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ON AN ELEMENTARY INEQUALITY AND ITS APPLICATION IN THE THEORY OF INTEGRAL EQUATIONS

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Abstract. An elementary inequality is proved and some special cases of that inequality are discussed. Moreover, the usefulness of that inequality in the theory of some classes of nonlinear integral equations is shown.

1. Introduction

Inequalities of various kinds play a significant role in almost all branches of mathematics. With help of inequalities we express numerous estimates in mathematical analysis, functional analysis, numerical analysis, probability theory, geometry and so on. The mentioned estimates allow us to deduce a lot of important properties of functions describing investigated phenomena and other important quantities considered in connection with real world events, geometric relations and other parameters investigated in natural sciences (cf. [1], [2], [3], [4], for example).

In this paper we focus on an inequality of an elementary type. It seems that the mentioned inequality cannot be deduced in a standard way with help of the methods and tools of classical mathematical analysis (cf. [5], [6], [7]). It turns out that the inequality in question is very useful in the theory of nonlinear integral equations. Namely, applying that inequality we can derive the solvability of a lot of classes of nonlinear integral equations such as quadratic integral equations of Fredholm type and nonlinear Volterra-Wiener-Hopf integral equations.

Obviously, we can also obtain existence results concerning nonlinear integral equations of Hammerstein or Urysohn type but in the present paper we restrict ourselves to the classes of integral equations mentioned above.

In our considerations we will use the existence results proved in other, earlier published papers [8] and [9], but we will illustrate those results by new classes of integral equations being more general than those included in papers [8], [9].

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2. An inequality and its special cases

In order to set the stage for our considerations we establish first some notation. Namely, denote by $\mathbb{R}$ the set of real numbers and put $\mathbb{R}_+ = [0,\infty)$. The symbol $\mathbb{N}$ will stand for the set of natural numbers.

In what follows we will consider the inequality being the principal object of the study in the paper.

THEOREM 2.1. *Let p,q be arbitrary real numbers such that $1 \leqslant q < p$. Moreover, let a be an arbitrarily fixed nonnegative number. Then, the following inequality is satisfied*

$$\left|(x^q+a)^{\frac{1}{p}} - (y^q+a)^{\frac{1}{p}}\right| \leqslant |x-y|^{\frac{q}{p}} \tag{2.1}$$

for all $x,y \in \mathbb{R}$.

REMARK 2.2. Observe that using the notation of the generalized root of an arbitrary degree p $(p>0)$, i.e. putting

$$\sqrt[p]{x} = x^{\frac{1}{p}} \tag{2.2}$$

for $x \in \mathbb{R}_+$, we can represent inequality (2.1) in a more transparent form

$$\left|\sqrt[p]{x^q+a} - \sqrt[p]{y^q+a}\right| \leqslant \sqrt[p]{|x-y|^q}. \tag{2.3}$$

Proof of Theorem 2.1. It is easily seen that the function $f : \mathbb{R}_+ \to \mathbb{R}$ defined by the formula

$$f(x) = (x^q+a)^{\frac{1}{p}}$$

is increasing on the interval $\mathbb{R}_+$. Indeed, we have

$$f'(x) = \frac{q}{p}\,(x^q+a)^{\frac{1}{p}-1}\,x^{q-1}.$$

Hence we see that $f'(x) > 0$ for $x > 0$. Moreover, notice that $f(0) = a^{\frac{1}{p}} \geqslant 0$. Thus, applying standard tools of the classical analysis we deduce that f is a self-mapping of $\mathbb{R}_+$ and f is increasing (more precisely: strictly increasing) on $\mathbb{R}_+$. In view of the above established facts we infer that we can restrict ourselves to the proof of inequality (2.1) in the case when $y < x$.

Thus, fix arbitrarily a number $y \geqslant 0$ and take x such that $x > y$. For convenience we will write $x = y + \alpha$, where $\alpha > 0$. Then we can rewrite inequality (2.1) (or (2.3)) in the form

$$\sqrt[p]{(y+\alpha)^q+a} - \sqrt[p]{y^q+a} \leqslant \alpha^{\frac{q}{p}}. \tag{2.4}$$

Our aim is to show that inequality (2.4) is satisfied for $\alpha \in \mathbb{R}_+$, where a and y are arbitrary nonnegative numbers.

