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ON SOME MULTILINEAR COMMUTATORS IN VARIABLE LEBESGUE SPACES

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Abstract. In this paper, the authors obtain some characterizations of BMO in terms of commutators of multilinear fractional integrals and Caldrón-Zygmund singular integrals on variable Lebesgue spaces. The corresponding weighted estimates for vector-valued commutators and multilinear commutators with vector symbol on variable Lebesgue spaces are also considered.

1. Introduction

In 1976, Coifman, Rochberg and Weiss [4] studied the L^p boundedness of linear commutators generated by the Calderón-Zygmund singular integral operator and $b \in \mathrm{BMO}$. Jason [22] and Uchiyama [40] independently established characterization of BMO in terms of the commutators of singular integral operators. In addition, Chanillo [3] characterized BMO functions using the commutators of fractional integral operators. The boundedness of the commutators of multilinear operators has also been studied already in [11] and [33]. Recently Chaffee [2] characterized BMO in terms of the boundedness of the commutators of various bilinear singular integral operators with pointwise multiplication. On the other hand, due to its applications to partial differential equations and the calculus of variations, variable exponent function spaces theory have been attracted by many authors, see [10, 19, 23, 29, 37, 38, 39] for the theory of function spaces with variable exponents. We point out that variable Lebesgue spaces were first established by Orlicz [32] in 1931. In the early 1950's, Nakano [30, 31] first systematically studied modular function spaces which include the variable Lebesgue spaces as specific examples. It is natural to ask whether the boundedness of the commutator on variable Lebesuge space implys that the function is in BMO. The main purpose of this paper is to characterize BMO via the boundedness of the commutators of multilinear fractional and singular integral operators. Using the theory of Rubio de Francia extrapolation extended by Cruz-Uribe and Wang [9], some weighted norm inequalities of the commutators on variable Lebesgue spaces and the vector-valued inequalities are also considered. Before stating our results, we need some notation.

We denote by $\mathscr{D}$ the set of all C^∞ functions with compact support on $\mathbb{R}^n$. For a measurable subset $E \subset \mathbb{R}^n$, we denote $p^-(E) = \inf_{x\in E} p(x)$ and $p^+(E) = \sup_{x\in E} p(x)$.

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Especially, we denote $p^- = p^-(\mathbb{R}^n)$ and $p^+ = p^+(\mathbb{R}^n)$. Let $p(\cdot)$: $\mathbb{R}^n \to (0,\infty)$ be a measurable function with $0 < p^- \leqslant p^+ < \infty$ and $\mathscr{P}^0(\mathbb{R}^n)$ be the set of all these $p(\cdot)$. Let $\mathscr{P}$ be the set of all measurable functions $p(\cdot) : \mathbb{R}^n \to [1,\infty)$ such that $1 < p^- \leqslant p^+ < \infty$. Let $\mathscr{B}(X,Y)$ be the class of all bounded sublinear operators from a Banach space X to a Banach space Y and let $\|A\|_{\mathscr{B}(X,Y)}$ denote the operator norm of $A \in \mathscr{B}(X,Y)$. Especially, we abbreviate $\|A\|_{\mathscr{B}(X_1\times X_2\times\ldots\times X_n,Y)}$ to $\|A\|_{\mathscr{B}(\prod_{i=1}^n X_i,Y)}$.

The variable Lebesgue space $L^{p(\cdot)}$ is defined as the set of all measurable functions f for which the quantity $\int_{\mathbb{R}^n} |\varepsilon f(x)|^{p(x)}dx$ is finite for some $\varepsilon > 0$ and we define

$$\|f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \leqslant 1\right\}.$$

As a special case of the theory of Nakano and Luxemberg, we see that $L^{p(\cdot)}$ is a quasi-normed space. Especially, when $p^- \geqslant 1$, $L^{p(\cdot)}$ is a Banach space.

Given a measurable function $w > 0$, for $1 < p < \infty$, it is said that $w \in A_p$ if

$$[w]_{A_p} = \sup_B \left(\frac{1}{|B|}\int_B w(x)dx\right)\left(\frac{1}{|B|}\int_B w(x)^{1-p'}dx\right)^{p-1} < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$.

Similarly, $w \in A_{p(\cdot)}$ if

$$\sup_B |B|^{-1}\|w\chi_B\|_{p(\cdot)}\|w^{-1}\chi_B\|_{p'(\cdot)} < \infty.$$

Given a weight w and $p(\cdot) \in \mathscr{P}$, define the weighted variable Lebesgue space $L^{p(\cdot)}(w)$ to be the set of all measurable functions f such that $fw \in L^{p(\cdot)}$. We say that T is a bounded operator on $L^{p(\cdot)}(w)$ if $\|(Tf)w\|_{L^{p(\cdot)}} \leqslant \|fw\|_{L^{p(\cdot)}}$ for all $f \in L^{p(\cdot)}(w)$.

We say that $p(\cdot) \in LH$, if $p(\cdot)$ satisfies

$$|p(x)-p(y)| \leqslant \frac{C}{-\log(|x-y|)}, \quad |x-y| \leqslant 1/2$$

and

$$|p(x)-p(y)| \leqslant \frac{C}{\log|x|+e}, \quad |y| \geqslant |x|.$$

Let $\mathfrak{B}$ be the set of $p(\cdot) \in \mathscr{P}$ such that the Hardy-littlewood maximal operator M is bounded on $L^{p(\cdot)}$. It is well known that $p(\cdot) \in \mathfrak{B}$ if $p(\cdot) \in \mathscr{P} \cap LH$. Furthermore, let $W\mathfrak{B}$ be the set of $p(\cdot) \in \mathscr{P}$ such that the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(w)$. In [9], the authors have proved that $p(\cdot) \in W\mathfrak{B}$ if $p(\cdot) \in \mathscr{P} \cap LH$.

For any $1 \leqslant j \leqslant m$, we can define the commutator of multilinear integral operator by

$$[b,T]_j(\vec{f})(x) := bT(\vec{f})(x) - T(f_1,\ldots,bf_j,\ldots,f_m)(x),$$

where b is a locally integral function and T is an m-linear integral operator.

