Journal of Mathematical Inequalities
Volume 11, Number 3 (2017), 683–694
doi:10.7153/jmi-2017-11-54

ADDITIVE WEIGHTED L_p ESTIMATES OF SOME CLASSES OF INTEGRAL OPERATORS INVOLVING GENERALIZED OINAROV KERNELS

A. M. ABYLAYEVA, A. O. BAIARYSTANOV, L.-E. PERSSON AND P. WALL

(*Communicated by R. Oinarov*)

Abstract. Inequalities of the form

$$\|u\mathscr{K}f\|_q \leqslant C(\|\rho f\|_p + \|vHf\|_p),\ f \geqslant 0,$$

are considered, where $\mathscr{K}$ is an integral operator of Volterra type and H is the Hardy operator. Under some assumptions on the kernel $\mathscr{K}$ we give necessary and sufficient conditions for such an inequality to hold.

1. Introduction

Let $I=(0,+\infty)$, $1 \leqslant p,q<\infty$. Let $u(\cdot)$, $v(\cdot)$ and $\rho(\cdot)$ be weighted functions, i.e. positive measurable functions on I. Let $\mathscr{K}^+$, $\mathscr{K}^-$, H^+ and H^- be integral operators of the form

$$\mathscr{K}^+f(x)=\int_0^x K(x,s)f(s)ds,\quad \mathscr{K}^-f(x)=\int_x^\infty K(t,x)f(t)dt,$$

$$H^+f(x)=\int_0^x f(s)ds,\quad H^-f(x)=\int_x^\infty f(s)ds,\quad x>0,$$

where $K(x,s)\geqslant 0$ as $x\geqslant s\geqslant 0$.

Denote by L_p the set of all measurable functions f such that

$$\|f\|_p := \left(\int_0^\infty |f(x)|^p\,dx\right)^{\frac{1}{p}} < \infty.$$

Inequalities of the form

$$\|uHf\|_q \leqslant C\|vf\|_p, \tag{1.1}$$

Mathematics subject classification (2010): 26D10, 39B62.
Keywords and phrases: Weighted Hardy inequalities, weighted functions, fractional order operator.

Paper JMI-11-54

where H is some of the operators H^+, H^-, K^+ and K^- are called Hardy type inequalities in the literature. For the standard Hardy operators H^+ and H^- almost everything is nowadays known, see e.g. the books [4], [5], [12] and [3] and the references given there. However, for the case with a general positive kernel $k(x,y)$ a characterization of the weights so that (1.1) holds for K^+ or K^- is a long standing open question. However, for some kernels and parameters the answer of this open question is known. The most typical such example is when $k(x,y)$ is a so called Oinarov kernel (in particular satisfying (1.4) below) and when $1 < p \leqslant q < \infty$ or $0 < q < p < \infty$, $p \geqslant 1$. See especially Chapter 2 in [4] and the references therein. Later on R.Oinarov [9] generalized such results to cover also the case with so called generalized Oinarov conditions, for definitions and some of these results see Section 2.

In this paper we consider the following more general additive weighted inequalities

$$\|u\mathscr{K}^+ f\|_q \leqslant C\left(\|\rho f\|_p + \|vH^+ f\|_p\right),\ f \geqslant 0, \tag{1.2}$$

and

$$\|u\mathscr{K}^- f\|_q \leqslant C\left(\|\rho f\|_p + \|vH^- f\|_p\right),\ f \geqslant 0. \tag{1.3}$$

In particular, our results give new information related to the open question mentioned above.

Inequalities of the form (1.2)–(1.3) were considered in [6, 7, 10, 11, 8]. In [8] the inequalities (1.2)–(1.3) have been studied assuming that the kernels $K(\cdot,\cdot)$ of the operators $\mathscr{K}^+$, $\mathscr{K}^-$ satisfy "Oinarov's condition", i.e., that there exist a number $d \geqslant 1$ such that the relation

$$d^{-1}\left(K(x,t) + K(t,s)\right) \leqslant K(x,s) \leqslant d\left(K(x,t) + K(t,s)\right) \tag{1.4}$$

holds for $x \geqslant t \geqslant s > 0$.

In this paper we study the inequalities (1.2)–(1.3) when the kernels of the operators $\mathscr{K}^+$ and $\mathscr{K}^-$ satisfy weaker conditions than the conditions (1.4), namely, we assume that the kernels of the operators $\mathscr{K}^+$ and $\mathscr{K}^-$ belong to the classes $\mathscr{O}_n^+$, $\mathscr{O}_n^-$, $n \geqslant 0$, respectively, which was first introduced in [9]. (for definitions see Section 2)

This paper is organized as follows: In Section 3 we present our main results with proofs. In order not to disturb our presentations we present some Preliminaries of independent interest in Section 2.

