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TURÁN TYPE INEQUALITIES FOR GENERALIZED MITTAG–LEFFLER FUNCTION

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Abstract. In the present paper, we introduce a generalization of Mittag-Leffler function by considering the p-gamma function. Some Turán type inequalities for generalized Mittag-Leffler function were obtained.

1. Introduction

The Mittag-Leffler function is defined by

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)},\quad z,\alpha,\beta\in\mathbb{C},\mathrm{Re}(\alpha)>0,\mathrm{Re}(\beta)>0,\tag{1.1}$$

where $\Gamma(\cdot)$ is classical gamma function. The function was introduced by Swedish mathematician Mittag-Leffler for $\beta=1$.

The Mittag-Leffler function plays an important role in several branches of mathematics and engineering sciences, such as statistics, chemistry, mechanics, quantum physics, informatics and others. In particular, it is an explicit formula for the resolvent of Riemann-Liouville fractional integrals by Hille and Tamarkin. The more properties and applications of Mittag-Leffler are collected, for instance, in references [1],[2]. We also refer to the references [3],[4],[5].

The work on this paper has been inspired by a preprinted article by K. Mehrez and S. M. Sitnik [6, 7] in 2016. They obtained some Turán type inequalities for Mittag-Leffler function by considering monotonicity for special ratio of sections for series of Mittag-Leffler function.

In this paper, we consider the following generalized Mittag-Leffler function defined by

$$E_{\alpha,\beta,p}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma_p(\alpha n+\beta)},\quad \alpha,\beta,z\in\mathbb{C},p\in(0,\infty),\mathrm{Re}(\alpha)>0,\mathrm{Re}(\beta)>0,\tag{1.2}$$

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where $\Gamma_p(x)$ is classical p-gamma function defined by

$$\Gamma_p(x) = \frac{p!p^x}{x(x+1)\cdots(x+p)}.$$

It is known that $\lim_{p\to\infty}\Gamma_p(x) = \Gamma(x)$. The logarithmic derivative of p-gamma function

$$\psi_p(x) = \frac{d}{dx}\log\Gamma_p(x) = \frac{\Gamma_p'(x)}{\Gamma_p(x)}$$

is known as generalized digamma function. Its derivatives $\psi_p^{(n)}(x)$ are known as the generalized polygamma function. These functions have the following representations of series

$$\psi_p^{(n)}(x) = \sum_{k=0}^{p}\frac{(-1)^{n-1}n!}{(x+k)^{n+1}}. \tag{1.3}$$

Due to the generalized Mittag-Leffler functions, our main results read as follows.

THEOREM 1.1 *For $\alpha,\beta,p>0$ and fixed $z>0$, the function $f:\beta\mapsto\Gamma_p(\beta)E_{\alpha,\beta,p}(z)$ is strictly log-covex on $(0,\infty)$. As a result, we have the following inequality*

$$E^2_{\alpha,\beta+1,p}(z) < \frac{(\beta+1)(\beta+p+1)}{\beta(\beta+p+2)}E_{\alpha,\beta,p}(z)E_{\alpha,\beta+2,p}(z). \tag{1.4}$$

COROLLARY 1.1 *For $\alpha,p>0,\beta_2>\beta_1>0$ and fixed $z\in(0,\infty)$, we have*

$$\frac{E_{\alpha,\beta_1+1,p}(z)}{E_{\alpha,\beta_1,p}(z)} < \frac{\beta_2(\beta_1+p+1)}{\beta_1(\beta_2+p+1)}\frac{E_{\alpha,\beta_2+1,p}(z)}{E_{\alpha,\beta_2,p}(z)}. \tag{1.5}$$

Putting

$$E^n_{\alpha,\beta,p}(z) = E_{\alpha,\beta,p}(z) - \sum_{k=0}^{n}\frac{z^k}{\Gamma_p(\alpha k+\beta)} = \sum_{k=n+1}^{\infty}\frac{z^k}{\Gamma_p(\alpha k+\beta)}, \tag{1.6}$$

we have the following results.

THEOREM 1.2 *For $n\in\mathbb{N},\alpha,\beta,z>0$, we have*

$$E^n_{\alpha,\beta,p}(z)E^{n+2}_{\alpha,\beta,p}(z) \leqslant \left[E^{n+1}_{\alpha,\beta,p}(z)\right]^2. \tag{1.7}$$

THEOREM 1.3 *For $\alpha,\beta,z>0$ and $n\in\mathbb{N}$, the function*

$$g_n : z\mapsto g_n(\alpha,\beta,p,z) = \frac{E^n_{\alpha,\beta,p}(z)E^{n+2}_{\alpha,\beta,p}(z)}{\left[E^{n+1}_{\alpha,\beta,p}(z)\right]^2} \tag{1.8}$$

is increasing on $(0,\infty)$. *As a result, we have*

$$E^{n}_{\alpha,\beta,p}(z)E^{n+2}_{\alpha,\beta,p}(z) \geqslant \prod_{j=0}^{p} \frac{(n\alpha+\alpha+\beta+j)(n\alpha+3\alpha+\beta+j)}{(n\alpha+2\alpha+\beta+j)^2} \left[E^{n+1}_{\alpha,\beta,p}(z)\right]^2. \tag{1.9}$$

