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ON A NEW FAMILY OF BIVARIATE MEANS

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Abstract. A new family of bivariate means is defined and investigated. Members of that family are generated by the Schwab-Borchardt mean. Comparison results involving new means and the second Neuman mean are established. In particular, two means introduced and studied by J. Sándor and Z. Yang belong to a new class of means.

1. Introduction

In recent years means of two variables and their inequalities have attracted attention of several researchers. A complete list of research papers which deal with this subject is too long to be included here.

In this paper we introduce and study a family of bivariate means whose definition is included below. In what follows the letters a and b will always stand for positive and unequal numbers. A generic strict mean of a and b will be by denoted by $m(a,b) \equiv m$. It satisfies the double inequality

$$\min(a,b) < m(a,b) < \max(a,b). \tag{1.1}$$

We will always assume that the mean m is homogeneous of degree one in its variables.

In the sequel we choose as the mean m the Schwab-Borchardt mean which is defined as follows:

$$SB(a,b) \equiv SB = \begin{cases} \dfrac{\sqrt{b^2-a^2}}{\cos^{-1}(a/b)} & \text{if } a < b, \\ \dfrac{\sqrt{a^2-b^2}}{\cosh^{-1}(a/b)} & \text{if } b < a \end{cases} \tag{1.2}$$

(see, e.g., [2], [3]). This mean has been studied extensively in [15], [16], and in [14]. It is well known that the mean SB is strict, nonsymmetric and homogeneous of degree one in its variables.

Mean SB can also be represented in terms of the degenerated completely symmetric elliptic integral of the first kind (see, e.g., [13]). It has been pointed out in [15] that

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some well known bivariate means such as logarithmic mean and two Seiffert means (see [21, 22]) can be represented as the Schwab-Borchardt mean of two simpler means such as geometric and arithmetic means or as the Schwab-Borchardt mean of arithmetic and the square root mean. This idea was employed lately by this author and other researchers as well. For more details see [4, 5, 6, 7, 8, 9, 10, 14, 17, 18, 24, 25]

Another bivariate mean utilized in this paper is defined as follows:

$$N(a,b) \equiv N = \frac{1}{2}\left(a + \frac{b^2}{SB(a,b)}\right) \tag{1.3}$$

(see [13]). It's easy to see that mean N is also strict, nonsymmetric and homogeneous of degree one in its variables. Some authors call this mean, Neuman mean of the second kind (see, e.g., [4, 5, 17, 18, 24, 25]). Mean N can be represented in terms of the degenerated completely symmetric elliptic integral of the second kind (see, e.g., [13]). By taking the N mean of two other means one can generate several new bivariate means. This idea was partially explored in [13].

This paper can be regarded as continuation of investigations initiated in author's earlier papers [6, 7, 8, 9, 10, 11, 13, 14] and is organized as follows. Preliminaries are given in Section 2. Definition of a new family of bivariate means is included in Section 3. Therein bounds for the new mean are derived. Main results of this paper are established in Section 4.

2. Preliminaries

First of all we will give new formulas for means SB and N. We have [12]

$$SB(a,b) \equiv SB = \begin{cases} b\dfrac{\sin r}{r} = a\dfrac{\tan r}{r} & \text{if } a<b, \\ b\dfrac{\sinh s}{s} = a\dfrac{\tanh s}{s} & \text{if } b<a, \end{cases} \tag{2.1}$$

where

$$\cos r = a/b \quad \text{if} \quad a<b \quad \text{and} \quad \cosh s = a/b \quad \text{if} \quad a>b. \tag{2.2}$$

Clearly

$$0 < r < \pi/2 \tag{2.3}$$

and

$$0 < s < \infty. \tag{2.4}$$

For the later use let us record similar formulas for the mean N. It follows from [12] that

$$N(a,b) \equiv N = \frac{1}{2}b\left(\cos r + \frac{r}{\sin r}\right) = \frac{1}{2}a\left(1 + \frac{r}{\sin r \cos r}\right) \tag{2.5}$$

provided $a<b$. Similarly, if $a>b$, then

$$N(a,b) \equiv N = \frac{1}{2}b\left(\cosh s + \frac{s}{\sinh s}\right) = \frac{1}{2}a\left(1 + \frac{s}{\sinh s \cosh s}\right). \tag{2.6}$$

Here the domains for r and s are the same as these in (2.3) and (2.4).