Conventions: If A and B are functionals, then $A \ll B$ means that there exist a constant $C > 0$ independent of the arguments of the functionals A and B and the inequality $A \leqslant CB$ holds. In the case $A \ll B \ll A$ we write $A \approx B$.

2. Preliminaries

In [9] the classes $\mathscr{O}_n^+$ and $\mathscr{O}_n^-$ of the kernels of the form $\mathscr{K}^+$, $\mathscr{K}^-$ are defined for each $n \geqslant 0$. We agree to write $K(\cdot,\cdot) \equiv K_n^{\pm}(\cdot,\cdot)$, if $K(\cdot,\cdot) \in \mathscr{O}_n^{\pm}$.

Let $K^+(\cdot,\cdot)$ and $K^-(\cdot,\cdot)$ be nonnegative measurable functions in $\Omega = \{(x,s) : x \geqslant s \geqslant 0\}$ and besides the function $K^+(\cdot,\cdot)$ is non-decreasing in the first argument and $K^-(\cdot,\cdot)$ is non-increasing in the second argument.

We say that the function $K(\cdot,\cdot) \equiv K_0^{\pm}(\cdot,\cdot)$ belongs to the class $\mathscr{O}_0^{\pm}(\Omega)$ if only if $K_0^+(x,s) = v(s) \geqslant 0$, $K_0^-(x,s) = u(x) \geqslant 0$ for all $(x,s) \in \Omega$.

The classes $\mathscr{O}_n^{\pm}$, $n = 1,2,\dots$ are defined recursively as follows: Let the classes $\mathscr{O}_i^{\pm}(\Omega)$, $i = 0,1,\dots,n-1$, $n \geqslant 1$ be defined. Then $K(\cdot,\cdot) \equiv K_n(\cdot,\cdot) \in \mathscr{O}_n^{\pm}(\Omega)$ if and only if there exist functions $K_i^{\pm}(\cdot,\cdot) \in \mathscr{O}_i^{\pm}(\Omega)$, $i = 0,1,\dots,n-1$ such that

$$K_n^+(x,s) \approx \sum_{i=0}^{n} K_{n,i}^+(x,t)K_i^+(t,s), \tag{2.1}$$

$$K_n^-(x,s) \approx \sum_{i=0}^{n} K_i^-(x,t)K_{i,n}^-(t,s), \tag{2.2}$$

when $0 < s \leqslant t \leqslant x < \infty$ and $K_{n,n}^{\pm}(\cdot,\cdot) \equiv 1$, where the functions $K_{n,i}^+(\cdot,\cdot)$, $K_{i,n}^-(\cdot,\cdot)$, $i = 0,1,\dots,n-1$, generally speaking, are arbitrary nonnegative measurable functions defined on Ω, satisfying the conditions (2.1) or (2.2), respectively. In fact, these functions can be defined in the following form (see [9]):

$$K_{n,i}^+(x,t) = \inf_{0<s\leqslant t} \frac{K_n^+(x,s)}{K_i^+(t,s)},$$
$$K_{i,n}^-(t,s) = \inf_{t<x} \frac{K_n^-(x,s)}{K_i^-(x,t)}, \quad i = 0,1,\dots,n-1.$$

From (2.1) and (2.2) we have for $n = 1$ that the functions $K_1^+(\cdot,\cdot)$, $K_1^-(\cdot,\cdot)$ belong to the classes $\mathscr{O}_1^+$, $\mathscr{O}_1^-$, respectively, if there exist functions $v_1 \geqslant 0$ and $u_1 \geqslant 0$ such that

$$K_1^+(x,s) \approx K_{1,0}^+(x,t)v_1(s) + K_1^+(t,s),$$
$$K_1^-(x,s) \approx K_1^-(x,t) + K_{0,1}^-(t,s)u_1(x),$$

respectively, for all $x \geqslant t \geqslant s > 0$.