The constant $\prod_{j=0}^{p} \frac{(n\alpha+\alpha+\beta+j)(n\alpha+3\alpha+\beta+j)}{(n\alpha+2\alpha+\beta+j)^2}$ *is the best possible for which the inequality* (1.9) *holds true.*

2. Lemmas

LEMMA 2.1 ([8]) *Let* $\{a_n\}$ *and* $\{b_n\},(n=0,1,2,\cdots)$ *be real numbers such that* $b_n > 0$ *and* $\{\frac{a_n}{b_n}\}_{n\geqslant 0}$ *is increasing(decreasing), then* $\{\frac{a_0+a_1+\cdots+a_n}{b_0+b_1+\cdots+b_n}\}$ *is increasing(decreasing).*

LEMMA 2.2 ([9]) *Let* $\{a_n\}$ *and* $\{b_n\},(n=0,1,2,\cdots)$ *be real numbers and let the power series* $A(x)=\sum_{n=0}^{\infty} a_n x^n$ *and* $B(x)=\sum_{n=0}^{\infty} b_n x^n$ *be convergent if* $|x|<r$. *If* $b_n>0,(n=0,1,2,\cdots)$ *and the sequence* $\{\frac{a_n}{b_n}\}_{n\geqslant 0}$ *is (strictly)increasing(decreasing), then the function* $\frac{A(x)}{B(x)}$ *is also (strictly) increasing(decreasing) on* $[0,r)$.

3. Proofs of Theorems

Proof of Theorem 1.1.
Simple computation yields

$$\frac{\partial}{\partial\beta}\left(\log\frac{\Gamma_p(\beta)}{\Gamma_p(\alpha k+\beta)}\right)=\psi_p(\beta)-\psi_p(\alpha k+\beta)$$

and

$$\frac{\partial^2}{\partial\beta^2}\left(\log\frac{\Gamma_p(\beta)}{\Gamma_p(\alpha k+\beta)}\right)=\psi_p'(\beta)-\psi_p'(\alpha k+\beta)<0,$$

where we apply that the function $\psi_p(x)$ is concave on $\mathbb{R}$. Therefore, we get that the function $\beta \mapsto \frac{\Gamma_p(\beta)}{\Gamma_p(\alpha k+\beta)}$ is strictly log-convex on $(0,\infty)$. Using the fact that the sum of log-convex functions is also log-convex, we obtain that the function f is strictly log-convex on $(0,\infty)$.

Due to inequality (1.4), we easily know

$$\log f\left(\frac{\beta+\beta+2}{2}\right) < \frac{\log f(\beta)+\log f(\beta+2)}{2}.$$

That is

$$E^2_{\alpha,\beta+1,p}(z) < \frac{\Gamma_p(\beta)\Gamma_p(\beta+2)}{[\Gamma_p(\beta+1)]^2}E_{\alpha,\beta,p}(z)E_{\alpha,\beta+2,p}(z).$$

Using the definition of $\Gamma_p(x)$, we easily obtain

$$\frac{\Gamma_p(\beta)\Gamma_p(\beta+2)}{[\Gamma_p(\beta+1)]^2}=\frac{(\beta+1)(\beta+p+1)}{\beta(\beta+p+2)}.$$

The proof of Theorem 1.1 is complete.

Proof of Corollary 1.1.

Since the function $f(\beta)$ is strictly log-convex, we obtain that the function

$$\frac{f(\beta+1)}{f(\beta)}=\frac{\Gamma_p(\beta+1)E_{\alpha,\beta+1,p}(z)}{\Gamma_p(\beta)E_{\alpha,\beta,p}(z)}$$

is strictly increasing on $(0,\infty)$. Taking $0<\beta_1<\beta_2$, we have

$$\frac{\Gamma_p(\beta_1+1)E_{\alpha,\beta_1+1,p}(z)}{\Gamma_p(\beta_1)E_{\alpha,\beta_1,p}(z)}<\frac{\Gamma_p(\beta_2+1)E_{\alpha,\beta_2+1,p}(z)}{\Gamma_p(\beta_2)E_{\alpha,\beta_2,p}(z)}.$$

Considering the formula

$$\frac{\Gamma_p(\beta_2+1)}{\Gamma_p(\beta_2)}\cdot\frac{\Gamma_p(\beta_1)}{\Gamma_p(\beta_1+1)}=\frac{\beta_2(\beta_1+p+1)}{\beta_1(\beta_2+p+1)},$$

we complete the proof.

Proof of Theorem 1.2.