3. Definition and properties of the new family of bivariate means

The family of means under discussion, denoted by

$$R(a,b) \equiv R$$

is defined as follows:

$$R(a,b) = be^{a/SB(a,b)-1}. \tag{3.1}$$

Clearly function $R : \mathbb{R}^2_+ \to \mathbb{R}$ is nonsymmetric and homogeneous of degree one in its variables.

For the later use let us record two formulas for the mean under discussion

$$R(a,b) = be^{r\cot r-1} \tag{3.2}$$

if $a < b$. Recall that $\cos r = a/b$. Also,

$$R(a,b) = be^{s\coth s-1} \tag{3.3}$$

provided $a > b$. Here $\cosh s = a/b$. Both formulas follow immediately from (3.1), (2.1) and (2.2).

The main result of this section reads as follows:

THEOREM 3.1. *The following inequalities*

$$\min(a,b) < \frac{ab}{SB(a,b)} < R(a,b) < \max(a,b) \tag{3.4}$$

are satisfied.

Proof. For the proof of the second inequality in (3.4) we utilize the following inequality

$$\ln t < t-1 \tag{3.5}$$

$(t > 0,\ t \neq 1)$. Inequality (3.5) follows easily from [1, 4.1.33]

$$\ln(1+x) < x$$

$(x > -1,\ x \neq 0)$ by letting $x = t-1$. To obtain the desired result we substitute in (3.5)

$$t = \frac{R}{b} = e^{a/SB-1}$$

to obtain

$$\frac{a}{SB} - 1 < e^{a/SB-1} - 1 = \frac{R}{b} - 1.$$

This yields the desired result. We shall establish now the first inequality in (3.4). Consider the case when $a < b$. Clearly $SB < b$ and $ab/SB > a$. The first inequality in (3.4) follows. For the proof of the third inequality in (3.4) let us note that $a < SB$ implies $a/SB - 1 < 0$. This in conjunction with (3.1) yields $R < be^0 = b$.

We shall establish now the first and third inequalities in (3.4) when $a > b$. In this case one has $b < SB < a$. This in conjunction with (3.4) implies

$$R(a,b) > \frac{ab}{SB(a,b)} > \frac{SB(a,b)b}{SB(a,b)} = b.$$

In order to show that $R(a,b) < a$ let us note that this inequality together with (3.3) implies

$$e^{s\coth s - 1} < \cosh s$$

or what is the same that

$$s\coth s - 1 < \log(\cosh s) \tag{3.6}$$

holds for all $s > 0$. For the proof of the inequality (3.6) let us introduce the following function $f(s) = \log(\cosh s) - s\coth s + 1$. Differentiation yields

$$(\sinh^2 s)f'(s) = s - \tanh s.$$

This in conjunction with the inequality $s - \tanh s > 0$ implies that $f'(s) > 0$ is valid for all $s > 0$. Taking into account that $f(0^+) = 0$ we conclude that the function $f(s)$ is strictly increasing on the positive semi-axis. Inequality (3.6) has been established. Thus $R(a,b) < a$. The proof is complete. $\square$

4. Main results

For the sake of presentation we include definitions of several bivariate means used in this section.

Let x and y be positive numbers. In order to avoid trivialities we will always assume that $x \neq y$. The unweighted arithmetic mean A of x and y is defined as

$$A = \frac{x+y}{2}.$$

For the reader's convenience we also recall definitions of the first and the second Seiffert means, denoted respectively by P and T

$$P = A\frac{v}{\sin^{-1} v}, \quad T = A\frac{v}{\tan^{-1} v} \tag{4.1}$$

(see [21], [22]), where

$$v = \frac{x-y}{x+y}. \tag{4.2}$$

Clearly $0 < |v| < 1$. Other unweighted bivariate means used in this paper are the geometric mean G and the root-square mean Q which are defined as follows

$$G = \sqrt{xy}, \quad Q = \sqrt{\frac{x^2+y^2}{2}}. \tag{4.3}$$

One can easily verify that the means listed in (4.3) all can be expressed in terms of A and v. We have

$$G = A\sqrt{1-v^2}, \quad Q = A\sqrt{1+v^2} \tag{4.4}$$

All the means mentioned above are comparable. It is known that

$$G < P < A < T < Q \tag{4.5}$$

(see, e.g., [15]).

