). As we have seen in the proofs of the above cases, to prove (1.4) is equivalent to prove, for instance, that

$$\left(\int_a^b (Hf)(x))^q w_g(x)dx \right)^{\frac{1}{q}} \leqslant C \left(\int_a^b f^{p_1} w_1 \right)^{\frac{1}{p_1}}, \tag{3.5}$$

where

$$w_g(x) = (H(g/||g||_{p_2,w_2})(x))^q w(x).$$

On the other hand, we know that, since $p_1 > q$, (3.5) is equivalent to

$$B^{p_1,q}_{(w_1,w_g)}(a,b) = \left\{ \int_a^b \left(\tilde{H}w_g(x)\right)^{\frac{r_1}{q}} (H\sigma_1(x))^{\frac{r_1}{q'}} \sigma_1(x)dx \right\}^{\frac{1}{r_1}} < \infty.$$

Let us see that $B^{p_1,q}_{(w_1,w_g)}(a,b) < \infty$:

$$B^{p_1,q}_{(w_1,w_g)}(a,b) \leqslant \left\{ \int_a^b \left(\int_x^b \left(\int_a^x g(s)/||g||_{p_2,w_2}\, ds \right)^q w(t)\, dt \right)^{\frac{r_1}{q}} (H\sigma_1(x))^{\frac{r_1}{q'}} \sigma_1(x) dx \right\}^{\frac{1}{r_1}}$$
$$+ \left\{ \int_a^b \left(\int_x^b \left(\int_x^t g(s)/||g||_{p_2,w_2}\, ds \right)^q w(t)\, dt \right)^{\frac{r_1}{q}} (H\sigma_1(x))^{\frac{r_1}{q'}} \sigma_1(x) dx \right\}^{\frac{1}{r_1}}$$
$$= I + II.$$

Observe that

$$I = \left\{ \int_a^b (H(g/||g||_{p_2,w_2})(x))^{r_1} u(x)\, dx \right\}^{\frac{1}{r_1}},$$

where

$$u(x) = \left(\int_x^b w(t)\, dt \right)^{r_1/q} (H\sigma_1(x))^{\frac{r_1}{q'}} \sigma_1(x).$$

Then $I \leqslant C$ if we show that the pair of weights (w_2,u) verifies condition $A^{p_2,r_1}_{(w_2,u)}(a,b) < \infty$, due to $p_2 \leqslant r_1$. But this holds because this condition is equivalent to condition $\mathscr{A}\mathscr{B}^{p_2;p_1,q}_{w_2;(w_1,w)} < \infty$. On the other hand, if we set $v_1(x) = (H\sigma_1(x))^{\frac{r_1}{q'}} \sigma_1(x)$, to prove $II \leqslant C$ is equivalent to prove that

$$\left\{ \int_a^b \left(\int_x^b \left(\int_x^t g(s)\, ds \right)^q w(t)\, dt \right)^{\frac{r_1}{q}} v_1(x) dx \right\}^{\frac{1}{r_1}} \leqslant C||g||_{p_2,w_2}.$$

Since $1 < q < p_2$ and $1 < p_2 \leqslant r_1 < \infty$, using Theorem 1.1 with $u = v_1$, $v = w_2$, $r = r_1$ and $p = p_2$ we get that the above inequality is equivalent to the condition $\mathscr{A}\mathscr{B}^{p_1;p_2,q}_{w_1;(w_2,w)} < \infty$. In fact, $p = p_2$ implies that $\theta = r_2$ and since $u = v_1$ and $r = r_1$ we get that

$$\left(\int_a^x u \right)^{1/r} = \left(\int_a^x (H\sigma_1(t))^{\frac{r_1}{q'}} \sigma_1(t) \right)^{1/r_1}$$
$$= \left(\int_a^x \sigma_1(t)\, dt \right)^{\left(\frac{r_1}{q'}+1\right)\frac{1}{r_1}} = (H\sigma_1(x))^{1/p_1'}.$$

The necessity of condition $\mathscr{A}\mathscr{B}^{p_1;p_2,q}_{w_1;(w_2,w)} < \infty$ follows as in the above case. Finally, the necessity of condition $\mathscr{A}\mathscr{B}^{p_2;p_1,q}_{w_2;(w_1,w)} < \infty$ follows as in [1].