Using the formulas

$$E^n_{\alpha,\beta,p}(z)=E^{n+1}_{\alpha,\beta,p}(z)+\frac{z^{n+1}}{\Gamma_p[\alpha(n+1)+\beta]}$$

and

$$E^{n+2}_{\alpha,\beta,p}(z)=E^{n+1}_{\alpha,\beta,p}(z)-\frac{z^{n+2}}{\Gamma_p[\alpha(n+2)+\beta]},$$

we have

$$\begin{aligned}
&E^n_{\alpha,\beta,p}(z)E^{n+2}_{\alpha,\beta,p}(z)-[E^{n+1}_{\alpha,\beta,p}(z)]^2\\
&=E^{n+1}_{\alpha,\beta,p}(z)\left[\frac{z^{n+1}}{\Gamma_p[\alpha(n+1)+\beta]}-\frac{z^{n+2}}{\Gamma_p[\alpha(n+2)+\beta]}\right]-\frac{z^{2n+3}}{\Gamma_p[\alpha(n+1)+\beta]\Gamma_p[\alpha(n+2)+\beta]}\\
&=\sum_{k=n+3}^{\infty}\frac{z^{n+k+1}}{\Gamma_p(\alpha k+\beta)\Gamma_p[\alpha(n+1)+\beta]}-\sum_{k=n+3}^{\infty}\frac{z^{n+k+1}}{\Gamma_p[\alpha(k-1)+\beta]\Gamma_p[\alpha(n+2)+\beta]}\\
&=\sum_{k=n+3}^{\infty}\frac{\Gamma_p[\alpha(k-1)+\beta]\Gamma_p[\alpha(n+2)+\beta]-\Gamma_p(\alpha k+\beta)\Gamma_p[\alpha(n+1)+\beta]}{\Gamma_p(\alpha k+\beta)\Gamma_p[\alpha(n+1)+\beta]\Gamma_p[\alpha(k-1)+\beta]\Gamma_p[\alpha(n+2)+\beta]}z^{n+k+1}.
\end{aligned}$$

Since the function $\Gamma_p(x)$ is log-convex on $(0,\infty)$, we know that the function $x\mapsto \frac{\Gamma_p(x+a)}{\Gamma_p(x)}(a>0)$ is increasing on $(0,\infty)$. Thus, with $a=\alpha, x=\alpha(n+1)+\beta<\alpha(n+1)+\beta+\alpha(k-(n+2))$, we obtain

$$\frac{\Gamma_p[\beta+\alpha(n+1)+\alpha]}{\Gamma_p[\beta+\alpha(n+1)]}\leqslant\frac{\Gamma_p[\beta+\alpha(n+1)+\alpha+\alpha(k-(n+2))]}{\Gamma_p[\beta+\alpha(n+1)+\alpha(k-(n+2))]}.$$

That is

$$\frac{\Gamma_p[\alpha(n+2)+\beta]}{\Gamma_p[\alpha(n+1)+\beta]} \leqslant \frac{\Gamma_p(\alpha k+\beta)}{\Gamma_p[\alpha(k-1)+\beta]}.$$

It follows that

$$E^{n}_{\alpha,\beta,p}(z)E^{n+2}_{\alpha,\beta,p}(z) - \left[E^{n+1}_{\alpha,\beta,p}(z)\right]^2 \geqslant 0.$$

Proof of Theorem 1.3.

Easy calculation results in

$$g_n(\alpha,\beta,p,z) = \frac{\sum_{k=n+1}^{\infty}\frac{z^k}{\Gamma_p(\alpha k+\beta)}\sum_{k=n+3}^{\infty}\frac{z^k}{\Gamma_p(\alpha k+\beta)}}{[\sum_{k=n+2}^{\infty}\frac{z^k}{\Gamma_p(\alpha k+\beta)}]^2}$$

$$= \frac{\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\frac{1}{\Gamma_p[\alpha(n+1+j)+\beta]\Gamma_p[\alpha(n+3+k-j)+\beta]}\right)z^{2n+2+k}}{\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\frac{1}{\Gamma_p[\alpha(n+2+j)+\beta]\Gamma_p[\alpha(n+2+k-j)+\beta]}\right)z^{2n+2+k}}.$$

We write the sequence $\{\lambda_j\}_{j\geqslant 0}$, defined by

$$\lambda_j = \frac{\Gamma_p[\alpha(n+2+j)+\beta]\,\Gamma_p[\alpha(n+2+k-j)+\beta]}{\Gamma_p[\alpha(n+1+j)+\beta]\,\Gamma_p[\alpha(n+3+k-j)+\beta]}.$$

Using the fact that the function $\frac{\Gamma_p(x+a)}{\Gamma_p(x)}(a>0)$ is increasing on $(0,\infty)$ again, we have

$$\frac{\lambda_{j+1}}{\lambda_j} \geqslant 1.$$

Using Lemma 2.1 and Lemma 2.2, we easily obtain that $z \mapsto g_n(\alpha,\beta,p,z)$ is increasing on $z\in(0,\infty)$. This implies

$$g_n(\alpha,\beta,p,z) \geqslant \lim_{z\to 0} g_n(\alpha,\beta,p,z)$$

$$= \frac{\Gamma_p^2[\alpha(n+2)+\beta]}{\Gamma_p[\alpha(n+1)+\beta]\,\Gamma_p[\alpha(n+3)+\beta]}$$

$$= \prod_{j=0}^{p}\frac{(n\alpha+\alpha+\beta+j)(n\alpha+3\alpha+\beta+j)}{(n\alpha+2\alpha+\beta+j)^2}.$$

The proof is complete.

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