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INEQUALITIES INVOLVING CIRCULAR, HYPERBOLIC AND EXPONENTIAL FUNCTIONS

YOGESH J. BAGUL

Dedicated to my mothers Vimal and Vandana

(*Communicated by E. Neuman*)

Abstract. This paper is aimed at obtaining some new lower and upper bounds for the functions $\cos x$, $\sin x/x$, $x/\cosh x$, thus establishing inequalities involving circulr, hyperbolic and exponential functions.

1. Introduction

The well-known Jordan's inequality [1], [6] is stated as follows:

$$\frac{2}{\pi} < \frac{\sin x}{x} < 1, \quad x \in (0,1). \tag{1}$$

During the past few years the Jordan's inequality has been in the focus of studies on the trigonometric and hyperbolic inequalities and many refinements have been proved [1–3], [5–11].

R. Klén, M. Visuri, and M. Vuorinen [7, Thm. 1.2] have proved the inequality

$$\left(\frac{1}{\cosh x}\right)^{1/2} < \frac{x}{\sinh x} < \left(\frac{1}{\cosh x}\right)^{1/4}, \quad x \in (0,1). \tag{2}$$

This work is motivated by these studies and we aim to improve the bounds given in the above inequality (2) using exponential functions.

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2. The main results and proofs

The following l'Hôpital's rule for monotonicity [4, Thm. 1.25] is a standard tool for the above mentioned study.

LEMMA 1. (The monotone form of l'Hôpital's rule [4]) *Let $f,g:[a,b]\to\mathbb{R}$ be two continuous functions which are differentiable on (a,b) and $g'\neq 0$ in (a,b). If f'/g' is increasing (or decreasing) on (a,b), then the functions $\frac{f(x)-f(a)}{g(x)-g(a)}$ and $\frac{f(x)-f(b)}{g(x)-g(b)}$ are also increasing (or decreasing) on (a,b).*

If f'/g' is strictly monotone, then the monotonicity in the conclusion is also strict.

The main results and their proofs are given as follows.

In [8, Thm. 2], the bounds of $\cos x$ by using hyperbolic functions are given as below:

$$\left(\frac{1}{\cosh x}\right)^{\frac{1}{a}} < \cos x < \frac{1}{\cosh x}, \quad x\in\left(0,\frac{\pi}{4}\right) \quad \text{and} \quad a\approx 0.811133. \tag{3}$$

Now we improve the upper bound of $\cos x$ by using exponential function.

THEOREM 1. *If $x\in(0,1)$ then*

$$e^{-ax^2} < \cos x < e^{-x^2/2} \tag{4}$$

where $a\approx 0.615626$.

Proof. Let $e^{-ax^2}<\cos x<e^{-bx^2}$, which implies that $b<\frac{\log(\cos x)}{-x^2}<a$.
Then

$$f(x)=\frac{-\log(\cos x)}{x^2}=\frac{f_1(x)}{f_2(x)},$$

where $f_1(x)=-\log(\cos x)$, and $f_2(x)=x^2$, with $f_1(0)=f_2(0)=0$. By differentiation we get

$$\frac{f_1'(x)}{f_2'(x)}=\frac{\tan x}{2x}=\frac{f_3(x)}{f_4(x)},$$

where $f_3(x)=\tan x$ and $f_4(x)=2x$ with $f_3(0)=f_4(0)=0$. Differentiation gives $\frac{f_3'(x)}{f_4'(x)}=\frac{sec^2 x}{2}$, which is clearly increasing in $(0,1)$. By Lemma 1, $f(x)$ is strictly increasing in $(0,1)$. Clearly, $a=f(1-)=-\log(\cos 1)\approx 0.615626$ and $b=f(0+)=\frac{1}{2}=0.5$ by l'Hôpital's rule. $\square$

The bounds of $\frac{\sin x}{x}$ in [2, Thm. 2.1], by using hyperbolic functions are given by the following inequality:

$$\frac{1}{(\cosh x)^p}<\frac{\sin x}{x}<\frac{1}{(\cosh x)^q}, \quad \text{where} \quad p\approx 0.49, \quad q=\frac{1}{3}\approx 0.3333. \tag{5}$$

In [7, Thm.2.4], the bounds of $\frac{\sin x}{x}$ by using trigonometric functions are given as follows:

$$\cos^2\left(\frac{x}{2}\right) \leqslant \frac{\sin x}{x} \leqslant \cos^3\left(\frac{x}{3}\right), \qquad x \in (-\sqrt{27/5}, \sqrt{27/5}) \tag{6}$$

Recently in [9, Thm. 1], Cătălin Barbu and Laurian-Ioan Pişcoran established the upper bound for $\frac{\sin x}{x}$ by using hyperbolic cosine which is given as follows:

$$\frac{\sin x}{x} < \sqrt{\cosh x}, \quad x \in (0, \infty). \tag{7}$$

We improve both the bounds of (5), the lower bound of (6) and the upper bound of (7) by using exponential functions.