In particular, we note that each function, satisfying the condition (1.4), belong to $\mathscr{O}_1^+$ and $\mathscr{O}_1^-$. However, functions from $\mathscr{O}_1^+$ and $\mathscr{O}_1^-$ need not to satisfy the condition (1.4). For example, the functions $K_1^+(x,s) = x^\beta - (x-s)^\beta$ and $K_1^+(x,s) = \ln^\gamma \frac{(x+1)^\beta}{s}$, $x \geqslant s > 0$, $\gamma > 0$, $\beta > 1$, do not satisfy the condition (1.4). However, they belong to the class $\mathscr{O}_1^+(\Omega)$ since

$$x^\beta - (x-s)^\beta \approx (x-t)^{\beta-1}s + t^\beta - (t-s)^\beta, \ x \geqslant t \geqslant s > 0,$$

and

$$\ln^\gamma \frac{(x+1)^\beta}{s} \approx \ln^\gamma \frac{x+1}{t+1} + \ln^\gamma \frac{(t+1)^\beta}{s}, \ x \geqslant t \geqslant s > 0.$$

Consider the inequality (1.1) with $H = K^+$ or $H = K^-$, i.e.

$$\|u\mathscr{K} f\|_q \leqslant C\|vf\|_p, \tag{2.3}$$

where $\mathscr{K}$ is one of the operators $\mathscr{K}^+$ or $\mathscr{K}^-$. The following Theorems were proved in [9]:

THEOREM A^+. *Let $1 < p \leqslant q < \infty$ and the kernel of the operator $\mathscr{K}^+$ belong to the class $\mathscr{O}_n^+(\Omega)$, $n \geqslant 0$. Then the inequality (2.3) holds for the operator $\mathscr{K}^+$ if and only if one of the conditions*

$$A_1^+ = \sup_{z>0}\left(\int_z^\infty u^q(x)\left(\int_0^z \left|K^+(x,s)v^{-1}(s)\right|^{p'} ds\right)^{\frac{q}{p'}} dx\right)^{\frac{1}{q}} < \infty,$$

$$A_2^+ = \sup_{z>0}\left(\int_0^z v^{-p'}(s)\left(\int_z^\infty \left|K^+(x,s)u(x)\right|^{q} dx\right)^{\frac{p'}{q}} ds\right)^{\frac{1}{p'}} < \infty$$

holds and for the best constant $C > 0$ in (2.3) holds the relation $A_1^+ \approx C \approx A_2^+$.

THEOREM A^-. *Let $1 < p \leqslant q < \infty$ and the kernel of the operator $\mathscr{K}^-$ belongs to the class $\mathscr{O}_n^-(\Omega)$, $n \geqslant 0$. Then the inequality (2.3) holds for the operator $\mathscr{K}^-$ if and only if one of the conditions:*

$$A_1^- = \sup_{z>0}\left(\int_0^z u^q(x)\left(\int_z^\infty \left|K^-(x,s)v^{-1}(s)\right|^{p'} ds\right)^{\frac{q}{p'}} dx\right)^{\frac{1}{q}} < \infty,$$

$$A_2^- = \sup_{z>0}\left(\int_z^\infty v^{-p'}(s)\left(\int_0^z \left|K^-(x,s)u(x)\right|^{q} dx\right)^{\frac{p'}{q}} ds\right)^{\frac{1}{p'}} < \infty$$

holds and $A_1^- \approx C \approx A_2^-$, where $C > 0$ is the best constant from (2.3).

Let $1 < p < \infty$. We introduce the functions

$$\varphi(x) = \left\{\inf_{0<t<x}\left[\left(\int_t^x \rho^{-p'}(s)ds\right)^{-\frac{1}{p'}} + \left(\int_t^\infty v^p(s)ds\right)^{\frac{1}{p}}\right]\right\}^{-1},$$

and

$$\psi(x) = \left\{\inf_{x<t}\left[\left(\int_x^t \rho^{-p'}(s)ds\right)^{-\frac{1}{p'}} + \left(\int_0^t v^p(s)ds\right)^{\frac{1}{p}}\right]\right\}^{-1}.$$

The following result was proved in [8]:

THEOREM B^+. *Let $1 < p < \infty$, g is a nonnegative non-increasing function and the functions ρ, v satisfy the conditions $\rho^{-1} \in L_{p'}^{loc}(I)$, $v \in L_p(t,\infty)$, $t > 0$, and $\varphi(0) =$*

0. *Then*

$$\sup_{f\geqslant 0}\frac{\int\limits_0^\infty f(s)g(s)ds}{\|\rho f\|_p+\|vH^+f\|_p}\approx\left(\int\limits_0^\infty g^{p'}(s)d\varphi^{p'}(s)\right)^{\frac{1}{p'}}, \tag{2.4}$$

where $\varphi(0)=\lim\limits_{x\to 0}\varphi(x)$.

