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ON THE REVERSE YOUNG AND HEINZ INEQUALITIES

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Abstract. In this paper, we study further improvements of the reverse Young and Heinz inequalities for positive real numbers. We use these modified inequalities to obtain corresponding operator inequalities and matrix inequalities on the Hilbert–Schmidt norm.

1. Introduction

Let $a,b \geqslant 0$ and $v \in [0,1]$. The classical Young inequality for scalars a,b is known as

$$a^{1-v}b^{v} \leqslant (1-v)a+vb, \tag{1}$$

with equality if and only if $a=b$. If $v=\frac{1}{2}$, we obtain the arithmetic-geometric mean inequality

$$\sqrt{ab} \leqslant \frac{a+b}{2}.$$

The supplemental Young inequality

$$a^{1-v}b^{v} \geqslant (1-v)a+vb, \tag{2}$$

holds when $v \notin [0,1]$, the proof can be found in [1].

Heinz mean, introduced in [2], defined as

$$H_v(a,b) = \frac{a^{1-v}b^{v}+a^{v}b^{1-v}}{2}$$

for $a,b \geqslant 0$ and $v \in [0,1]$, interpolates between the arithmetic and geometric means. It is easy to see that

$$\sqrt{ab} \leqslant H_v(a,b) \leqslant \frac{a+b}{2},$$

which are called Heinz inequalities. Improvements of Young and Heinz inequalities and their reverses have been generalized, extended and strengthened in various directions.

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The reader can find them in [3, 4, 9, 10, 12, 16, 18, 20]. Kittaneh and Manasrah [13] gave refinements of Young and Heinz inequalities respectively as follows

$$a^{1-v}b^{v}+r(\sqrt{a}-\sqrt{b})^{2}\leqslant(1-v)a+vb, \tag{3}$$

$$H_{v}(a,b)+r(\sqrt{a}-\sqrt{b})^{2}\leqslant\frac{a+b}{2}, \tag{4}$$

where $r=\min\{v,1-v\}$ and $v\in[0,1]$. Subsequently, reverses of the inequalities (3) and (4) were obtained in [14] respectively as follows

$$a^{1-v}b^{v}+s(\sqrt{a}-\sqrt{b})^{2}\geqslant(1-v)a+vb, \tag{5}$$

$$H_{v}(a,b)+s(\sqrt{a}-\sqrt{b})^{2}\geqslant\frac{a+b}{2}, \tag{6}$$

where $s=\max\{v,1-v\}$ and $v\in[0,1]$.

Let $B(H)$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space H. In the case of dim $H=n$, we identify $B(H)$ with the matrix Algebra $M_n(\mathbb{C})$ of all $n\times n$ matrices with entries in the complex field $\mathbb{C}$. An operator $A\in B(H)$ is called positive and we write $A\geqslant 0$ if $\langle Ax,x\rangle\geqslant 0$ for all $x\in H$. The set of all positive invertible operators is denoted by $B^{++}(H)$. If $A\in B^{++}(H)$, we write $A>0$. We say $A\geqslant B$ if $A-B\geqslant 0$. The Hilbert-Schmidt norm of $A=[a_{ij}]\in M_n(\mathbb{C})$ is defined by

$$\|A\|_2=\left(\sum_{i,j=1}^{n}|a_{ij}|^2\right)^{\frac{1}{2}}.$$

This norm is unitarily invariant in the sense that $\|UAV\|_2=\|A\|_2$ for all unitary matrices $U,V\in M_n(\mathbb{C})$.

Let $A,B\in B^{++}(H)$ and $v\in[0,1]$. v-weighted geometric mean of A and B, denoted by $A\sharp_v B$, is defined as

$$A\sharp_v B=A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{v}A^{\frac{1}{2}}$$

and v-weighted arithmetic mean of A and B, denoted by $A\nabla_v B$, is

$$A\nabla_v B=(1-v)A+vB.$$

When $v=\frac{1}{2}$, $A\sharp_{\frac{1}{2}}B$ and $A\nabla_{\frac{1}{2}}B$ are called geometric mean and arithmetic mean and denoted by $A\sharp B$ and $A\nabla B$ respectively [17]. One can easily show that if $v\in[0,1]$, then

$$A\sharp_v B=B\sharp_{1-v}A. \tag{7}$$

It is well known that if $A,B\in B^{++}(H)$ and $v\in[0,1]$, then

$$A\nabla_v B\geqslant A\sharp_v B,$$

which is the operator version of the scalar Young inequality (1) [5, 6].