Then $[b,I_\alpha]_j(\vec{f})$ is defined by

$$[b,I_\alpha]_j(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} \frac{(b(x)-b(y_j))\prod_{i=1}^m f_i(y_i)}{(\sum_{i=1}^m |x-y_i|)^{nm-\alpha}} \prod_{i=1}^m dy_i,$$

while the multilinear fractional integral operator is defined by

$$I_\alpha(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} \frac{\prod_{i=1}^m f_i(y_i)}{(\sum_{i=1}^m |x-y_i|)^{nm-\alpha}} \prod_{i=1}^m dy_i.$$

Throughout this paper, C denotes a positive constant that may vary at each occurrence but is independent to the main parameter, and $A \sim B$ means that there are constants $C_1 > 0$ and $C_2 > 0$ independent of the the main parameter such that $C_1 B \leqslant A \leqslant C_2 B$. Given a measurable set $S \subset \mathbb{R}^n$, $|S|$ denotes the Lebesgue measure and χ_S means the characteristic function.

First, we characterize BMO via the boundedness of the commutators of multilinear fractional integrals $I_\alpha(\vec{f})$ as follows. We note that multilinear fractional integral operators have been studied by many authors, see [14, 15, 24, 25, 28].

THEOREM 1.1. *Suppose that $b \in L^1_{\mathrm{loc}}$, $0 < \alpha < mn$ and $p_i(\cdot) \in LH \cap \mathscr{P}$, $i = 1,2,\ldots,m$. Suppose further that $q(\cdot) \in \mathscr{P}$ satisfies*

$$\sum_{i=1}^m \frac{1}{p_i(x)} - \frac{\alpha}{n} = \frac{1}{q(x)} < 1, \quad x \in \mathbb{R}^n.$$

For any $1 \leqslant j \leqslant m$, then $[b,I_\alpha]_j$ is bounded from $L^{p_1(\cdot)} \times L^{p_2(\cdot)} \times \ldots \times L^{p_m(\cdot)}$ to $L^{q(\cdot)}$ if and only if $b \in BMO$. Furthermore,

$$\|b\|_* \sim \|[b,I_\alpha]_j\|_{\mathscr{B}(\prod_{i=1}^m L^{p_i(\cdot)}, L^{q(\cdot)})}.$$

Next we recall multilinear Calderón-Zygmund operator introduced by Grafakos and Torres in [17]. Let $K(x,y_1,\ldots,y_m)$ be a locally integrable function defined away from the diagonal $x = y_1 = \ldots = y_m$. If for some positive parameters A and ε,

$$|K(y_0,\ldots,y_m)| \leqslant \frac{A}{(\sum_{k,l=0}^m |y_k - y_l|)^{mn}}$$

and

$$|K(y_0,\ldots,y_j,\ldots,y_m) - K(y_0,\ldots,y_j',\ldots,y_m)| \leqslant \frac{A|y_j - y_j'|^\varepsilon}{(\sum_{k,l=0}^m |y_k - y_l|)^{mn+\varepsilon}}$$

when $0 \leqslant j \leqslant m$ and $|y_j - y_j'| \leqslant 1/2 \max_{0\leqslant k\leqslant m} |y_j - y_k|$, then K is an m-linear Calderón-Zygmund kernel. Now let

$$T(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} K(x,y_1,\ldots,y_m) \prod_{i=1}^m f_i(y_i) d\vec{y}$$

for all $f_i \in \mathscr{D}$ and all $x \notin \cap_1^m \mathrm{supp}(f_i)$, where $d\vec{y} = dy_1 \dots dy_m$. If

$$T : L^{p_1} \times \dots \times L^{p_m} \to L^p$$

for some $1 < p_1, \dots, p_m < \infty$, where $\frac{1}{p} = \sum_{i=1}^m \frac{1}{p_i}$, T is called an m-linear Caldrón-Zygmund operator. If $K(x, y_1, \dots, y_m)$ is of form $K(x - y_1, \dots, x - y_m)$, then we say the operator T is of convolution type.

THEOREM 1.2. *Suppose $b \in L^1_{\mathrm{loc}}$, $p_i(\cdot) \in LH \cap \mathscr{P}$, $i = 1, 2, \dots, m$, and let T be an m-linear Calderón-Zygmund operator of convolution type such that $K(\lambda y_1, \dots, \lambda y_m) = \lambda^{-mn} K(y_1, \dots, y_m)$. Also suppose that for some ball B in $(\mathbb{R}^n)^m$, the Fourier series of $1/K$ is absolutely convergent. Let $q(\cdot)$ satisfy*

$$\sum_{i=1}^m \frac{1}{p_i(x)} = \frac{1}{p(x)} < 1, \quad x \in \mathbb{R}^n.$$

For any $1 \leqslant j \leqslant m$, then $[b, T]_j$ is a bounded operator from $L^{p_1(\cdot)} \times L^{p_2(\cdot)} \times \dots L^{p_m(\cdot)}$ to $L^{p(\cdot)}$ if and only if $b \in BMO$. Furthermore,

$$\|b\|_* \sim \|[b, T]_j\|_{\mathscr{B}(\prod_{i=1}^m L^{p_i(\cdot)}, L^{p(\cdot)})}.$$

We also can obtain some boundedness of the following maximal operator $T_*(\vec{f})$, where

$$T_*(\vec{f})(x) = \sup_{\delta > 0} \left| \int_{\sum_1^m |x - y_i|^2 > \delta^2} K(x, y_1, \dots, y_m) \prod_{i=1}^m f_i(y_i) d\vec{y} \right|.$$

THEOREM 1.3. *Let T_* be a maximal Calderón-Zygmund operator and $p(\cdot)$ satisfy*

$$\sum_{i=1}^m \frac{1}{p_i(x)} = \frac{1}{p(x)}, \quad x \in \mathbb{R}^n.$$

Then for $p_i(\cdot) \in LH \cap \mathscr{P}$, $i = 1, 2, \dots, m$, we have

$$\|T_*(\vec{f})\|_{p(\cdot)} \leqslant C \prod_{i=1}^m \|f_i\|_{p_i(\cdot)}.$$

Next we will show that weighted $L^{p(\cdot)}$ boundenness of the commutator of a linear operator T and the BMO function.