Also the next result was formulated in [8]:

THEOREM B^-. *Let* $1<p<\infty$, g *is a nonnegative non-decreasing function and the functions* ρ, v *satisfy the conditions* $\rho^{-1}\in L_{p'}^{loc}(I)$, $v\in L_p(t,\infty)$, $\forall t>0$, *and* $\psi(\infty)=0$. *Then*

$$\sup_{f\geqslant 0}\frac{\int\limits_0^\infty f(s)g(s)ds}{\|\rho f\|_p+\|vH^-f\|_p}\approx\left(\int\limits_0^\infty g^{p'}(s)d(-\psi^{p'}(s))\right)^{\frac{1}{p'}}, \tag{2.5}$$

where $\psi(\infty)=\lim\limits_{x\to\infty}\psi(x)$.

REMARK. The assertion in Theorem B^- was given without proof in [8]. However, this result is crucial for the proof of one of our main result so for completeness we present a proof also of Theorem B^- as a part of our main results given in the next Section.

3. The main results

Our first main result reads:

THEOREM 3.1. *Let* $1<p\leqslant q<\infty$, $\varphi(0)=0$, $\rho^{-1}\in L_{p'}^{loc}(I)$, $v\in L_p(0,t)$, $t>0$, *and the kernel of the operator* $\mathscr{K}^+$ *belongs to the class* $\mathscr{O}_n^-(\Omega)$, $n\geqslant 0$. *Then the inequality* (1.2) *holds if and only if one of the conditions*

$$E_1^+=\sup_{z>0}\left(\int\limits_z^\infty\left(\int\limits_0^z K^{p'}(x,s)d\varphi^{p'}(s)\right)^{\frac{q}{p'}}u^q(x)dx\right)^{\frac{1}{q}}<\infty,$$

$$E_2^+=\sup_{z>0}\left(\int\limits_0^z\left(\int\limits_z^\infty K^q(x,s)u^q(x)dx\right)^{\frac{p'}{q}}d\varphi^{p'}(s)\right)^{\frac{1}{p'}}<\infty$$

holds. Moreover, for the sharp constant $C>0$ *in* (1.2) *it holds that* $E_1^+\approx E_2^+\approx C$.

The corresponding main result for the operator $\mathscr{K}^-$ reads:

THEOREM 3.2. *Let* $1 < p \leqslant q < \infty$, $\psi(\infty) = 0$, $\rho^{-1} \in L_{p'}^{loc}(I)$, $v \in L_p(t,\infty)$, $t > 0$, *and the kernel of the operator* $\mathcal{K}^-$ *belongs to the class* $\mathcal{O}_n^+(\Omega)$, $n \geqslant 0$. *Then the inequality* (1.3) *holds if and only if one of the conditions*

$$E_1^- = \sup_{z>0} \left(\int_z^\infty \left(\int_0^z K^q(x,s) u^q(s) ds \right)^{\frac{p'}{q}} d\left(-\psi^{p'}(x)\right) \right)^{\frac{1}{p'}} < \infty,$$

$$E_2^- = \sup_{z>0} \left(\int_0^z \left(\int_z^\infty K^{p'}(x,s) d\left(-\psi^{p'}(x)\right) \right)^{\frac{q}{p'}} u^q(s) ds \right)^{\frac{1}{q}} < \infty$$

holds. In this case $E_1^- \approx E_2^- \approx C$, *where* $C > 0$ *is the sharp constant in* (1.3).

We will begin by proving Theorem 3.2. However, since this proof heavily depends on the (unproved) Theorem B^- we first prove this Theorem.

Proof of Theorem B^-. First we assume that the inequalities

$$\left(\int_0^\infty \left(\int_t^\infty f ds \right)^{p-1} f(t) \psi^{-p}(t) dt \right)^{\frac{1}{p}} \ll \left(\|\rho f\|_p + \|vH^- f\|_p \right), \ f \geqslant 0 \tag{3.1}$$

and

$$\left(\|\rho f\|_p + \|vH^- f\|_p \right) \ll \left(\int_0^\infty |f(t)|^p (\psi)^{-1} \left| \frac{d\psi}{dt} \right|^{1-p} dt \right)^{\frac{1}{p}} \tag{3.2}$$

hold.