An operator version of Heinz mean was introduced in [12] by

$$H_v(A,B) = \frac{A\sharp_v B + A\sharp_{1-v} B}{2}.$$

It is easy to see that Heinz operator mean interpolates the arithmetic-geometric operator mean inequality

$$A\sharp B \leqslant H_v(A,B) \leqslant A\nabla B,$$

which are called the Heinz operator inequalities [11, 12].

The operator versions of the refined reverse Young and Heinz inequalities (5) and (6) were obtained in [14] as follows

$$A\nabla_v B \leqslant A\sharp_v B + 2s(A\nabla B - A\sharp B),$$
$$A\nabla B \leqslant H_v(A,B) + 2s(A\nabla B - A\sharp B),$$

where $A,B \in B^{++}(H)$, $s = \max\{v, 1-v\}$ and $v \in [0,1]$.

Recently, Zhao and Wu [19] presented some new refinements of the reverse Young inequality (5). It is interesting to ask whether there exist further refinements and improvements of the reverse Young and Heinz inequalities. This is a main motivation for the present paper and we are concerned with these inequalities in this paper. In section 2, we give the whole series of new refinements of the scalar reverse Young inequalities which help us to obtain refined Heinz inequalities. In section 3, we extend inequalities proved in section 2 from the scalar setting to a Hilbert space operator setting. In section 4, the corresponding Hilbert-Schmidt norm inequalities are established.

2. Improved reverse Young and Heinz inequalities for scalars

We start from the following improved reverse Young inequalities given in [19].

LEMMA 1. ([19]) *Let $a,b \geqslant 0$, $v \in [0,1]$, $r = \min\{v, 1-v\}$ and $r_0 = \min\{2r, 1-2r\}$.*

(i) If $0 \leqslant v \leqslant \frac{1}{2}$, then

$$(1-v)a + vb \leqslant a^{1-v}b^v + (1-v)(\sqrt{a}-\sqrt{b})^2 - r_0(\sqrt[4]{ab}-\sqrt{b})^2.$$

(ii) If $\frac{1}{2} < v \leqslant 1$, then

$$(1-v)a + vb \leqslant a^{1-v}b^v + v(\sqrt{a}-\sqrt{b})^2 - r_0(\sqrt[4]{ab}-\sqrt{a})^2.$$

The following corollary is a direct consequence of Lemma 1. Notice that Corollary 2 is deduced by Corollary 1.

COROLLARY 1. *Let* $a,b \geqslant 0$ *and* $v \in [0,1]$.

(i) If $0 \leqslant v \leqslant \frac{1}{4}$, *then*

$$(1-v)a+vb \leqslant a^{1-v}b^{v}+(1-v)(\sqrt{a}-\sqrt{b})^{2}-2v(\sqrt[4]{ab}-\sqrt{b})^{2}.$$

(ii) If $\frac{1}{4} \leqslant v \leqslant \frac{1}{2}$, *then*

$$(1-v)a+vb \leqslant a^{1-v}b^{v}+(1-v)(\sqrt{a}-\sqrt{b})^{2}+(2v-1)(\sqrt[4]{ab}-\sqrt{b})^{2}.$$

(iii) If $\frac{1}{2} < v \leqslant \frac{3}{4}$, *then*

$$(1-v)a+vb \leqslant a^{1-v}b^{v}+v(\sqrt{a}-\sqrt{b})^{2}-(2v-1)(\sqrt[4]{ab}-\sqrt{a})^{2}.$$

(iv) If $\frac{3}{4} \leqslant v \leqslant 1$, *then*

$$(1-v)a+vb \leqslant a^{1-v}b^{v}+v(\sqrt{a}-\sqrt{b})^{2}+(2v-2)(\sqrt[4]{ab}-\sqrt{a})^{2}.$$

COROLLARY 2. *Let* $a,b \geqslant 0$ *and* $v \in [0,1]$.