To this end consider the function $g : \mathbb{R}_+ \to \mathbb{R}$ defined by the formula

$$g(\alpha) = \sqrt[p]{(y+\alpha)^q+a} - \alpha^{\frac{q}{p}} - \sqrt[p]{y^q+a}.$$

We have $g(0)=0$ and

$$
\begin{aligned}
g'(\alpha) &= \frac{1}{p}\left[(y+\alpha)^q+a\right]^{\frac{1}{p}-1}(y+\alpha)^{q-1}q-\frac{q}{p}\alpha^{\frac{q}{p}-1}\\
&= \frac{q}{p}\left\{\frac{(y+\alpha)^{q-1}}{\left[(y+\alpha)^q+a\right]^{1-\frac{1}{p}}}-\frac{1}{\alpha^{1-\frac{q}{p}}}\right\}\\
&= \frac{q}{p}\,\frac{\alpha^{1-\frac{q}{p}}(y+\alpha)^{q-1}-\left[(y+\alpha)^q+a\right]^{1-\frac{1}{p}}}{\alpha^{1-\frac{q}{p}}\left[(y+\alpha)^q+a\right]^{1-\frac{1}{p}}}.
\end{aligned} \tag{2.5}
$$

In what follows let us observe that applying the well known binomial expansion, we get

$$
\left[(y+\alpha)^q+a\right]^{p-1}=(y+\alpha)^{q(p-1)}+\binom{p-1}{1}(y+\alpha)^{q(p-2)}a+\dots. \tag{2.6}
$$

On the other hand we have the following obvious inequality

$$
\alpha^{p-q}(y+\alpha)^{p(q-1)}\leqslant(y+\alpha)^{p-q}(y+\alpha)^{p(q-1)}=(y+\alpha)^{q(p-1)}. \tag{2.7}
$$

Combining (2.6) and (2.7), we obtain

$$
\alpha^{p-q}(y+\alpha)^{p(q-1)}<\left[(y+\alpha)^q+a\right]^{p-1}.
$$

Further, raising both sides of the above inequality to the power $\frac{1}{p}$ we derive the following estimate

$$
\alpha^{1-\frac{q}{p}}(y+\alpha)^{q-1}<\left[(y+\alpha)^q+a\right]^{1-\frac{1}{p}}
$$

or, equivalently

$$
\alpha^{1-\frac{q}{p}}(y+\alpha)^{q-1}-\left[(y+\alpha)^q+a\right]^{1-\frac{1}{p}}<0. \tag{2.8}
$$

Now, linking expression (2.5) with estimate (2.8) we conclude that $g'(\alpha)<0$ for any $\alpha>0$. Taking into account this assertion and the equality $g(0)=0$ we deduce that the function $g=g(\alpha)$ is decreasing on $\mathbb{R}_+$ and $g(\alpha)\leqslant 0$ for $\alpha\geqslant 0$. This proves inequality (2.1) (or, equivalently, (2.3)) and completes the proof. $\square$

Observe that in the case when q is a natural even number inequality (2.1) can be extended to the whole real axis $\mathbb{R}$ i.e., if $q=2n$, where $n\in\mathbb{N}$, then for an arbitrary number $p>2n$ the following inequality is satisfied

$$
\left|(x^{2n}+a)^{\frac{1}{p}}-(y^{2n}+a)^{\frac{1}{p}}\right|\leqslant|x-y|^{\frac{2n}{p}}, \tag{2.9}
$$

for arbitrary $x,y\in\mathbb{R}$ and for each fixed $a\geqslant 0$.

Notice that under convention (2.2) the above inequality can be written in the form

$$
\left|\sqrt[p]{x^{2n}+a}-\sqrt[p]{y^{2n}+a}\right|\leqslant\sqrt[p]{(x-y)^{2n}}, \tag{2.10}
$$

where $x,y\in\mathbb{R}$ and $a\geqslant 0$.