THEOREM 1.4. *Suppose that $1 < p_0 < \infty$. The operator T is defined by*

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y) f(y) dy,$$

and the commutator $[b, T]$ of T and b is defined by

$$[b, T]f(x) = \int_{\mathbb{R}^n} (b(x) - b(y)) K(x, y) f(y) dy.$$

Given $w_0 \in A_{p_0}$ and $p(\cdot) \in LH \bigcap \mathscr{P}$, if T is bounded on $L^{p_0}(w_0)$, then for $b \in BMO$ and $w \in A_{p(\cdot)}$,

$$\|[b,T]fw\|_{p(\cdot)} \leqslant C\|fw\|_{p(\cdot)}.$$

Moreover, for every $1 < q < \infty$ and sequence $\{f_j\}$,

$$\left\| \left(\sum_j |[b,T]f_j|^q \right)^{\frac{1}{q}} w \right\|_{p(\cdot)} \leqslant C \left\| \left(\sum_j |f_j|^q \right)^{\frac{1}{q}} w \right\|_{p(\cdot)}.$$

Finally, we consider the multilinear commutators with the vector symbol $\vec{b} = (b_1, \ldots, b_m)$ defined by

$$T_{\vec{b}}f(x) = \int_{\mathbb{R}^n} \left[\prod_{i=1}^m (b_i(x) - b_i(y)) \right] K(x,y)f(y)dy,$$

where K is a Calderón-Zygmund kernel.

Hereafter we consider the following of symbols: for $r \geqslant 1$ and for any $b \in L^1_{\text{loc}}$, define

$$\|b\|_{\mathrm{osc}_{\mathrm{expL^r}}} = \sup_Q \|b - b_Q\|_{\exp L^r, Q},$$

which is the supremum taken over all the cubes Q with sides parallel to the axes. Here the Φ-average of b over a cube Q with respect to the Young function $\Phi(t) = e^{t^r} - 1$ is defined by

$$\|b\|_{\exp L^r, Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q \left(\exp\left(|b(x)|/\lambda \right)^r - 1 \right) dx \leqslant 1 \right\}.$$

In the particular case of $r = 1$, $osc_{\exp L^1}$ coincides with BMO by the John-Nirenberg theorem. Furthermore, assume that $b_i = b$, then

$$T_{\vec{b}}f(x) = \int_{\mathbb{R}^n} (b(x) - b(y))^m K(x,y)f(y)dy.$$

See the work of Sawano, Sugano and Tanaka [36] for more details on m-fold commutators.

THEOREM 1.5. *Let $b_i \in \mathrm{osc}_{\mathrm{expL^{r_i}}}$ and $r_i \geqslant 1, 1 \leqslant i \leqslant m$. Then for $p(\cdot) \in LH \bigcap \mathscr{P}$ and $w \in A_{p(\cdot)}$, we have*

$$\|(T_{\vec{b}}f)w\|_{p(\cdot)} \leqslant C\|fw\|_{p(\cdot)}.$$

Furthermore, for every $1 < q < \infty$ and sequence $\{(T_{\vec{b}}f_j, f_j)\}$,

$$\left\| \left(\sum_j |T_{\vec{b}}f_j|^q \right)^{1/q} w \right\|_{p(\cdot)} \leqslant C \left\| \left(\sum_j |f_j|^q \right)^{1/q} w \right\|_{p(\cdot)}.$$

2. Some preliminaries

In this section, we state some results about the variable L^p spaces. The following generalized Hölder inequality on variable Lebesgue spaces can be found in in [6].

LEMMA 2.1. *Given exponent function* $p_i(\cdot) \in \mathscr{P}^0$, *define* $p(\cdot) \in \mathscr{P}^0$ *by*

$$\frac{1}{p(x)} = \sum_{i=1}^{m} \frac{1}{p_i(x)},$$

where $i = 1, 2, \ldots, m$. *Then for all* $f_i \in L^{p_i(\cdot)}$ *and* $f_i \in L^{p(\cdot)}$ *and*

$$\|\prod_{i=1}^{m} f_i\|_{p(\cdot)} \leqslant C \prod_{i=1}^{m} \|f_i\|_{p_i(\cdot)}.$$

Let $0 \leqslant \alpha < n$, we define

$$M_\alpha f(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\alpha/n}} \int_Q |f(y)| dy.$$

LEMMA 2.2. ([1]) *Given* $p(\cdot) \in LH \cap \mathscr{P}$. *Define the exponent function* $q(\cdot) \in LH$ *by*

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha}{n}, \; x \in \mathbb{R}^n,$$

then

$$\|M_\alpha f\|_{q(\cdot)} \leqslant C \|f\|_{p(\cdot)}.$$

LEMMA 2.3. ([13]) *Suppose that* $p(\cdot) \in LH$ *and* $0 < p^- \leqslant p^+ < \infty$.

(1) For all cubes (or balls) $|Q| \leqslant 2^n$ *and any* $x \in Q$, *we have*

$$\|\chi_Q\|_{p(\cdot)} \sim |Q|^{1/p(x)}.$$

(2) For all cubes (or balls) $|Q| \geqslant 1$, *we have*

$$\|\chi_Q\|_{p(\cdot)} \sim |Q|^{1/p_\infty},$$

where $p_\infty = \lim_{x \to \infty} p(x)$.

LEMMA 2.4. ([20, 21]) *If a variable exponent* $p(\cdot) \in LH \cap \mathscr{P}$, *then*

$$\|b\|_* \sim \sup_Q \frac{1}{\|\chi_Q\|_{p(\cdot)}} \|(b - b_Q)\chi_Q\|_{p(\cdot)}$$

holds for all $b \in BMO$. *Furthermore,*

$$\|b\|_* \sim \sup_Q \inf_{c \in \mathbb{C}} \frac{1}{\|\chi_Q\|_{p(\cdot)}} \|(b - c)\chi_Q\|_{p(\cdot)}$$

holds for all $b \in BMO$.