(i) If $0 \leqslant v \leqslant \frac{1}{4}$, *then*

$$\frac{a+b}{2} \leqslant H_{v}(a,b)+(1-v)(\sqrt{a}-\sqrt{b})^{2}-v\left[(\sqrt[4]{ab}-\sqrt{b})^{2}+(\sqrt[4]{ab}-\sqrt{a})^{2}\right].$$

(ii) If $\frac{1}{4} \leqslant v \leqslant \frac{1}{2}$, *then*

$$\frac{a+b}{2} \leqslant H_{v}(a,b)+(1-v)(\sqrt{a}-\sqrt{b})^{2}+(v-\frac{1}{2})\left[(\sqrt[4]{ab}-\sqrt{b})^{2}+(\sqrt[4]{ab}-\sqrt{a})^{2}\right].$$

(iii) If $\frac{1}{2} < v \leqslant \frac{3}{4}$, *then*

$$\frac{a+b}{2} \leqslant H_{v}(a,b)+v(\sqrt{a}-\sqrt{b})^{2}-(v-\frac{1}{2})\left[(\sqrt[4]{ab}-\sqrt{b})^{2}+(\sqrt[4]{ab}-\sqrt{a})^{2}\right].$$

(iv) If $\frac{3}{4} \leqslant v \leqslant 1$, *then*

$$\frac{a+b}{2} \leqslant H_{v}(a,b)+v(\sqrt{a}-\sqrt{b})^{2}+(v-1)\left[(\sqrt[4]{ab}-\sqrt{b})^{2}+(\sqrt[4]{ab}-\sqrt{a})^{2}\right].$$

Next we give our first main Theorem which concerns improved reverse Young inequalities.

THEOREM 1. *Let $a,b \geqslant 0$ and $v \in [0,1]$.*

(i) If $v \notin [\frac{1}{2}, \frac{3}{4}]$, then

$$(1-v)a + vb \leqslant a^{1-v}b^{v} + (1-v)(\sqrt{a}-\sqrt{b})^2 + (2v-1)(\sqrt[4]{ab}-\sqrt{b})^2. \qquad (8)$$

(ii) If $v \notin [\frac{1}{4}, \frac{1}{2}]$, then

$$(1-v)a + vb \leqslant a^{1-v}b^{v} + v(\sqrt{a}-\sqrt{b})^2 - (2v-1)(\sqrt[4]{ab}-\sqrt{a})^2. \qquad (9)$$

Proof. *(i)* Notice that $(3-4v) \notin [0,1]$ where $v \notin [\frac{1}{2}, \frac{3}{4}]$. Compute

$$\begin{aligned}
&(1-v)a + vb + (v-1)(\sqrt{a}-\sqrt{b})^2 - (2v-1)(\sqrt[4]{ab}-\sqrt{b})^2 \\
&= (1-v)a + vb + (v-1)(a - 2\sqrt{ab} + b) - (2v-1)(\sqrt{ab} - 2\sqrt[4]{ab^3} + b) \\
&= (3-4v)\sqrt{ab} + (4v-2)\sqrt[4]{ab^3} \\
&\leqslant (\sqrt{ab})^{3-4v}(\sqrt[4]{ab^3})^{4v-2} = a^{1-v}b^{v} \qquad \text{(by (2))}
\end{aligned}$$

and so

$$(1-v)a + vb + (v-1)(\sqrt{a}-\sqrt{b})^2 - (2v-1)(\sqrt[4]{ab}-\sqrt{b})^2 \leqslant a^{1-v}b^{v},$$

which gives the inequality (8).

(ii) If $v \notin [\frac{1}{4}, \frac{1}{2}]$, then $(4v-1) \notin [0,1]$. We have

$$\begin{aligned}
&(1-v)a + vb - v(\sqrt{a}-\sqrt{b})^2 + (2v-1)(\sqrt[4]{ab}-\sqrt{a})^2 \\
&= (1-v)a + vb - v(a - 2\sqrt{ab} + b) + (2v-1)(\sqrt{ab} - 2\sqrt[4]{a^3b} + a) \\
&= (4v-1)\sqrt{ab} + (2-4v)\sqrt[4]{a^3b} \\
&\leqslant (\sqrt{ab})^{4v-1}(\sqrt[4]{a^3b})^{2-4v} = a^{1-v}b^{v} \qquad \text{(by (2))}
\end{aligned}$$

so we get the desired inequality (9) as follows

$$(1-v)a + vb - v(\sqrt{a}-\sqrt{b})^2 + (2v-1)(\sqrt[4]{ab}-\sqrt{a})^2 \leqslant a^{1-v}b^{v}.$$

This completes the proof of Theorem. □

REMARK 1. We here give advantage of Theorem 1 in comparison with Corollary 1.