The proof of inequality (2.10) (or (2.9)) follows immediately from inequality (2.3) (or (2.1), respectively). Indeed, for arbitrarily fixed $x,y\in\mathbb{R}$, we have

$$\begin{aligned}\left|\sqrt[p]{x^{2n}+a}-\sqrt[p]{y^{2n}+a}\right| &= \left|\sqrt[p]{|x|^{2n}+a}-\sqrt[p]{|y|^{2n}+a}\right| \\ &\leqslant \sqrt[p]{\big||x|-|y|\big|^{2n}}.\end{aligned}$$

Hence, in view of the fact that the function $h(x)=\sqrt[p]{x}$ is nondecreasing on $\mathbb{R}_+$, we obtain

$$\left|\sqrt[p]{x^{2n}+a}-\sqrt[p]{y^{2n}+a}\right|\leqslant\sqrt[p]{|x-y|^{2n}}=\sqrt[p]{(x-y)^{2n}}$$

which proves our assertion.

Particularly, taking in (2.10) $p=3$ and $n=1$, we obtain the following inequality

$$\left|\sqrt[3]{x^{2}+a}-\sqrt[3]{y^{2}+a}\right|\leqslant\sqrt[3]{(x-y)^{2}}$$

which was proved in [10].

REMARK 2.3. In the case when $a=0$ we have that $f(x)=x^{\frac{q}{p}}$ (cf. the proof of Theorem 2.1). Applying the standard methods of mathematical analysis (second derivative, the concavity and the subadditivity of the function f) we can easily show that

$$\left|x^{\frac{q}{p}}-y^{\frac{q}{p}}\right|\leqslant|x-y|^{\frac{q}{p}}$$

for $x,y\in\mathbb{R}_+$. Unfortunately, such an approach fails to work in the case $a>0$.

3. Application to a class of Volterra-Wiener-Hopf integral equations

In this section we present a result on the existence of solutions of a nonlinear integral equation of Volterra-Wiener-Hopf type which was obtained in [8]. Next, using the inequality proved in Section 2, we indicate a class of integral equations of Volterra-Wiener-Hopf type for which the mentioned result can be applied.

For further purposes denote by $BC(\mathbb{R}_+)$ the Banach space consisting of real functions defined, continuous and bounded on the half-axis $\mathbb{R}_+$. This space will be furnished with the standard supremum norm, i.e. for $x\in BC(\mathbb{R}_+)$ we put

$$||x||=\sup\left\{|x(t)|:t\in\mathbb{R}_+\right\}.$$

The object of our study in this section is the following nonlinear Volterra-Wiener-Hopf integral equation

$$x(t)=a(t)+\int_0^t k(t-s)f(s,x(s))ds, \tag{3.1}$$

where $t\in\mathbb{R}_+$.

Now, we recall the existence result concerning equation (3.1) which was obtained in [8].

THEOREM 3.1. *Assume that the functions involved in equation (3.1) satisfy the following conditions:*

(i) *The function* $a = a(t)$ *belongs to the space* $BC(\mathbb{R}_+)$ *and is such that there exists the limit* $\lim_{t\to\infty} a(t)$ *(obviously, this limit is finite).*

(ii) $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ *is continuous and there exists a function* $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *which is nondecreasing,* $\phi(0) = 0$, $\lim_{t\to 0} \phi(t) = 0$ *and such that*

$$|f(t,x) - f(t,y)| \leqslant \phi(|x-y|)$$

for all $t \in \mathbb{R}_+$ *and* $x, y \in \mathbb{R}$.

(iii) *The function* $t \to f(t,0)$ *belongs to the space* $BC(\mathbb{R}_+)$.

(iv) *The function* $k(u) = k : \mathbb{R}_+ \to \mathbb{R}_+$ *is nonincreasing and integrable on* $\mathbb{R}_+$.