By virtue of (3.2) and the principle of duality in L_p spaces we have

$$\sup_{f \geqslant 0} \frac{\int_0^\infty f(s) g(s) ds}{\|\rho f\|_p + \|vH^- f\|_p} \gg \sup_{f \geqslant 0} \frac{\int_0^\infty f(s) g(s) ds}{\left(\int_0^\infty |f|^p \psi^{-1} |\frac{d\psi}{dt}|^{1-p} \right)^{\frac{1}{p}}}$$

$$= \left(\int_0^\infty g^{p'} \left(\psi^{-1} \left| \frac{d\psi}{dt} \right|^{1-p} \right)^{1-p'} dt \right)^{\frac{1}{p'}} = \left(\int_0^\infty g^{p'} \psi^{p'-1} \frac{d\psi}{dt} dt \right)^{\frac{1}{p'}}$$

$$= \left(\frac{1}{p'} \right)^{\frac{1}{p}} \left(\int_0^\infty g^{p'}(t) d\psi^{p'}(t) \right)^{\frac{1}{p'}}. \tag{3.3}$$

Moreover, from the results of [1] the inequality

$$\int_0^\infty fgds \leqslant \left(\int_0^\infty \left(\int_t^\infty f(s)ds\right)^{p-1} f(t)\psi^{-p}(t)dt\right)^{\frac{1}{p}} \left(\int_0^\infty g^{p'}(s)d\psi^{p'}(s)\right)^{\frac{1}{p'}}, \ f \geqslant 0, \tag{3.4}$$

holds for all functions g, which are non-negative and non-decreasing.

Therefore, according to (3.1) and (3.4), we have

$$\sup_{f\geqslant 0} \frac{\int_0^\infty f(s)g(s)ds}{\|\rho f\|_p + \|vH^- f\|_p} \ll \sup_{f\geqslant 0} \frac{\int_0^\infty f(s)g(s)ds}{\left(\int_0^\infty \left(\int_t^\infty fds\right)^{p-1} f(t)\psi^{-p}(t)dt\right)^{\frac{1}{p}}}$$
$$\leqslant \left(\int_0^\infty g^{p'}(s)d\psi^{p'}(s)\right)^{\frac{1}{p'}}.$$

This estimate combined with (3.3) implies (2.5). And now we prove (3.1). First, we note that by definition ψ is a non-increasing function. Let $f \geqslant 0$ and $k \in Z$. Assume that $T_k = \{x \in I : \int_x^\infty f(s)ds \leqslant 2^{-k}\}$, $x_k = \inf T_k$, if $T_k \neq 0$ and $x_k = \infty$, if $T_k = \varnothing$. Let $Z_0 = \{k \in Z : x_k < \infty\}$. From the definition x_k it follows that $2^{-(k+1)} \leqslant \int_x^\infty f(s)ds \leqslant 2^{-k}$ for $x_k \leqslant x \leqslant x_{k+1}$, $k \in Z_0$, $\int_{x_k}^{x_{k+1}} f(s)ds = 2^{-(k+1)}$, $I = \bigcup_{k\in Z_0} [x_k, x_{k+1})$.

Thus

$$\left(\int_0^\infty \left(\int_t^\infty f(s)ds\right)^{p-1} f(t)\psi^{-p}(t)dt\right)^{\frac{1}{p}}$$
$$= \left(\sum_{k\in Z_0} \int_{x_k}^{x_{k+1}} \left(\int_t^\infty f(s)ds\right)^{p-1} f(t)\psi^{-p}(t)dt\right)^{\frac{1}{p}}$$
$$\leqslant \left(\sum_k \psi^{-p}(x_{k+1}) \int_{x_k}^{x_{k+1}} \left(\int_t^\infty f(s)ds\right)^{p-1} f(t)dt\right)^{\frac{1}{p}}$$
$$\leqslant \left(\sum_k \left[\left(\int_{x_{k+1}}^{x_{k+2}} \rho^{-p'}ds\right)^{-\frac{1}{p'}} + \left(\int_0^{x_{k+2}} v^p ds\right)^{\frac{1}{p}}\right]^p 2^{-k(p-1)} \cdot 2^{-(k+1)}\right)^{\frac{1}{p}}$$

$$\ll \left(\sum_k \left(\int_{x_{k+1}}^{x_{k+2}} \rho^{-p'} ds \right)^{-\frac{p}{p'}} 2^{-kp} \right)^{\frac{1}{p}} + \left(\sum_k 2^{-kp} \int_0^{x_{k+2}} v^p ds \right)^{\frac{1}{p}} := I_1 + I_2. \quad (3.5)$$

We estimate I_1 and I_2 separately.