(a) Firstly, inequality (8) corresponds to *(ii)* of Corollary 1 when $v \in [\frac{1}{4}, \frac{1}{2}]$. Notice that the range of *(i)* of Theorem 1 is wider than *(ii)* of Corollary 1. Namely, *(i)* of Theorem 1 also holds in the cases such as $v \in [0, \frac{1}{4}]$ and $v \in [\frac{3}{4}, 1]$.

(a1) For the case of $v \in [0, \frac{1}{4}]$, we easily find that the right hand side of *(i)* of Theorem 1 is less than or equal to the right hand side of *(i)* of Corollary 1. This means that *(i)* of Theorem 1 is tighter bound of $(1-v)a+vb$ than *(i)* of Corollary 1 where $v \in [0, \frac{1}{4}]$.

(a2) For the case of $v \in [\frac{3}{4}, 1]$, we can claim that the right hand side of *(i)* of Theorem 1 is less than or equal to the right hand side of *(iv)* of Corollary 1. Since the inequality

$$v(\sqrt{a}-\sqrt{b})^2+2(v-1)(\sqrt[4]{ab}-\sqrt{a})^2 \geqslant (1-v)(\sqrt{a}-\sqrt{b})^2+(2v-1)(\sqrt[4]{ab}-\sqrt{b})^2$$

is equivalent to the inequality

$$(t^{1/4}-1)^2\left\{(4v-2)t^{1/4}+4v-3\right\} \geqslant 0,$$

for $t > 0$ and $v \in [\frac{3}{4}, 1]$, this is obviously true.

(b) Secondly, inequality (9) corresponds to *(iii)* of Corollary 1 when $v \in [\frac{1}{2}, \frac{3}{4}]$. Notice that *(ii)* of Theorem 1 also holds in the cases such as $v \in [0, \frac{1}{4}]$ and $v \in [\frac{3}{4}, 1]$.

(b1) For the case of $v \in [\frac{3}{4}, 1]$, we easily find that the right hand side of *(ii)* of Theorem 1 is less than or equal to the right hand side of *(iv)* of Corollary 1.

(b2) For the case of $v \in [0, \frac{1}{4}]$, we can claim that the right hand side of *(ii)* of Theorem 1 is less than or equal to the right hand side of *(i)* of Corollary 1. Since the inequality

$$(1-v)(\sqrt{a}-\sqrt{b})^2-2v(\sqrt[4]{ab}-\sqrt{b})^2 \geqslant v(\sqrt{a}-\sqrt{b})^2-(2v-1)(\sqrt[4]{ab}-\sqrt{a})^2$$

is equivalent to the inequality

$$t^{1/4}(t^{1/4}-1)^2\left\{(1-4v)t^{1/4}+2-4v\right\} \geqslant 0,$$

for $t > 0$ and $v \in [0, \frac{1}{4}]$, this is obviously true.

Thus for all cases, the right hand sides of both inequalities (8) and (9) in Theorem 1 give tighter upper bounds of v-weighted arithmetic mean than those in Corollary 1.

As a direct consequence of Theorem 1, we have the following improved reverse Heinz inequalities

COROLLARY 3. *Let $a, b \geqslant 0$ and $v \in [0, 1]$.*

(i) If $v \notin [\frac{1}{2}, \frac{3}{4}]$, then

$$\frac{a+b}{2} \leqslant H_v(a,b)+(1-v)(\sqrt{a}-\sqrt{b})^2+\left(v-\frac{1}{2}\right)\left[(\sqrt[4]{ab}-\sqrt{b})^2+(\sqrt[4]{ab}-\sqrt{a})^2\right].$$

(ii) If $v \notin [\frac{1}{4}, \frac{1}{2}]$, then

$$\frac{a+b}{2} \leqslant H_v(a,b)+v(\sqrt{a}-\sqrt{b})^2-\left(v-\frac{1}{2}\right)\left[(\sqrt[4]{ab}-\sqrt{b})^2+(\sqrt[4]{ab}-\sqrt{a})^2\right].$$

REMARK 2. The importance of Corollary 3 in comparison with Corollary 2 is same to the advantage of Theorem 1 in comparison with Corollary 1.

COROLLARY 4. *Let* $a,b \geqslant 0$ *and* $v \in [0,1]$.