(v) *There exists a positive solution of the inequality*

$$||a|| + (\phi(r) + F_1)\overline{k} \leqslant r,$$

where $F_1 = \sup\left\{|f(t,0)| : t \in \mathbb{R}_+\right\}$ *and* $\overline{k} = \int_0^\infty k(u)du$.

Then there exists at least one solution $x = x(t)$ *of equation (3.1) in the space* $BC(\mathbb{R}_+)$ *which has a limit at infinity.*

In the sequel we will investigate the class of Volterra-Wiener-Hopf integral equations having the form

$$x(t) = a(t) + \int_0^t k(t-s)\left(x^{2n}(s) + b(s)\right)^{\frac{1}{p}} ds, \tag{3.2}$$

where $t \in \mathbb{R}_+$ and n is a natural number. Moreover, we assume that p is a fixed real number such that $p > 2n$. Additionally, we require that the function $a = a(t)$ satisfies assumption (i) and the function $k = k(u)$ fulfills assumption (iv) of Theorem 3.1. Further, we impose the following assumption

(vi) *The function* $b : \mathbb{R}_+ \to \mathbb{R}_+$ *is a member of* $BC(\mathbb{R}_+)$.

Then we can formulate the following result.

THEOREM 3.2. *Under assumptions* (i), (iv) *and* (vi) *equation (3.2) has at least one solution in the space* $BC(\mathbb{R}_+)$ *which has a limit at infinity.*

Proof. Observe that equation (3.2) is a particular case of the Volterra-Wiener-Hopf equation (3.1), where

$$f(t,x) = \left(x^{2n} + b(t)\right)^{\frac{1}{p}}. \tag{3.3}$$

Thus, we need only check that there are satisfied assumptions (ii), (iii) and (v) of Theorem 3.1.

To verify assumption (ii) let us take into account the fact that the function $f = f(t,x)$ appearing in equation (3.2) has the form (3.3). So, applying Theorem 2.1 (in the special case indicated by (2.10)), for arbitrarily fixed $x,y \in \mathbb{R}$ and $t \in \mathbb{R}_+$, we obtain

$$\begin{aligned} \left|f(t,x)-f(t,y)\right| &= \left|\left(x^{2n}+b(t)\right)^{\frac{1}{p}} - \left(y^{2n}+b(t)\right)^{\frac{1}{p}}\right| \\ &\leqslant \sqrt[p]{(x-y)^{2n}} = (x-y)^{\frac{2n}{p}}. \end{aligned}$$

This shows that the function ϕ appearing in assumption (ii) can be accepted in the form $\phi(r) = r^{\frac{2n}{p}}$. In order to check assumption (iii) of Theorem 3.1 let us observe that $f(t,0) = (b(t))^{\frac{1}{p}}$. Obviously this function is a member of the space $BC(\mathbb{R}_+)$ on the basis of assumption (vi). Further, we have

$$F_1 = \sup\left\{|f(t,0)| : t \in \mathbb{R}_+\right\} = ||b||^{\frac{1}{p}}.$$

Finally, let us notice that the inequality from assumption (v) of Theorem 3.1 has the form

$$||a|| + \left(r^{\frac{2n}{p}} + ||b||^{\frac{1}{p}}\right)\overline{k} \leqslant r.$$

The above inequality can be written in a more transparent form as follows

$$\alpha + \overline{k} r^{\frac{2n}{p}} \leqslant r, \tag{3.4}$$

where $\alpha = ||a|| + \overline{k}||b||^{\frac{1}{p}}$.

Taking into account the concavity of the function $\phi(r) = r^{\frac{2n}{p}}$ we conclude that inequality (3.4) has positive solution. More precisely, there exists $r_0 > 0$ such that inequality (3.4) is satisfied for each $r \geqslant r_0$. This shows that assumption (v) of Theorem 3.1 is satisfied. Finally, in view of Theorem 3.1 we complete the proof of our theorem. $\square$

No, we illustrate the result of Theorem 3.2 by an example.