By the Hölder inequality we have

$$\begin{aligned} I_1 &= \left(\sum_k 2^{2p} \left(\int_{x_{k+1}}^{x_{k+2}} \rho^{-p'} ds \right)^{1-p} \left(\int_{x_{k+1}}^{x_{k+2}} f(t) dt \right)^p \right)^{\frac{1}{p}} \\ &\ll \left(\sum_k \int_{x_{k+1}}^{x_{k+2}} |\rho f|^p dt \right)^{\frac{1}{p}} \leqslant \|\rho f\|_p \end{aligned} \quad (3.6)$$

and

$$\begin{aligned} I_2 &= \left(\sum_{k \in Z_0} 2^{-kp} \sum_{i \leqslant k} \int_{x_{i+1}}^{x_{i+2}} v^p ds \right)^{\frac{1}{p}} \leqslant \left(\sum_i \int_{x_{i+1}}^{x_{i+2}} v^p ds \sum_{k \geqslant i} 2^{-kp} \right)^{\frac{1}{p}} \\ &\ll \left(\sum_i \int_{x_{i+1}}^{x_{i+2}} v^p ds 2^{-(i+2)p} \right)^{\frac{1}{p}} \leqslant \left(\sum_i \int_{x_{i+1}}^{x_{i+2}} v^p ds \left(\int_s^{\infty} f(t) dt \right)^p \right)^{\frac{1}{p}} \\ &\leqslant \|vH^- f\|_p. \end{aligned}$$

This inequality together with (3.5) and (3.6) implies (3.1).

Finally, we prove (3.2). Let $0 < x < z$. From the definition of ψ we find

$$\begin{aligned} \psi^{p'}(x) &\leqslant \sup_{x<t<z} \frac{\int_x^t \rho^{-p'} ds}{\left[1 + \left(\int_x^t \rho^{-p'}(s) ds \right)^{\frac{1}{p'}} \left(\int_0^z v^p ds \right)^{\frac{1}{p}} \right]^p} \\ &\quad + \sup_{z<t} \frac{\int_x^z \rho^{-p'} ds + \int_z^t \rho^{-p'} ds}{\left[1 + \left(\int_x^z \rho^{-p'}(s) ds + \int_z^t \rho^{-p'} ds \right)^{\frac{1}{p'}} \left(\int_0^t v^p ds \right)^{\frac{1}{p}} \right]^p} \\ &\leqslant 2 \int_x^z \rho^{-p'} ds + \psi^{p'}(z). \end{aligned}$$

We note that $0 < \psi^{p'}(x) - \psi^{p'}(z) \leqslant 2\int_x^z \rho^{-p'}ds$. Hence, the function ψ is locally absolutely continuous and

$$p'\psi^{p'-1}(z)\left(-\frac{d\psi}{dz}\right) = \lim_{x\to z}\frac{\psi^{p'}(x)-\psi^{p'}(z)}{z-x} \leqslant 2\lim_{x\to z}\frac{1}{z-x}\int_x^z \rho^{-p'}ds = 2\rho^{-p'}(z).$$

for almost all $z \in I$. Therefore,

$$\rho^p(z)\psi(z)\left|\frac{d\psi}{dz}\right|^{p-1} \ll 1$$

or

$$\rho^p(z) \ll \psi^{-1}(z)\left|\frac{d\psi}{dz}\right|^{1-p} \quad \text{a.e. } z \in I. \tag{3.7}$$

According to (3.7) we have

$$\|f\rho\|_p \ll \left(\int_0^\infty |f|^p\psi^{-1}(z)\left|\frac{d\psi}{dz}\right|^{1-p}dz\right)^{\frac{1}{p}}. \tag{3.8}$$

By the Hardy inequality (see e.g. [4]) we obtain

$$\|vH^-f\|_p \ll \left(\int_0^\infty |f|^p\psi^{-1}(z)\left|\frac{d\psi}{dz}\right|^{1-p}dz\right)^{\frac{1}{p}} \tag{3.9}$$

since

$$\sup_{z>0}\left(\int_0^z v^p ds\right)^{\frac{1}{p}}\left(\int_z^\infty \psi^{p'-1}(t)(-\psi'(t))dt\right)^{\frac{1}{p'}} = \left(\frac{1}{p'}\right)^{\frac{1}{p'}}\sup_{z>0}\left(\int_0^z v^p ds\right)^{\frac{1}{p}}\psi(z) \ll 1.$$