LEMMA 2.5. ([9]) *Let* $\mathscr{F}$ *denote a family of ordered pairs of non-negative measurable functions* (f,g). *Assume that for some* p_0 *with* $0<p_0<\infty$ *and every weight* $w_0\in A_{p_0}$,

$$\int_{\mathbb{R}^n}|f(x)|^{p_0}w_0dx\leqslant\int_{\mathbb{R}^n}|g(x)|^{p_0}w_0dx,\quad (f,g)\in\mathscr{F}.$$

If $p(\cdot)\in LH\bigcap\mathscr{P}$, *then for any* $(f,g)\in\mathscr{F}$ *and* $f\in L^{p(\cdot)}(w)$, *we have*

$$\|fw\|_{L^{p(\cdot)}}\leqslant C\|gw\|_{L^{p(\cdot)}}.$$

3. Proofs of main results

First, we prove Theorem 1.1.

Proof. To show Theorem 1.1, we need the following sharp maximal estimate,

$$M^{\sharp}\left([b,I_\alpha]_j(\vec{f})\right)(x)\leqslant C\|b\|_*\left[(M(|I_\alpha(\vec{f})|^r)^{1/r}(x)+\prod_{i=1}^m(M_{\alpha_i s_i}|f_i|^{s_i})^{1/s_i}(x)\right],$$

where $\sum_{i=1}^m\alpha_i=\alpha$, $0<\alpha_i<n$, $1<s_i<p_j^-$, $1<r<q^-$ and $i=1,2,\cdots,n$.

To do this, denote that $f_j^0=f_j\chi_{2Q}$ and $f_j^\infty=f_j-f_j^0$. Fix a cube Q and set

$$\begin{aligned}
[b,I_\alpha]_j(\vec{f})(x) &= \int_{(\mathbb{R}^n)^m}\frac{[(b(x)-b_Q)+(b_Q-b(y_j))]\prod_{i=1}^m f_i(y_i)}{(\sum_{i=1}^m|x-y_i|)^{nm-\alpha}}\prod_{i=1}^m dy_i\\
&= (b(x)-b_Q)I_\alpha(\vec{f})(x)-I_\alpha(f_1,\ldots,(b-b_Q)f_j,\cdots,f_m)(x)\\
&= (b(x)-b_Q)I_\alpha(\vec{f})(x)-I_\alpha(f_1^0,\ldots,(b-b_Q)f_j^0,\ldots,f_m^0)(x)\\
&\quad -I_\alpha(f_1^\infty,\ldots,(b-b_Q)f_j^\infty,\ldots,f_m^\infty)(x)\\
&\quad -\sum I_\alpha(f_1^{r_1},\ldots,(b-b_Q)f_j^{r_j},\ldots,f_m^{r_m})(x)\\
&=: A_1-A_2-A_3-A_4,
\end{aligned}$$

where in the last sum each $r_i=0$ or ∞ and in each term there is at least one $r_j=0$ and $r_l=\infty$.

By Hölder's inequality, we have

$$\begin{aligned}
\frac{1}{|Q|}\int_Q|A_1(z)|dz &\leqslant\left(\frac{1}{|Q|}\int_Q|b(z)-b_Q|^{r'}dz\right)^{1/r'}\left(\frac{1}{|Q|}\int_Q|I_\alpha(\vec{f})(z)|^r dz\right)^{1/r}\\
&\leqslant C\|b\|_*(M(|I_\alpha(\vec{f})|)^r)^{1/r}(x).
\end{aligned}$$

Since $1<s_i<p_j^-$, we can choose $\gamma,\beta_i\geqslant 1$ such that $r\beta_i=s_i$. Thus $1<\beta_i<p_j^-$. Then there exists $u>1$ such that $\frac{1}{u}=\sum_{i=1}^m\frac{1}{\beta_i}-\frac{\alpha}{n}$. In fact, we can choose β_i and α_i such that $0<\frac{1}{\beta_i}-\frac{\alpha_i}{n}<\frac{1}{m}$. Applying Hölder's inequality and boundedness of the multilinear fractional integrals (see [24]), we have

$$
\begin{aligned}
\frac{1}{|Q|}\int_Q |A_2(z)|dz &= \left\{\frac{1}{|Q|}\int_Q |I_\alpha(f_1^0,\ldots,(b-b_Q)f_j^0,\ldots,f_m^0)(z)|^u dz\right\}^{1/u}\\
&\leqslant \frac{C}{|Q|^{1/u}}\|(b-b_Q)f_j^0\|_{\beta_j}\prod_{i\neq j}\|f_i^0\|_{\beta_i}\\
&\leqslant C|Q|^{-1/u}\left(\int_{2Q}|b-b_Q|^{r'\beta_j}dy_j\right)^{1/r'\beta_j}\left(\int_{2Q}|f_j(y_j)|^{r\beta_j}dy_i\right)^{1/r\beta_j}\\
&\quad\times\prod_{i\neq j}\left(\int_{2Q}|f_i(y_i)|^{\beta_i}dy_i\right)^{1/\beta_i}\\
&\leqslant C\|b\|_*|Q|^{\alpha/n}\prod_{i=1}^m\left(\frac{1}{|2Q|}\int_{2Q}|f_i(y_i)|^{r\beta_i}dy_i\right)^{1/r\beta_i}\\
&\leqslant C\|b\|_*\prod_{i=1}^m\left(\frac{1}{|Q|^{1-\frac{\alpha_i s_i}{n}}}\int_{2Q}|f_i(y_i)|^{s_i}dy_i\right)^{1/s_i}\\
&\leqslant C\|b\|_*\prod_{i=1}^m(M_{\alpha_i s_i}|f_i|^{s_i})^{1/s_i}(x).
\end{aligned}
$$