EXAMPLE 3.3. Consider integral equation (3.2), where the functions involved have the form:

$$\begin{aligned} a(t) &= \arctan t, \\ b(t) &= \frac{t^2+1}{2t^2+3}, \\ k(u) &= \frac{u+1}{e^u}. \end{aligned}$$

Moreover, assume that $n = 3$ and $p = 7$.

Then, the function $f(t,x)$ has the form

$$f(t,x) = \sqrt[7]{x^6 + \frac{t^2+1}{2t^2+3}}.$$

Apart form this we have that $||a|| = \frac{\pi}{2}$, $||b|| = \frac{1}{2}$ and $\phi(r) = r^{\frac{6}{7}}$. We can also calculate that

$$\overline{k} = \int_0^\infty k(u)du = \int_0^\infty (u+1)e^{-u}du = 2.$$

Hence we see that there are satisfied assumptions (i), (iv) and (vi) of Theorem 3.2. On the other hand notice that the inequality from assumption (v) of Theorem 3.1 has the form

$$\frac{\pi}{2} + 2\left(r^{\frac{6}{7}} + \frac{1}{\sqrt[7]{2}}\right) \leqslant r.$$

It is easily seen that each real number $r \geqslant r_0$, where $r_0 \cong 150$, satisfies the above inequality.

4. Solvability of a quadratic Fredholm integral equation in the Hölder space

The considerations of this section are located in the class of functions satisfying the Hölder condition. More precisely, if α is a fixed number such that $0 < \alpha \leqslant 1$, then the symbol $H_\alpha([a,b])$ will denote the set of all real functions $x = x(t)$ defined on the interval $[a,b]$ and satisfying on $[a,b]$ the Hölder condition with the exponent α. This means that $x \in H_\alpha([a,b])$ if and only if there exists a nonnegative constant H_x^α (depending on x) such that

$$|x(t) - x(s)| \leqslant H_x^\alpha |t-s|^\alpha$$

for all $t,s \in [a,b]$.

Observe that the set $H_\alpha([a,b])$ forms a linear space. Moreover, $H_\alpha([a,b])$ endowed with the norm

$$||x||_\alpha = |x(a)| + \sup\left\{\frac{|x(t)-x(s)|}{|t-s|^\alpha} : t,s \in [a,b], t \neq s\right\}$$

is a Banach space.

The investigations concerning the space $H_\alpha([a,b])$ and its properties were conducted in [11] (cf. also [9]) and we will not recall here involved details.

For our purposes we recall an existence result proved in [9] and concerning the following quadratic integral equation of Fredholm type

$$x(t) = p(t) + x(t)\int_a^b k(t,\tau)x(\tau)d\tau, \tag{4.1}$$

where $t \in [a,b]$. In order to present the mentioned result we formulate first appropriate assumptions under which our considerations will be conducted.

(i) The function $p = p(t)$ belongs to the Hölder space $H_\beta([a,b])$, where β is a fixed number in the interval $(0,1]$.

(ii) $k : [a,b] \times [a,b] \to \mathbb{R}$ is a continuous function such that it satisfies the Hölder condition with the exponent β with respect to the first variable, that is, there exists a constant $k_\beta > 0$ such that

$$\left|k(t,\tau) - k(s,\tau)\right| \leqslant k_\beta |t-s|^\beta$$

for all $t,s,\tau \in [a,b]$.

In what follows, on the basis of the above assumptions, we can define the constant K by putting

$$K = \sup\left\{\int_a^b |k(t,\tau)|d\tau : t \in [a,b]\right\}.$$

Thus, we are prepared to formulate our last assumption.

(iii) The following inequality is satisfied

$$||p||_\beta \left(\max\left\{1,(b-a)^\beta\right\}\right)^2 \left(2K + k_\beta(b-a)\right) < \frac{1}{4}, \tag{4.2}$$

where $||p||_\beta$ denotes the norm of the function p in the space $H_\beta([a,b])$.

Now, we are prepared to present the above announced existence result.