By combining (3.8) and (3.9) we get (3.2). Theorem B^- is proved. □

Proof of Theorem 3.2. Let $C > 0$ be the sharp constant in (1.3). Then, by using the duality principle in L_q, $1 < q < \infty$, we have

$$C = \sup_{f\geqslant 0}\frac{\|u\mathscr{K}^-f\|_q}{\|\rho f\|_p + \|vH^-f\|_p} = \sup_{f\geqslant 0}\sup_{0\leqslant g\in L_{q'}}\frac{\int_0^\infty gu\mathscr{K}^-f ds}{(\|\rho f\|_p + \|vH^-f\|_p)\|g\|_{q'}}$$
$$= \sup_{g\geqslant 0}\frac{1}{\|g\|_{q'}}\sup_{f\geqslant 0}\frac{\int_0^\infty f(x)(\mathscr{K}^+gu)dx}{\|\rho f\|_p + \|vH^-f\|_p}.$$

Hence, by using the fact that the function $(\mathscr{K}^+gu)(x)$ is non-decreasing we can apply Theorem B^- to obtain that

$$C \approx \sup_{0\leqslant g\in L_{q'}} \frac{\left(\int\limits_0^\infty (\mathscr{K}^+gu)^{p'}(x)d(-\psi^{p'}(x))\right)^{\frac{1}{p'}}}{\|g\|_{q'}} = \widetilde{C}.$$

Therefore, the inequality (1.3) is equivalent to the inequality

$$\left(\int\limits_0^\infty (\mathscr{K}^+gu)^{p'}(x)d(-\psi^{p'}(x))\right)^{\frac{1}{p'}} \leqslant \widetilde{C}\left(\int\limits_0^\infty |g(t)|^{q'}dt\right)^{\frac{1}{q'}}, \ g \geqslant 0,$$

or the inequality

$$\left(\int\limits_0^\infty (\mathscr{K}^+g)^{p'}(x)d(-\psi^{p'}(x))\right)^{\frac{1}{p'}} \leqslant \widetilde{C}\left(\int\limits_0^\infty \left|u^{-1}g\right|^{q'}dt\right)^{\frac{1}{q'}}, \ g \geqslant 0, \tag{3.10}$$

and $C \approx \widetilde{C}$.

The inequality (3.10) is the inequality of the form (2.3). Since $1 < p \leqslant q < \infty$ implies that $1 < q' \leqslant p' < \infty$, then applying Theorem A^+ to the inequality (3.10), we get that the inequality (3.10) holds if and only if one of the conditions

$$A_1^* = \sup_{z>0}\left(\int\limits_z^\infty \left(\int\limits_0^z \left|K^+(x,s)u(s)\right|^q ds\right)^{\frac{p'}{q}} d\left(-\psi^{p'}(x)\right)\right)^{\frac{1}{p'}} = E_1^- < \infty,$$

$$A_2^* = \sup_{z>0}\left(\int\limits_0^z u^q(s)\left(\int\limits_z^\infty \left|K^+(x,s)\right|^{p'} d\left(-\psi^{p'}(x)\right)\right)^{\frac{q}{p'}}\right)^{\frac{1}{q}} = E_2^- < \infty$$

holds and, moreover, $\widetilde{C} \approx E_1^- \approx E_2^-$. But $C \approx \widetilde{C}$ and, thus, also $C \approx E_1^- \approx E_2^-$. The proof is complete. $\square$

Proof of Theorem 3.1. The proof is similar to that of Theorem 3.2 so we omit the details. We only remark that in this case we use Theorem B^+ and Theorem A^- instead of Theorem B^- and Theorem A^+, respectively.

Finally, we will consider the case $p = 1$. In this case for $f \geqslant 0$ we have

$$\begin{aligned}\|\rho f\|_1 + \|vH^+f\|_1 &= \int\limits_0^\infty \rho(t)f(t)dt + \int\limits_0^\infty v(t)\int\limits_0^t f(s)dsdt \\ &= \int\limits_0^\infty \rho(t)f(t)dt + \int\limits_0^\infty f(s)\int\limits_s^\infty v(t)dtds\end{aligned}$$

$$= \int_0^\infty f(s)\left(\rho(s) + \int_s^\infty v(t)dt\right)ds = \int_0^\infty w^+(s)f(s)ds;$$

where

$$w^+(s) \equiv \rho(s) + \int_s^\infty v(t)dt,$$

and

$$\|\rho f\|_1 + \|vH^- f\|_1 = \int_0^\infty \rho(t)f(t)dt + \int_0^\infty v(t)\int_t^\infty f(s)dsdt$$