It has been pointed out in [15] that

$$P = SB(G,A), \quad T = SB(A,Q). \tag{4.6}$$

Two bivariate means introduced in [19, 20, 23]

$$X = Ae^{G/P-1} \tag{4.7}$$

and

$$Y = Qe^{A/T-1} \tag{4.8}$$

are called the Sándor - Yang means. It follows from (3.1) and (4.6) that

$$X = R(G,A) \qquad \text{and} \qquad Y = R(A,Q) \tag{4.9}$$

It is now obvious why mean R has been defined as in (3.1).

We are in a position to prove the following:

THEOREM 4.1. *If $a < b$, then the following inequality*

$$R(a,b) < R(b,a). \tag{4.10}$$

is valid. The inequality (4.10) is reversed if $a > b$.

Proof. Assume that $a < b$. Then inequality (4.10) can be written, using (3.2) and (3.3), as

$$e^{\lambda} < (\cos r)e^{\mu}, \tag{4.11}$$

where $\lambda = r\cot r$ with r defined as $\cos r = a/b$. Also, $\mu = s\coth s$ where s is defined implicitly as follows $\cosh s = b/a$. Since $\cosh s = 1/\cos r$, $\coth s = 1/\sin r$. Also, $s = \cosh^{-1}(1/\cos r) = \tanh^{-1}(\sin r)$. Taking logarithms on both sides of (4.11) and multiplying sides of the resulting inequality by $\sin r$ we obtain

$$r\cos r < (\sin r)\ln(\cos r) + \tanh^{-1}(\sin r) \tag{4.12}$$

where $r \in (0, \pi/2)$. In order to establish inequality (4.12) we introduce a function

$$f(r) = -r\cos r + (\sin r)\ln(\cos r) + \tanh^{-1}(\sin r). \tag{4.13}$$

Differentiation yields

$$f'(r) = (\cos r)\ln(\cos r) + r\sin r.$$

Further, let

$$g(r) := f'(r)/\cos r.$$

Clearly $g(r) = \ln(\cos r) + r\tan r$ and

$$g'(r) = \frac{r}{\cos^2 r}.$$

Since $g(0^+) = 0$ and $g'(r) > 0$ the function $g(r)$ is strictly increasing on $(0, \pi/2)$. This leads to the conclusion that $g(r) > 0$ and in consequence that $f(r) > 0$ for all $r \in (0, \pi/2)$. Thus the inequality (4.12) is valid and in consequence we see that inequality (4.11) holds true. This completes the proof when $a < b$. If $a > b$, then the statement of the theorem can be established by interchanging a with b. The proof is complete. □

Results similar to that of Theorem 4.1 are valid for means SB and N. For the reader's convenience we recall that

$$SB(b,a) < SB(a,b)$$

if $a < b$ with inequality reversed if $a > b$ (see [15, Proposition 2.1]). With mean SB replaced by mean N the same result, as the last one, is valid. See [13, Theorem 4.1].

We shall demonstrate now that means R, SB and N are comparable.

THEOREM 4.2. *If $a < b$, then*

$$R(a,b) < SB(a,b) < N(a,b). \tag{4.14}$$

Otherwise, if $a > b$, then

$$SB(a,b) < N(a,b) < R(a,b). \tag{4.15}$$

Proof. The second inequality in (4.14) and the first one in (4.15) are established in [13, (4.18)]. We shall establish the first inequality in (4.14). Making use of (3.2) and (2.1) we see that the inequality in question reads as follows

$$e^{r\cot r - 1} < \frac{\sin r}{r},$$

where $r \in (0, \pi/2)$. Taking logarithms on both sides we can rewrite the last inequality in the form $f(r) > 0$, where

$$f(r) = \ln\frac{\sin r}{r} - r\cot r + 1. \tag{4.16}$$

Differentiation yields

$$f'(r) = \left[\left(\frac{r}{\sin r}\right)^2 - 1\right]/r.$$

Since $(r/\sin r)^2 > 1$, the inequality $f'(r) > 0$ is valid for all $r \in (0, \pi/2)$. This in conjunction with $f(0^+) = 0$ implies that the function $f(r)$ is positive on the stated domain. This completes the proof of the first inequality in (4.14). We shall establish now the second inequality in (4.15). Making use of (3.3) and (2.6) we see that the inequality in question is equivalent to