THEOREM 4.1. *Under assuptions (i)–(iii) equation (4.1) has at least one solution belonging to the space $H_\alpha([a,b])$, where α is arbitrarily fixed number such that $0 < \alpha < \beta$.*

In the sequel we distinguish a class of quadratic integral equations of Fredholm type which have solutions in view of Theorem 4.1. This class will be created on the basis of the inequality presented in Section 2.

Namely, consider the quadratic Fredholm integral equation having the form

$$x(t) = p(t) + x(t)\int_a^b \left(q|t|^\gamma + r(\tau)\right)^\delta x(\tau)d\tau, \tag{4.3}$$

where $t \in [a,b]$ and the function $p = p(t)$ satisfies assumption (i), i.e. p satisfies the Hölder condition with the exponent $\beta \in (0,1)$, while the function $r : [a,b] \to \mathbb{R}_+$ is continuous on $[a,b]$. Moreover, we assume that γ, δ, q are positive constants such that $\delta \in (0,1)$, $\gamma > 1$ and $\gamma^\delta = \beta$. Apart form this we impose the following assumption:

(iv) Inequality (4.2) is satisfied with $k_\beta = q^\delta$ and the constant K can be evaluated in the following way

$$K \leqslant q^\beta \max\left\{|a|^\beta, |b|^\beta\right\}(b-a) + \int_a^b \left(r(\tau)\right)^\beta d\tau. \tag{4.4}$$

Then we have the following theorem.

THEOREM 4.2. *Under assumptions* $(i),(iv)$ *and the above formulated requirements the quadratic integral equation (4.3) has a solution in the space* $H_\alpha([a,b])$, *where* α *is an arbitrarily fixed number such that* $0<\alpha<\beta\gamma$.

Proof. Notice that equation (4.3) is a particular case of equation (4.1), where

$$k(t,\tau)=\left(q|t|^\gamma+r(\tau)\right)^\delta$$

for $t,\tau\in[a,b]$. Obviously, in order to prove our theorem it is sufficient to show that there is satisfied assumption (ii) of Theorem 4.1 and estimate (4.4) holds.

Thus, fix arbitrary numbers $t,s,\tau\in[a,b]$. Thus, in view of inequality (2.1) (with $p=\frac{1}{\delta}$) we obtain

$$\begin{aligned}\left|k(t,\tau)-k(s,\tau)\right| &= \left|\left(q|t|^\gamma+r(\tau)\right)^\delta-\left(q|s|^\gamma+r(\tau)\right)^\delta\right|\\ &= \left|\left((q^{\frac{1}{\gamma}}|t|)^\gamma+r(\tau)\right)^\delta-\left((q^{\frac{1}{\gamma}}|s|)^\gamma+r(\tau)\right)^\delta\right|\\ &\leqslant \left|q^{\frac{1}{\gamma}}|t|-q^{\frac{1}{\gamma}}|s|\right|^{\gamma\delta}=\left(q^{\frac{1}{\gamma}}\right)^{\gamma\delta}\left||t|-|s|\right|^{\gamma\delta}\\ &\leqslant q^\delta|t-s|^\beta.\end{aligned}$$

Hence, we see that our function $k=k(t,\tau)$ satisfies assumption (ii) with $k_\beta=q^\delta$.

Further, we show the validity of estimate (4.4). To this end let us observe that in virtue of the fact that the function $z(t)=t^\delta$ is concave (hence subadditive) on $\mathbb{R}_+$, we get

$$\begin{aligned}K&=\sup\left\{\int_a^b|k(t,\tau)|d\tau:t\in[a,b]\right\}=\sup\left\{\int_a^b\left(q|t|^\gamma+r(\tau)\right)^\delta d\tau:t\in[a,b]\right\}\\ &\leqslant\sup\left\{\int_a^b q^\delta|t|^{\gamma\delta}d\tau+\int_a^b\left(r(\tau)\right)^\delta d\tau:t\in[a,b]\right\}\\ &\leqslant q^\delta\max\left\{|a|^\beta,|b|^\beta\right\}(b-a)+\int_a^b\left(r(\tau)\right)^\delta d\tau.\end{aligned}$$