(iv). For the sufficiency of the conditions, we have to prove, as in the proof of (iii), that

$$I = \left\{ \int_a^b (H(g/||g||_{p_2,w_2})(x))^{r_1} u(x)\, dx \right\}^{\frac{1}{r_1}} < \infty,$$

where

$$u(x) = \left(\int_x^b w(t)\,dt\right)^{r_1/q} (H\sigma_1(x))^{\frac{r_1}{q'}} \sigma_1(x),$$

and

$$\left\{\int_a^b \left(\int_x^b \left(\int_x^t g(s)\,ds\right)^q w(t)\,dt\right)^{\frac{r_1}{q}} v_1(x)dx\right\}^{\frac{1}{r_1}} \leqslant C||g||_{p_2,w_2}. \tag{3.6}$$

But now $p_2 > r_1$. For the boundedness of I, we have that the condition on the weights can be written as

$$B^{p_2,r_1}_{(w_2,u)}(a,b) = \left\{\int_a^b \left(\int_x^b u\right)^{s/r_1} \left(\int_a^x \sigma_2\right)^{s/r_1'} \sigma_2(x)\,dx\right\}^{1/s} < \infty,$$

where $1/s = 1/r_1 - 1/p_2$. This is condition $\mathscr{B}\mathscr{B}^{p_2;p_1,q}_{w_2;(w_1,w)} < \infty$ and then $I < \infty$ uniformly on g.

Notice that, since $p_2 > r_1$, the inequality (3.6) was characterized in Theorem 1.3. The condition in this theorem is the condition $\mathscr{B}\mathscr{B}^{p_1;p_2,q}_{w_1;(w_2,w)} < \infty$. In fact, taking $u = v_1$, $v = w_2$, $r = r_1$ and $p = p_2$ we get that $\theta = r_2$ and $\eta = s$. On the other hand, we have that

$$\begin{aligned} U(a,x) &= \left(\int_a^x u\right)^{1/r} \\ &= \left(\int_a^x (H\sigma_1(t))^{\frac{r_1}{q'}} \sigma_1(t)\right)^{1/r_1} \\ &= \left(\int_a^x \sigma_1(t)\,dt\right)^{\left(\frac{r_1}{q'}+1\right)\frac{1}{r_1}} = (H\sigma_1(x))^{1/p_1'}. \end{aligned}$$

Therefore

$$U(a,x)^{\frac{\eta}{p}} u(x) = (H\sigma_1(x))^{\frac{s}{pp_1'}} (H\sigma_1(x))^{\frac{r_1}{q'}} \sigma_1(x) = (H\sigma_1(x))^{\frac{s}{r_2'}} \sigma_1(x).$$

The necessity of condition $\mathscr{B}\mathscr{B}^{p_2;p_1,q}_{w_2;(w_1,w)} < \infty$ follows as in [1].

Acknowledgement. We would like to thank the referee for calling our attention to reference [9] and pointing out a mistake in the proof of one of the results.

REFERENCES

[1] M. I. AGUILAR CAÑESTRO, P. ORTEGA SALVADOR AND C. RAMÍREZ TORREBLANCA, *Weighted bilinear Hardy inequalities*, J. Math. Anal. Appl. **387** (2012), no. 1, 320–334.

[2] J. BRADLEY, *Hardy inequalities with mixed norms*, Canad. Math. Bull. **21** (1978), no. 4, 405–408.

[3] A. GOGATISHVILI AND L. PICK, *Discretization and anti-discretization of rearrangement-invariant norms*, Publ. Mat. **47** (2003), 311–358.

[4] A. KUFNER AND L. E. PERSSON, *Weighted inequalities of Hardy type*, World Scientific, 2003.

[5] Q. LAI, *Weighted modular inequalities for Hardy type operators*, Proc. London Math. Soc. **79** (1999), 649–672.
[6] V. G. MAZ'JA, *Sobolev Spaces*, Springer-Verlag, 1985.
[7] B. MUCKENHOUPT, *Hardy's inequality with weights*, Studia Math. **44** (1972), 31–38.
[8] R. OINAROV AND A. KALYBAY, *Three-parameter weighted Hardy type inequalities*, Banach J. Math. Anal. **2** (2008), no. 2, 85–93.
[9] R. OINAROV AND A. KALYBAY, *Weighted inequalities for a class of semiadditive operators*, Ann. Funct. Anal. **6** (2015), no. 4, 155–171.

(Received March 13, 2016)

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