(i) If $v \notin [\frac{1}{2}, \frac{3}{4}]$, *then*

$$(1-v)a^2 + vb^2 \leqslant \left(a^{1-v}b^v\right)^2 + (1-v)(a-b)^2 + (2v-1)(\sqrt{ab}-b)^2. \tag{10}$$

(ii) If $v \notin [\frac{1}{4}, \frac{1}{2}]$, *then*

$$(1-v)a^2 + vb^2 \leqslant \left(a^{1-v}b^v\right)^2 + v(a-b)^2 - (2v-1)(\sqrt{ab}-a)^2. \tag{11}$$

Proof. Replacing a and b by their squares in Theorem 1 gives the desired inequalities. □

THEOREM 2. *Let* $a,b \geqslant 0$ *and* $v \in [0,1]$.

(i) If $v \notin [\frac{1}{2}, \frac{3}{4}]$, *then*

$$((1-v)a + vb)^2 \leqslant \left(a^{1-v}b^v\right)^2 + (1-v)^2(a-b)^2 + (2v-1)(\sqrt{ab}-b)^2. \tag{12}$$

(ii) If $v \notin [\frac{1}{4}, \frac{1}{2}]$, *then*

$$((1-v)a + vb)^2 \leqslant \left(a^{1-v}b^v\right)^2 + v^2(a-b)^2 - (2v-1)(\sqrt{ab}-a)^2. \tag{13}$$

Proof. (i) By easy calculation, we have

$$\begin{aligned}
&((1-v)a+vb)^2 - (1-v)^2(a-b)^2 \\
&= (1-v)^2a^2 + v^2b^2 + 2v(1-v)ab - (1-v)^2a^2 - (1-v)^2b^2 + 2(1-v)^2ab \\
&= (1-v)a^2 + vb^2 - (1-v)(a-b)^2 \\
&\leqslant \left(a^{1-v}b^v\right)^2 + (2v-1)(\sqrt{ab}-b)^2 \qquad \text{(by (10))}
\end{aligned}$$

which gives the inequality (12).

(ii) According to the inequality (11), the proof can be completed by an argument similar to that used in *(i)*. □

3. Operator inequalities for the improved reverse Young and Heinz inequalities

In section 2, we obtained the improved reverse Young and Heinz inequalities for positive scalars a and b in Theorem 1 and Corollary 3. Now we are going to extend them for positive invertible operators.

LEMMA 2. ([7]) *Let $X \in B(H)$ be self-adjoint and let f and g be continuous functions such that $f(t) \leqslant g(t)$ for all t in the spectrum of X. Then $f(X) \leqslant g(X)$.*

LEMMA 3. ([19]) *Let $A,B \in B^{++}(H)$, $v \in [0,1]$, $r = \min\{v, 1-v\}$ and $r_0 = \min\{2r, 1-2r\}$.*

(i) if $0 \leqslant v \leqslant \frac{1}{2}$, then

$$A\nabla_v B \leqslant A\sharp_v B + 2(1-v)(A\nabla B - A\sharp B) - r_0(A\sharp B + B - 2A\sharp_{\frac{3}{4}} B).$$

(ii) If $\frac{1}{2} < v \leqslant 1$, then

$$A\nabla_v B \leqslant A\sharp_v B + 2v(A\nabla B - A\sharp B) - r_0(A\sharp B + A - 2A\sharp_{\frac{1}{4}} B).$$

COROLLARY 5. *Let $A,B \in B^{++}(H)$ and $v \in [0,1]$.*

(i) If $0 \leqslant v \leqslant \frac{1}{4}$, then

$$A\nabla_v B \leqslant A\sharp_v B + 2(1-v)(A\nabla B - A\sharp B) - 2v(A\sharp B + B - 2A\sharp_{\frac{3}{4}} B).$$

(ii) If $\frac{1}{4} \leqslant v \leqslant \frac{1}{2}$, then

$$A\nabla_v B \leqslant A\sharp_v B + 2(1-v)(A\nabla B - A\sharp B) + (2v-1)(A\sharp B + B - 2A\sharp_{\frac{3}{4}} B).$$

(iii) If $\frac{1}{2} < v \leqslant \frac{3}{4}$, then

$$A\nabla_v B \leqslant A\sharp_v B + 2v(A\nabla B - A\sharp B) - (2v-1)(A\sharp B + A - 2A\sharp_{\frac{1}{4}} B).$$