This proves estimate (4.4) and completes the proof. □

EXAMPLE 4.3. To illustrate the applicability of the above proved theorem let us consider the following special case of equation (4.3)

$$x(t)=\sqrt{ct+d}+x(t)\int_0^1\left(qt^{\frac{3}{2}}+\tau^3e^{-\tau}\right)^{\frac{1}{3}}d\tau \tag{4.5}$$

for $t\in[0,1]$,where c,d and q are positive constants. Observe that comparing equations (4.5) and (4.3) we see that $p(t)=\sqrt{ct+d}$ and

$$k(t,\tau)=\left(qt^{\frac{3}{2}}+\tau^3e^{-\tau}\right)^{\frac{1}{3}}.$$

Thus the function $p = p(t)$ satisfies the Hölder condition with the exponent $\beta = \frac{1}{2}$. Indeed, in view of inequality (2.1) we have

$$|p(t) - p(s)| = \left|(ct+d)^{\frac{1}{2}} - (cs+d)^{\frac{1}{2}}\right| \leqslant |ct - cs|^{\frac{1}{2}} = \sqrt{c}|t-s|^{\frac{1}{2}}.$$

Moreover, we have the following estimate

$$\begin{aligned}
||p||_\beta &= \sup\left\{\frac{|p(t)-p(s)|}{\sqrt{|t-s|}} : t,s \in [0,1], t \neq s\right\} \\
&= \sup\left\{\frac{\sqrt{ct+d} - \sqrt{cs+d}}{\sqrt{t-s}} : t,s \in [0,1], t > s\right\} \\
&= \sup\left\{\frac{c(t-s)}{\left(\sqrt{ct+d} + \sqrt{cs+d}\right)\sqrt{t-s}} : t,s \in [0,1], t > s\right\} \\
&= \sup\left\{\frac{c\sqrt{t-s}}{\sqrt{ct+d} + \sqrt{cs+d}} : t,s \in [0,1], t > s\right\} \leqslant \frac{c}{2\sqrt{d}}.
\end{aligned}$$

Further, for arbitrarily fixed $t, s, \tau \in [0,1]$, in view of inequality (2.1), we get

$$\begin{aligned}
\left|k(t,\tau) - k(s,\tau)\right| &= \left|(qt^{\frac{3}{2}} + \tau^3 e^{-\tau})^{\frac{1}{3}} - (qs^{\frac{3}{2}} + \tau^3 e^{-\tau})^{\frac{1}{3}}\right| \\
&= \left|\left[(q^{\frac{2}{3}}t)^{\frac{3}{2}} + \tau^3 e^{-\tau}\right]^{\frac{1}{3}} - \left[(q^{\frac{2}{3}}s)^{\frac{3}{2}} + \tau^3 e^{-\tau}\right]^{\frac{1}{3}}\right| \\
&\leqslant \left|q^{\frac{2}{3}}t - q^{\frac{2}{3}}s\right|^{\frac{1}{2}} = q^{\frac{1}{3}}|t-s|^{\frac{1}{2}}.
\end{aligned}$$

Hence we see that there is satisfied assumption (ii) with the exponent $\beta = \frac{1}{2}$ and the constant $k_\beta = q^{\frac{1}{3}}$.

Next, we estimate the constant K appearing in Theorem 4.1. In view of (4.4) we obtain

$$K = q^{\frac{1}{3}} + \int_0^1 \left(\tau^3 e^{-\tau}\right)^{\frac{1}{3}} d\tau = q^{\frac{1}{3}} + 9\left(1 - \frac{4}{3}e^{-\frac{1}{3}}\right) = q^{\frac{1}{3}} + 0.4013\ldots.$$

Thus, inequality (4.2) will be satisfied provided the following inequality is satisfied

$$\frac{c}{2\sqrt{d}}\left(2q^{\frac{1}{3}} + 0.8026\ldots + q^{\frac{1}{3}}\right) < \frac{1}{4}.$$

For example, if we take $c = \frac{1}{6}, d = 9$ and $q = 1$ then we can easily seen that the above inequality is satisfied.

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