Denote by x_0 the center of Q, then for $x\in Q$ and $y_i\in(2Q)^c$, then we have $|x-y_i|\sim|x_0-y_i|$. Since that $1<s_j<\infty$, we can choose $1<s_j'<\infty$ such that $\frac{1}{s_j'}+\frac{1}{s_j}=1$. We use Hölder's inequality with exponent s_j and s_j':

$$
\begin{aligned}
&|I_\alpha(f_1^\infty,\ldots,(b-b_Q)f_j^\infty,\ldots,f_m^\infty)(x)-I_\alpha(f_1^\infty,\ldots,(b-b_Q)f_j^\infty,\ldots,f_m^\infty)(x_0)|\\
&\leqslant \int_{(\mathbb{R}^n\setminus(2Q))^m}\frac{|Q|^{1/n}|b(y_j)-b_Q|\prod_{i=1}^m|f_i(y_i)|}{(\sum_{i=1}^m|x_0-y_i|)^{nm-\alpha+1}}\prod_{i=1}^m dy_i\\
&\leqslant \int_{\mathbb{R}^n\setminus 2Q}\frac{|Q|^{1/n}|b(y_j)-b_Q||f_j(y_j)|}{(\sum_{i=1}^m|x_0-y_i|)^{n-\alpha_j+1}}\\
&\quad\times\int_{(\mathbb{R}^n\setminus 2Q)^{m-1}}\frac{\prod_{i\neq j}|f_i(y_i)|}{(\sum_{i=1}^m|x_0-y_i|)^{n(m-1)-\sum_{i\neq j}\alpha_i+1}}\prod_{i=1}^m dy_i\\
&\leqslant \left(\int_{\mathbb{R}^n\setminus 2Q}\frac{|Q|^{1/n}|b(y_j)-b_Q|^{s_j'}}{(\sum_{i=1}^m|x_0-y_i|)^{n+1}}d_{y_j}\right)^{\frac{1}{s_j'}}\left(\int_{\mathbb{R}^n\setminus 2Q}\frac{|Q|^{1/n}|f_j(y_j)|^{s_j}}{(\sum_{i=1}^m|x_0-y_i|)^{n+1-\alpha_j}}d_{y_j}\right)^{\frac{1}{s_j}}\\
&\quad\times\int_{(\mathbb{R}^n\setminus 2Q)^{m-1}}\frac{\prod_{i\neq j}|f_i(y_i)|}{(\sum_{i=1}^m|x_0-y_i|)^{n(m-1)-\sum_{i\neq j}\alpha_i+1}}\prod_{i\neq j}dy_i\\
&\leqslant C\|b\|_*\prod_{i=1}^m(M_{\alpha_i s_i}|f_i|^{s_i})^{1/s_i}(x).
\end{aligned}
$$

We now deal with A_4. Without loss of generality, we may assume that $r_1=\ldots=r_d=0$ and $r_{d+1}=\ldots=r_m=\infty$. When $d+1\leqslant j\leqslant m$, applying the mean value theorem and Hölder's inequality, we have

$$\begin{aligned}
&|I_\alpha(f_1^{r_1},\ldots,(b-b_Q)f_j^{r_j},\ldots,f_m^{r_m})(x)-I_\alpha(f_1^{r_1},\ldots,(b-b_Q)f_j^{r_j},\ldots,f_m^{r_m})(x_0)|\\
&\leqslant \prod_{i=1}^{d}\int_{2Q}|f_i^{r_i}|\int_{(\mathbb{R}^n\setminus 2Q)^{m-d}}\frac{|Q|^{1/n}|b(y_j)-b_Q|\prod_{i=d+1}^m f_i^{r_i}(y_i)}{(\sum_{i=1}^m|x_0-y_i|)^{nm-\alpha+1}}\prod_{i=1}^m dy_i\\
&\leqslant \prod_{i=1}^{d}\int_{2Q}|f_i^{r_i}|dy_i\sum_{l=1}^{\infty}\int_{(2^{l+1}\setminus 2^lQ)^{m-d}}\frac{|Q|^{1/n}|b(y_j)-b_Q|\prod_{i=d+1}^m f_i^{r_i}(y_i)}{(2^l|Q|^{1/n})^{nm-\alpha+1}}\prod_{i=d+1}^m dy_i\\
&\leqslant C\|b\|_*\prod_{i=1}^m(M_{\alpha_i s_i}|f_i|^{s_i})^{1/s_i}(x).
\end{aligned}$$

When $1\leqslant j\leqslant d$, similarly we also can obtain

$$\begin{aligned}
&|I_\alpha(f_1^{r_1},\ldots,(b-b_Q)f_j^{r_j},\ldots,f_m^{r_m})(x)-I_\alpha(f_1^{r_1},\ldots,(b-b_Q)f_j^{r_j},\ldots,f_m^{r_m})(x_0)|\\
&\leqslant C\|b\|_*\prod_{i=1}^m(M_{\alpha_i s_i}|f_i|^{s_i})^{1/s_i}(x).
\end{aligned}$$

Note that $\|Mf\|_{p(\cdot)}\leqslant C\|M^\sharp f\|_{p(\cdot)}$ when $p(\cdot)\in LH\cap\mathscr{P}^0$ (see [7]). Thus, if $b\in BMO$, we have

$$\begin{aligned}
\|[b,I_\alpha]_j(\vec{f})\|_{q(\cdot)}&\leqslant \|M[b,I_\alpha]_j(\vec{f})\|_{q(\cdot)}\leqslant \|M^\sharp[b,I_\alpha]_j(\vec{f})\|_{q(\cdot)}\\
&\leqslant C\|b\|_*\left(\left\|M(|I_\alpha(\vec{f})|^r)^{1/r}\right\|_{q(\cdot)}+\left\|\prod_{i=1}^m(M_{\alpha_i s_i}|f_i|^{s_i})^{1/s_i}\right\|_{q(\cdot)}\right).
\end{aligned}$$

Note that $1<r<q^-$. It is easy to see that $q(\cdot)\in LH(\mathbb{R}^n)$, then M is of $(\frac{q(\cdot)}{r},\frac{q(\cdot)}{r})$. The weighted inequalities for multilinear fractional integral operators has been established by Moen in [27]. Motivated by this, we consider here the following variable exponent case,

$$\|I_\alpha(\vec{f})\|_{q(\cdot)}\leqslant C\prod_{i=1}^m\|f_i\|_{p_i(\cdot)},$$

where $\frac{1}{q(x)}=\sum_{i=1}^m\frac{1}{p_i(x)}-\frac{\alpha}{n}$, $x\in\mathbb{R}^n$ and $q,p_i^->1$.