$$\ln\left(\cosh s + \frac{s}{\sinh s}\right) < s\coth s - 1 + \ln 2.$$

In order to obtain the desired result we introduce a function

$$f(s) = s\coth s - 1 + \ln 2 - \ln\left(\cosh s + \frac{s}{\sinh s}\right), \tag{4.17}$$

where $s > 0$. Differentiation yields

$$f'(s) = \frac{s\left(\frac{\sinh 2s}{2s} - 1\right)}{(\sinh s)^2\left(\frac{\sinh 2s}{2s} + 1\right)}.$$

Taking into account that $\sinh x > x$ holds for all positive numbers x we conclude that $f'(s) > 0$ for all $s > 0$. Its easy to verify that $f(0^+) = 0$. Thus the function f is strictly positive on its domain. The desired result now follows. □

In our next theorem we provide an inequality connecting means SB and R with permuted arguments.

THEOREM 4.3. *The following inequality*

$$SB(a,b) < R(b,a) \tag{4.18}$$

is valid for all positive and unequal numbers a and b.

Proof. Assume that $a < b$. Then (4.18) is equivalent to

$$\ln\left(\frac{\tan r}{r}\right) < s\coth s - 1, \tag{4.19}$$

where $\cos r = a/b$ and $\cosh s = b/a$. Hence $s = \tanh^{-1}(\sin r)$ and $\coth s = 1/\sin r$. With

$$f(r) := \tanh^{-1}(\sin r) - (\sin r)\ln\left(\frac{\tan r}{r}\right) - \sin r \tag{4.20}$$

$(r \in (0, \pi/2))$ we see that the inequality (4.19) is equivalent to $f(r) > 0$. To prove the last claim we differentiate $f(r)$ to obtain

$$f'(r)/\cos r = \frac{\tan r}{r} - \ln\left(\frac{\tan r}{r}\right) - 1 =: g(r).$$

With $t = \dfrac{\tan r}{r}$ we have $g(r) = t - \ln t - 1$. Application of inequality (3.5) gives $g(r) > 0$. Clearly one has $f'(r) > 0$. This in conjunction with $f(0^+) = 0$ yields $f(r) > 0$. Hence the assertion follows.

Assume now that $a > b$. We employ again formulas (3.3) and (2.1) to write inequality (4.18) in the equivalent form

$$(\sin r)\ln\left(\frac{\tanh s}{s}\right) < r\cos r - \sin r. \tag{4.21}$$

Making use of $\cos r = 1/\cosh s$ we obtain $\sin r = \tanh s$ and also that $\cos^{-1}(1/\cosh s) = \sin^{-1}(\tanh s)$. Easy algebra allows to write (4.21) in the form $f(s) > 0$, where

$$f(s) = \sin^{-1}(\tanh s) - \sinh s - (\sinh s)\ln\left(\frac{\tanh s}{s}\right) \tag{4.22}$$

$(s > 0)$. Differentiating $f(s)$ we obtain

$$f'(s)/\cosh s = -\ln\left(\frac{\tanh s}{s}\right) + \frac{\tanh s}{s} - 1$$

We use again inequality (3.5) in the form

$$-\ln t > -t + 1$$

with $t = \tanh s/s$ to obtain

$$f'(s)/\cosh s > 0.$$

Thus $f'(s) > 0$ provided $s \neq 0$. Utilizing the fact that $f(0^+) = 0$ we obtain the asserted result. The proof is complete. □

A lower bound for the mean $R(a,b)$ can easily be derived from the last theorem.

COROLLARY 1. *If $a, b > 0$ with $a \neq b$, then*

$$a^{2/3}b^{1/3} < R(a,b). \tag{4.23}$$

Proof. It is known that

$$a^{1/3}b^{2/3} < SB(a,b)$$

(see [15]). This in conjunction with (4.18) yields

$$a^{1/3}b^{2/3} < SB(a,b) < R(b,a).$$

Interchanging a with b we obtain the desired inequality (4.23). □

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