$$= \int_0^\infty f(s)\left(\rho(s) + \int_0^s v(t)dt\right)ds = \int_0^\infty w^-(s)f(s)ds,$$

where

$$w^-(s) \equiv \rho(s) + \int_0^s v(t)dt.$$

Therefore, in the case $p = 1$ the inequalities (1.2) and (1.3) have the forms

$$\|u\mathcal{K}^+ f\|_q \leqslant C^+\|w^+ f\|_1,\ f \geqslant 0, \tag{3.11}$$

$$\|u\mathcal{K}^- f\|_q \leqslant C^-\|w^- f\|_1,\ f \geqslant 0, \tag{3.12}$$

respectively, i.e. the problem in this case reduces to the problem boundedness of the operators $\mathcal{K}^+$, $\mathcal{K}^-$ from $L_{1,w^\pm}$ to $L_{q,u}$. $\square$

Thus, on the basis of Theorem 4 of Chapter XI from [2], we have the following:

PROPOSITION 3.1. *Let $p = 1$ and $1 \leqslant q < \infty$. Then the inequalities* (1.2) *and* (1.3) *hold if and only if*

$$C^+ = \sup_{s>0}\left\{\left(\int_s^\infty \left|u(x)K^+(x,s)\right|^q dx\right)^{\frac{1}{q}}\left(\rho(s) + \int_s^\infty v(t)dt\right)^{-1}\right\} < \infty,$$

and

$$C^- = \sup_{x>0}\left\{\left(\int_0^x \left|u(s)K^-(x,s)\right|^q ds\right)^{\frac{1}{q}}\left(\rho(s) + \int_0^x v(t)dt\right)^{-1}\right\} < \infty$$

hold, respectively. Moreover, for the best constant C in (1.2) *and* (1.3)*, it yields that $C^+ \approx C$ and $C^- \approx C$, respectively.*

REFERENCES

[1] H. P. HEINIG AND V. D. STEPANOV, *Weighted Hardy inequalities for increasing functions*, Canad. J. Math. **45** (1993), no. 1, 104–116.

[2] L. V. KANTAROVICH AND G. R. AKILOV, *Functional analysis*, M. Nauka 1977 (in Russian).

[3] V. KOKILASHVILI, A. MESKHI AND L. E. PERSSON, *Weighted Norm Inequalities for Integral Transforms with Product Kernels*, Nova Scientific Publisher, INC., New York, 2009.

[4] A. KUFNER AND L. E. PERSSON, *Weighted Inequalities of Hardy Type*, World Scientific, New Jersey/London/Singapore/Hong Kong, 2003.

[5] A. KUFNER, L. MALIGRANDA AND L. E. PERSSON, *The Hardy Inequality. About its History and Some Related Results*, Vydavatelsky Servis Publishing House, Pilsen, 2007.

[6] R. OINAROV, *Weighted inequalities for one class of integral operators*, Soviet Math. Dokl., **44**, (1992), 291–293.

[7] R. OINAROV, *Three-weight generalization of Hardy's inequality*, Mathematical Notes, **54**, (1993), no. 2, 806–810.

[8] R. OINAROV, *Reversion of Hölder type inequalities for sums of weighted norms and additive weighted estimates of integral operators*, Math. Ineq. Appl. **6** (2003), no. 3, 421–436.

[9] R. OINAROV, *Boundedness and compactness of Volterra type integral operators*, Siberian Math. J. **48** (2007), no. 5, 884–896.

[10] R. OINAROV AND A. A. CHAGIROV, *Three-weighted inequalities with integral operators*, Doklady Nats. Akad. Nauk Resp. Kaz., **2** (1993), 13–16 (in Russian).

[11] R. OINAROV AND S. S. SAGINTAEVA, *On one Hardy type three-weighted inequality*, Science and education of South Kazakhstan, Series: Economics, Math., **6** (1997), 183–194 (in Russian).

[12] B. OPIC AND A. KUFNER, *Hardy type inequalities*, Pitman Research Notes in Mathematics, John Wiley, New York, (1990).

(Received June 7, 2016)

A. M. Abylayeva
L. N.Gumilev Eurasian National University
Kazakhstan
e-mail: abylayeva_b@mail.ru

A. O. Baiarystanov
L. N.Gumilev Eurasian National University
Kazakhstan
e-mail: oskar_62@mail.ru

L.-E. Persson
Luleå University of Technology
Sweden
and
UiT
The Artic University of Norway
Norway
e-mail: larserik@ltu.se

P. Wall
Luleå University of Technology
Sweden
e-mail: Peter.Wall@ltu.se