THEOREM 2. *If* $x \in (0,1)$ *then*

$$e^{-ax^2} < \frac{\sin x}{x} < e^{-x^2/6} \tag{8}$$

where $a \approx 0.172604$.

Proof. Let $e^{-ax^2} < \frac{\sin x}{x} < e^{-bx^2}$, which implies that, $b < \frac{-\log(\sin x/x)}{x^2} < a$. Then

$$f(x) = \frac{-\log(\frac{\sin x}{x})}{x^2} = \frac{f_1(x)}{f_2(x)}$$

where $f_1(x) = -\log\left(\frac{\sin x}{x}\right)$ and $f_2(x) = x^2$, with $f_1(0+) = f_2(0+) = 0$. Differentiation gives us

$$\frac{f_1'(x)}{f_2'(x)} = \frac{\sin x - x\cos x}{2x^2 \sin x} = \frac{f_3(x)}{f_4(x)}$$

where $f_3(x) = \sin x - x\cos x$ and $f_4(x) = 2x^2 \sin x$, with $f_3(0) = f_4(0) = 0$. Again by differentiation we have

$$\frac{f_3'(x)}{f_4'(x)} = \frac{1}{2\frac{x}{\tan x} + 4} = \frac{1}{2\frac{f_5(x)}{f_6(x)} + 4}$$

where $f_5(x) = x$ and $f_6(x) = \tan x$, with $f_5(0) = f_6(0) = 0$.

Then we get $\frac{f_5'(x)}{f_6'(x)} = \frac{1}{sec^2 x} = \cos^2 x$, which is clearly decreasing in $(0,1)$. By Lemma 1, $f(x)$ is strictly increasing in $(0,1)$. Consequently, $a = f(1-) = -\log(\sin 1)$ ≈ 0.172604. and $b = f(0+) = \frac{1}{6} \approx 0.166667$ by l'Hôpital's rule. $\square$

Next we improve the bounds of (2) by using exponential functions again.

THEOREM 3. *If* $x \in (0,1)$ *then*

$$e^{-x^2/6} < \frac{x}{\sinh x} < e^{-bx^2} \tag{9}$$

where $b \approx 0.161439$.

Proof. Let $e^{-ax^2} < \frac{x}{\sinh x} < e^{-bx^2}$, which implies that, $b < \frac{-\log\left(\frac{x}{\sinh x}\right)}{x^2} < a$. Then

$$f(x) = \frac{-\log\left(\frac{x}{\sinh x}\right)}{x^2} = \frac{f_1(x)}{f_2(x)},$$

where $f_1(x) = -\log\left(\frac{x}{\sinh x}\right)$ and $f_2(x) = x^2$, with $f_1(0+) = f_2(0+) = 0$. By differentiation we get

$$\frac{f_1'(x)}{f_2'(x)} = \frac{x\cosh x - \sinh x}{2x^2 \sinh x} = \frac{f_3(x)}{f_4(x)}$$

where $f_3(x) = x\cosh x - \sinh x$ and $f_4(x) = 2x^2 \sinh x$, with $f_3(0) = f_4(0) = 0$. Differentiation gives

$$\frac{f_3'(x)}{f_4'(x)} = \frac{1}{4 + 2\frac{x}{\tanh x}} = \frac{1}{4 + 2\frac{f_5(x)}{f_6(x)}}$$

where $f_5(x) = x$ and $f_6(x) = \tanh x$, with $f_5(0) = f_6(0) = 0$. Then we get $\frac{f_5'(x)}{f_6'(x)} = \frac{1}{sech^2 x} = \cosh^2 x$, which is clearly increasing in $(0,1)$. By Lemma 1, $f(x)$ is decreasing in $(0,1)$. Consequently, $a = f(0+) = \frac{1}{6} \approx 0.166667$ by l'Hôpital's rule and $b = -\log\left(\frac{1}{\sinh 1}\right) \approx 0.161439$. $\square$

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Yogesh J. Bagul
Department of Mathematics
K. K. M. College Manwath
Dist. Parbhani (M. S.) – 431505, India
e-mail: yjbagul@gmail.com