We can obtain the pointwise estimate of I_α. Here we use the techniques of Hedberg in [18] and of Welland in [41]. Fix ε, $\varepsilon=\sum_{i=1}^m\varepsilon_i$, $0<\varepsilon_i<\min\{\alpha_i,n-\alpha_i\}$. Then for all $f\in L^1_{loc}(\mathbb{R}^n)$ and any $Q\ni x$, we have

$$\begin{aligned}
I_\alpha f(x)&=\int_{Q^m}\frac{\prod_{i=1}^m f_i(y_i)}{(\sum_{i=1}^m|x-y_i|)^{mn-\alpha}}d\vec{y}+\int_{(\mathbb{R}^n\setminus Q)^m}\frac{\prod_{i=1}^m f_i(y_i)}{(\sum_{i=1}^m|x-y_i|)^{mn-\alpha}}d\vec{y}\\
&=I+II
\end{aligned}$$

Thus

$$
\begin{aligned}
|I| &\leqslant \sum_{i=1}^{\infty}\int_{(2^{-i}Q\backslash 2^{-i-1}Q)^m}\frac{\prod_{i=1}^m|f_i(y_i)|}{(\sum_{i=1}^m|x-y_i|)^{mn-\alpha}}d\vec{y}\\
&\leqslant \sum_{i=1}^{\infty}\frac{1}{(2^{-i-1}|Q|^{-\frac{1}{n}})^{nm-\alpha}}\int_{(2^{-i}Q)^m}\prod_{i=1}^m|f_i(y_i)|d\vec{y}\\
&\leqslant C\prod_{i=1}^m|Q|^{\frac{\varepsilon_i}{n}}M_{\alpha_i-\varepsilon_i}f_i(x)=C|Q|^{\frac{\varepsilon}{n}}\prod_{i=1}^m M_{\alpha_i-\varepsilon_i}f_i(x).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
|II| &\leqslant \sum_{i=0}^{\infty}\int_{(2^{i}Q\backslash 2^{i-1}Q)^m}\frac{\prod_{i=1}^m|f_i(y_i)|}{(\sum_{i=1}^m|x-y_i|)^{mn-\alpha}}d\vec{y}\\
&\leqslant \sum_{i=1}^{\infty}\frac{1}{(2^{i}|Q|^{-\frac{1}{n}})^{mn-\alpha}}\int_{(2^{i}Q)^m}\prod_{i=1}^m|f_i(y_i)|d\vec{y}\\
&\leqslant |Q|^{\frac{-\varepsilon}{n}}\prod_{i=1}^m M_{\alpha_i+\varepsilon_i}f_i(x).
\end{aligned}
$$

By choosing $|Q|^{\frac{2\varepsilon}{n}}=\frac{\prod_{i=1}^m M_{\alpha_i+\varepsilon_i}f_i(x)}{\prod_{i=1}^m M_{\alpha_i-\varepsilon_i}f_i(x)}$, we obtain that

$$
|I_\alpha(\vec{f})(x)|\leqslant C(\prod_{i=1}^m M_{\alpha_i+\varepsilon_i}f_i(x))^{1/2}(\prod_{i=1}^m M_{\alpha_i-\varepsilon_i}f_i(x))^{1/2}.
$$

In order to prove that

$$
\|I_\alpha(\vec{f})\|_{q(\cdot)}\leqslant C\prod_{i=1}^m\|f_i\|_{p_i(\cdot)},
$$

where $\frac{1}{q(x)}=\sum_{i=1}^m\frac{1}{p_i(x)}-\frac{\alpha}{n}$, $x\in\mathbb{R}^n$ and $p_i^->1$. Without loss of generality we may assume that $\|f_i\|_{p_i(\cdot)}=1$. We recall that $\|f\|_{p(\cdot)}\leqslant C$ if and only if $\int_{\mathbb{R}^n}|f(y)|^{p(y)}dy\leqslant C$, see ([1]). Since $q^+<\infty$, it will be sufficient to prove that $\int_{\mathbb{R}^n}|I_\alpha(\vec{f})(x)|^{q(x)}dx\leqslant C$.

Define $r(\cdot):\mathbb{R}^n\to[1,\infty)$ by

$$
r(x)=\frac{2}{\varepsilon q(x)/n+1}.
$$

Then for all $x\in\mathbb{R}^n$

$$
\sum_{i=1}^m\frac{1}{p_i(x)}-\frac{1}{r(x)q(x)/2}=\frac{\alpha-\varepsilon}{n}
$$

and

$$
\sum_{i=1}^m\frac{1}{p_i(x)}-\frac{1}{r'(x)q(x)/2}=\frac{\alpha+\varepsilon}{n}.
$$

Fix ε, $0<\varepsilon<\max\{\alpha,n-\alpha\}$ such that

$$\frac{2}{\frac{\varepsilon q^+}{n}+1}>1.$$

Then we have $r^->1$. By the pointwise estimate of $I^\alpha(\vec{f})$ and then Hölder's inequality for variable L^p with exponents $r(\cdot)$ and $r'(\cdot)$, we get

$$\begin{aligned}\int_{\mathbb{R}^n}|I_\alpha\vec{f}(x)|^{q(x)}dx &\leqslant C\int_{\mathbb{R}^n}\Big[\prod_{i=1}^m M_{\alpha_i+\varepsilon_i}f_i(x)\Big]^{q(x)/2}\Big[\prod_{i=1}^m M_{\alpha_i-\varepsilon_i}f_i(x)\Big]^{q(x)/2}dx\\ &\leqslant C\left\|\left[\prod_{i=1}^m M_{\alpha_i+\varepsilon_i}f_i\right]^{q(\cdot)/2}\right\|_{r'(\cdot)}\left\|\left[\prod_{i=1}^m M_{\alpha_i-\varepsilon_i}f_i\right]^{q(\cdot)/2}\right\|_{r(\cdot)}\end{aligned}$$