(iv) If $\frac{3}{4} \leqslant v \leqslant 1$, then

$$A\nabla_v B \leqslant A\sharp_v B + 2v(A\nabla B - A\sharp B) + (2v-2)(A\sharp B + A - 2A\sharp_{\frac{1}{4}} B).$$

Now we obtain the operator version of Theorem 1 as follows

THEOREM 3. *Let $A,B \in B^{++}(H)$ and $v \in [0,1]$.*

(i) If $v \notin [\frac{1}{2}, \frac{3}{4}]$, then

$$A\nabla_v B \leqslant A\sharp_v B + 2(1-v)(A\nabla B - A\sharp B) + (2v-1)(A\sharp B + B - 2A\sharp_{\frac{3}{4}} B). \tag{14}$$

(ii) If $v \notin [\frac{1}{4}, \frac{1}{2}]$, then

$$A\nabla_v B \leqslant A\sharp_v B + 2v(A\nabla B - A\sharp B) - (2v-1)(A\sharp B + A - 2A\sharp_{\frac{1}{4}} B). \tag{15}$$

Proof. *(i)* According to the inequality (8), we have the following inequality for $t \geqslant 0$

$$(1-v)+vt \leqslant t^v + (1-v)(1-\sqrt{t})^2 + (2v-1)(\sqrt[4]{t}-\sqrt{t})^2$$
$$= t^v + (1-v)\left(1+t-2t^{\frac{1}{2}}\right) + (2v-1)\left(t^{\frac{1}{2}}+t-2t^{\frac{3}{4}}\right),$$

if we replace t with $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ and then multiplying both sides of the inequality by $A^{\frac{1}{2}}$, we get

$$(1-v)A+vB \leqslant A\sharp_v B + (1-v)(A+B-2A\sharp B) + (2v-1)(A\sharp B+B-2A\sharp_{\frac{3}{4}}B),$$

since $(1-v)A+vB = A\nabla_v B$ and $\frac{A+B}{2} = A\nabla B$, so we have

$$A\nabla_v B \leqslant A\sharp_v B + 2(1-v)(A\nabla B - A\sharp B) + (2v-1)(A\sharp B + B - 2A\sharp_{\frac{3}{4}}B),$$

which is the desired inequality (14).

(ii) The line of proof is similar to the one presented in *(i)* by applying the inequality (9), thus we omit it. □

REMARK 3. According to the one-to-one correspondence between Theorem 1 and Theorem 3, the advantage of Theorem 3 in comparison with Corollary 5 is similar to the advantage of Theorem 1 in comparison with Corollary 1.

By applying (7), the operator version of Corollary 3 is deduced immediately by Theorem 3 as follows

COROLLARY 6. *Let* $A,B \in B^{++}(H)$ *and* $v \in [0,1]$.

(i) *If* $v \notin [\frac{1}{2},\frac{3}{4}]$, *then*

$$A\nabla B \leqslant H_v(A,B) + 2(1-v)(A\nabla B - A\sharp B) + (2v-1)(A\sharp B + A\nabla B - 2H_{\frac{3}{4}}(A,B)).$$

(ii) *If* $v \notin [\frac{1}{4},\frac{1}{2}]$, *then*

$$A\nabla B \leqslant H_v(A,B) + 2v(A\nabla B - A\sharp B) - (2v-1)(A\sharp B + A\nabla B - 2H_{\frac{1}{4}}(A,B)).$$

4. Improved reverse Young inequality for the Hilbert-Schmidt norm

In this section, we give the new improvement of the reverse Young inequality for the Hilbert-Schmidt norm based on Theorem 2.

LEMMA 4. ([19]) *Let* $A,B,X \in M_n(\mathbb{C})$, $A,B \geqslant 0$, $v \in [0,1]$, $r = \min\{v,1-v\}$ *and* $r_0 = \min\{2r,1-2r\}$.