Now we will estimate each term on the righthand side. We may assume that each is greater than 1, since otherwise it is nothing to prove. It is easy to see $(\frac{r'(\cdot)q(\cdot)}{2})^->1$, then we can choose exponent function $s_i(x)\geqslant s_i^->1$ such that $\sum_{i=1}^m\frac{1}{s_i(x)}=\frac{1}{\frac{r'(x)q(x)}{2}}$ and $\frac{1}{p_i(x)}-\frac{1}{s_i(x)}=\frac{\alpha_i+\varepsilon_i}{n}$, where $i=1,\cdots,m$ and $x\in\mathbb{R}^n$. By the definition of Luxembourg norm of variable Lebesgue spaces and Lemmas 2.1 and 2.2, we have

$$\begin{aligned}\left\|\left[\prod_{i=1}^m M_{\alpha_i+\varepsilon_i}f_i\right]^{q(\cdot)/2}\right\|_{r'(\cdot)} &\leqslant \left\|\prod_{i=1}^m M_{\alpha_i+\varepsilon_i}f_i\right\|_{q(\cdot)r'(\cdot)/2}^{q^+/2}\leqslant C\prod_{i=1}^m\|M_{\alpha_i+\varepsilon_i}f_i\|_{s_i(\cdot)}^{q^+/2}\\ &\leqslant C\prod_{i=1}^m\|f_i\|_{p_i(\cdot)}^{q^+/2}\leqslant C,\end{aligned}$$

where the first inequality follows from

$$\begin{aligned}\int_{\mathbb{R}^n}\left(\frac{\left(\prod_{i=1}^m M_{\alpha_i+\varepsilon_i}f_i(x)\right)^{q(x)/2}}{\lambda}\right)^{r'(x)}dx &= \int_{\mathbb{R}^n}\left(\frac{\left(\prod_{i=1}^m M_{\alpha_i+\varepsilon_i}f_i(x)\right)}{\lambda^{2/q(x)}}\right)^{r'(x)q(x)/2}dx\\ &\leqslant \int_{\mathbb{R}^n}\left(\frac{\left(\prod_{i=1}^m M_{\alpha_i+\varepsilon_i}f_i(x)\right)}{\lambda^{2/q^+}}\right)^{r'(x)q(x)/2}dx,\end{aligned}$$

since for $\lambda>1$, $\lambda^{2/q(x)}\geqslant\lambda^{2/q^+}$.

Similarly, we can also obtain

$$\left\|\left[\prod_{i=1}^m M_{\alpha_i-\varepsilon_i}f_i\right]^{q(\cdot)/2}\right\|_{r(\cdot)}\leqslant C.$$

Therefore, we have

$$\|(M(|I_\alpha(\vec{f})|)^r)^{1/r}\|_{q(\cdot)}\leqslant C\|(|I^\alpha(\vec{f})|^r)\|_{q(\cdot)/r}^{1/r}\leqslant C\prod_{i=1}^m\|f_i\|_{p_i(\cdot)}.$$

Then we can choose $q_i(x) > 1$ such that $\frac{1}{q(x)} = \sum_1^m \frac{1}{q_i(x)}$ and $\frac{1}{q_i(x)} = \frac{1}{p_i(x)} - \frac{\alpha_i}{n}$. Notice that $\frac{1}{q_i(x)/s_i} = \frac{1}{p_i(x)/s_i} - \frac{\alpha_i s_i}{n}$, where $1 \leqslant i \leqslant m$. By Lemma 2.1 and 2.2, we get

$$\left\| \prod_{i=1}^m (M_{\alpha_i s_i}|f_i|^{s_i})^{1/s_i} \right\|_{q(\cdot)} \leqslant \prod_{i=1}^m \|(M_{\alpha_i s_i}|f_i|^{s_i})^{1/s_i}\|_{q_i(\cdot)} \leqslant C \prod_{i=1}^m \|f_i\|_{p_i(\cdot)}.$$

Thus we have that

$$\|[b, I_\alpha]_j(\vec{f})\|_{q(\cdot)} \leqslant C\|b\|_* \prod_{i=1}^m \|f_i\|_{p_i(\cdot)}.$$

Next we will give the proof of the converse part. Its idea of the proof here comes from Chanillo [3] and Janson [22].

The reciprocal of the convolution kernel $K_\alpha(\vec{y})$ is smooth away from the origin. We can choose $(\vec{y'}) = (y'_1, y'_2, \ldots, y'_m) \in \mathbb{R}^{mn}$, $\delta > 0$ such that in the ball $B = B((\vec{y'}), \delta\sqrt{mn})$. Then we use the similar argument as Shirai [35]. Considering a smooth cut-off function which equal to 1 on B and equal to 0 outside $\tilde{B} \supset B$, $1/K_\alpha(\vec{y})$ can be represented in the ball B as a Fourier series which absolutely converges. That is

$$\frac{1}{K_\alpha(\vec{y})} \sim \left(\sum_{i=1}^m |y_i|^2 \right)^{(mn-\alpha)/2} = \sum_l a_l e^{\mathbf{i} v_l \cdot (\vec{y})},$$

whenever $\vec{y} \in B$. For any cube $Q = Q(y_0, r)$, let $\tilde{y}_i = y_0 - \frac{r y'_i}{\delta}$, and $Q'_i = Q(\tilde{y}_i, r)$, where $i = 1, 2, \cdot, m$. It is easy to see that $\frac{\delta}{r}(x - y_1, x - y_2, \ldots, x - y_m) \in B$ for any $x \in Q$ and $y_i \in Q'_i$.