(i) if $0 \leqslant v \leqslant \frac{1}{2}$, *then*

$$\|(1-v)AX+vXB\|_2^2 \leqslant \left\|A^{1-v}XB^v\right\|_2^2+(1-v)^2\|AX-XB\|_2^2 - r_0\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}-XB\right\|_2^2.$$

(ii) if $\frac{1}{2} < v \leqslant 1$, *then*

$$\|(1-v)AX+vXB\|_2^2 \leqslant \left\|A^{1-v}XB^v\right\|_2^2+v^2\|AX-XB\|_2^2 - r_0\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}-AX\right\|_2^2.$$

COROLLARY 7. *Let* $A,B,X \in M_n(\mathbb{C})$, $A,B \geqslant 0$ *and* $v \in [0,1]$.

(i) If $0 \leqslant v \leqslant \frac{1}{4}$, *then*

$$\|(1-v)AX+vXB\|_2^2 \leqslant \left\|A^{1-v}XB^v\right\|_2^2+(1-v)^2\|AX-XB\|_2^2 - 2v\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}-XB\right\|_2^2.$$

(ii) If $\frac{1}{4} \leqslant v \leqslant \frac{1}{2}$, *then*

$$\|(1-v)AX+vXB\|_2^2 \leqslant \left\|A^{1-v}XB^v\right\|_2^2+(1-v)^2\|AX-XB\|_2^2 + (2v-1)\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}-XB\right\|_2^2.$$

(iii) If $\frac{1}{2} < v \leqslant \frac{3}{4}$, *then*

$$\|(1-v)AX+vXB\|_2^2 \leqslant \left\|A^{1-v}XB^v\right\|_2^2+v^2\|AX-XB\|_2^2 - (2v-1)\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}-AX\right\|_2^2.$$

(iv) If $\frac{3}{4} \leqslant v \leqslant 1$, *then*

$$\|(1-v)AX+vXB\|_2^2 \leqslant \left\|A^{1-v}XB^v\right\|_2^2+v^2\|AX-XB\|_2^2 + (2v-2)\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}-AX\right\|_2^2.$$

THEOREM 4. *Let* $A,B,X \in M_n(\mathbb{C})$, $A,B \geqslant 0$ *and* $v \in [0,1]$.

(i) If $v \notin [\frac{1}{2}, \frac{3}{4}]$, *then*

$$\|(1-v)AX+vXB\|_2^2 \leqslant \left\|A^{1-v}XB^v\right\|_2^2+(1-v)^2\|AX-XB\|_2^2 + (2v-1)\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}-XB\right\|_2^2. \tag{16}$$

(ii) If $v \notin [\frac{1}{4}, \frac{1}{2}]$, then

$$\|(1-v)AX + vXB\|_2^2 \leqslant \left\|A^{1-v}XB^v\right\|_2^2 + v^2 \|AX - XB\|_2^2 - (2v-1)\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}} - AX\right\|_2^2. \tag{17}$$

Proof. By the spectral theorem, there are unitary matrices $U, V \in M_n(\mathbb{C})$ such that $A = UDU^*$ and $B = VEV^*$, where $D = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $E = diag(\gamma_1, \gamma_2, \ldots, \gamma_n)$ with $\lambda_i, \gamma_j \geqslant 0$ for $1 \leqslant i, j \leqslant n$. If $Y = U^*XV = [y_{ij}]$, then we have

$$\begin{aligned}
(1-v)AX + vXB &= U\left[((1-v)\lambda_i + v\gamma_j)y_{ij}\right]V^*, \\
A^{1-v}XB^v &= U\left[(\lambda_i^{1-v}\gamma_j^v)y_{ij}\right]V^*, \\
AX - XB &= U\left[(\lambda_i - \gamma_j)y_{ij}\right]V^*, \\
A^{\frac{1}{2}}XB^{\frac{1}{2}} - XB &= U\left[((\lambda_i\gamma_j)^{\frac{1}{2}} - \gamma_j)y_{ij}\right]V^*, \\
A^{\frac{1}{2}}XB^{\frac{1}{2}} - AX &= U\left[((\lambda_i\gamma_j)^{\frac{1}{2}} - \lambda_i)y_{ij}\right]V^*.
\end{aligned}$$

The ordering of scalar means is equivalent to the ordering of Hilbert-Schmidt norm inequality (see Proposition 2.5 in [15] or Exercise 5.1.5 in [8]). Now using this fact and the inequalities in Theorem 2 with $a = \lambda_i, b = \gamma_j$, we get the desired inequalities. □

REMARK 4. Theorem 2 is a consequence of Theorem 1. So according to Remark 1, the advantage of Theorem 4 in comparison with Corollary 7 follow such way as in Remark 1.

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