Set $s(x) = \operatorname{sgn}[b(x) - b_{Q'_j}]$ and $E = \{x : b(x) - b_{Q'_j} \neq 0\}$. Then we have

$$\begin{aligned}
|b(x) - b_{Q'_j}| &= s(x)(b(x) - b_{Q'_j}) \\
&= \frac{s(x)}{|Q|} \int_{Q'_j} (b(x) - b(y_j)) dy_j \\
&= \frac{s(x)}{|Q|^m} \int_{(\mathbb{R}^n)^m} \frac{b(x) - b(y_j)}{(\sum_{i=1}^m |x - y_i|)^{nm-\alpha}} \left(\sum_{i=1}^m |x - y_i| \right)^{mn-\alpha} \chi_Q(y) d\vec{y} \\
&= \sum_l a_l \delta^{-mn} s(x) \int_{(\mathbb{R}^n)^m} \frac{b(x) - b(y_j)}{(\sum_{i=1}^m |x - y_i|)^{mn-\alpha}} e^{\mathbf{i}\frac{\delta}{r} v_l \cdot (x, x, \ldots, x)} \\
&\quad \times \prod_{i=1}^m e^{-\mathbf{i}\frac{\delta}{r} v_l^i \cdot y_i} \chi_{Q'_i}(y_i) d\vec{y} \\
&=: \sum_l a_l \delta^{-mn} \int_{(\mathbb{R}^n)^m} \frac{b(x) - b(y_j)}{(\sum_{i=1}^m |x - y_i|)^{mn-\alpha}} h_l(x) \prod_{i=1}^m f_i(y_i) d\vec{y} \\
&= \sum_l a_l \delta^{-mn} h_l(x) \frac{1}{|Q|^{\frac{\alpha}{n}}} [b, I_\alpha]_j(\vec{f})(x),
\end{aligned}$$

where

$$f_i(y_i) = e^{-\mathbf{i}\frac{\delta}{r}v_l^i\cdot y_i}\chi_{Q_i'}(y_i),$$

$$h_l(x) = e^{\mathbf{i}\frac{\delta}{r}v_l\cdot(x,x,\dots,x)}s(x).$$

Observe that $|h_l(x)| = 1$ when $x \in E$ and otherwise $|h_l(x)| = 0$, then we obtain that

$$\begin{aligned}
&\|(b-b_{Q_j'})\chi_Q\|_{q(\cdot)}\\
&= \frac{1}{|Q|^{\frac{\alpha}{n}}}\Big\|\sum_l a_l\delta^{-mn}h_l[b,I_\alpha]_j(\vec{f})\chi_E\Big\|_{q(\cdot)} + \frac{1}{|Q|^{\frac{\alpha}{n}}}\Big\|\sum_l a_l\delta^{-mn}h_l[b,I_\alpha]_j(\vec{f})\chi_{E^c}\Big\|_{q(\cdot)}\\
&\leqslant \sum_l a_l\delta^{-mn}\frac{1}{|Q|^{\frac{\alpha}{n}}}\|[b,I_\alpha]_j\|_{\mathscr{B}(\prod_{i=1}^m L^{p_i(\cdot)},L^{q(\cdot)})}\prod_{i=1}^m\|f_i\|_{p_i(\cdot)}\\
&\leqslant C\frac{1}{|Q|^{\frac{\alpha}{n}}}\|[b,I_\alpha]_j\|_{\mathscr{B}(\prod_{i=1}^m L^{p_i(\cdot)},L^{q(\cdot)})}\prod_{i=1}^m\|\chi_{Q_i'}\|_{p_i(\cdot)}.
\end{aligned}$$

When $|Q| \leqslant 1$, choose a proper $\delta > 0$ such that $\bigcap_{i=1}^m Q_i' \neq \emptyset$, then by Lemma 2.3, then there exist $z \in \bigcap_{i=1}^m Q_i' \neq \emptyset$ such that

$$\begin{aligned}
\|Q\|_{q(\cdot)}^{-1}\|(b-b_{Q_j'})\chi_Q\|_{q(\cdot)} &\leqslant C\|Q\|_{q(\cdot)}^{-1}|Q|^{-\frac{\alpha}{n}}\|[b,I_\alpha]_j\|_{\mathscr{B}(\prod_{i=1}^m L^{p_i(\cdot)},L^{q(\cdot)})}\prod_{i=1}^m\|\chi_{Q_i'}\|_{p_i(\cdot)}\\
&\leqslant C\|[b,I_\alpha]_j\|_{\mathscr{B}(\prod_{i=1}^m L^{p_i(\cdot)},L^{q(\cdot)})}|Q|^{-\frac{1}{q(z)}+\sum_{i=1}^m\frac{1}{p_i(z)}-\frac{\alpha}{n}}\\
&\leqslant C\|[b,_\alpha]_j\|_{\mathscr{B}(\prod_{i=1}^m L^{p_i(\cdot)},L^{q(\cdot)})}.
\end{aligned}$$

When $|Q| > 1$, similarly we can get

$$\|Q\|_{q(\cdot)}^{-1}\|(b-b_{Q_j'})\chi_Q\|_{q(\cdot)} \leqslant C\|[b,I_\alpha]_j\|_{\mathscr{B}(\prod_{i=1}^m L^{p_i(\cdot)},L^{q(\cdot)})}.$$

Thus, Theorem 1.1 is proved according to Lemma 2.4. □

Then we give the proof of Theorem 1.2.

Proof. First we will get the following sharp maximal estimate,

$$M^\sharp\big([b,T]_j(\vec{f})\big)(x) \leqslant C\|b\|_*\Big[(M(|T(\vec{f})|^s))^{1/s}(x) + \prod_{i=1}^m(M|f_i|^{s_i})^{1/s_i}(x)\Big],$$

where $\frac{1}{s} = \sum_{i=1}^m\frac{1}{s_i} < 1$ and $1 < s_i < p_i^-$ with $i = 1,2,\cdots,n$.

Denote that $f_j^0 = f_j\chi_{2Q}$ and $f_j^\infty = f_j - f_j^0$. Fix a cube Q and set

$$\begin{aligned}
[b,T]_j(\vec{f})(x) &= \int_{(\mathbb{R}^n)^m}[(b(x)-b_Q)+(b_Q-b(y_j))]\prod_{i=1}^m f_i(y_i)K(x,\vec{y})d\vec{y}\\
&= (b(x)-b_Q)T(\vec{f})(x) - T(f_1,\dots,(b-b_Q)f_j,\dots,f_m)(x)
\end{aligned}$$

$$
\begin{aligned}
&= (b(x)-b_Q)T(\vec{f})(x) - T(f_1^0,\ldots,(b-b_Q)f_j^0,\ldots,f_m^0)(x)\\
&\quad -T(f_1^\infty,\ldots,(b-b_Q)f_j^\infty,\ldots,f_m^\infty)(x)\\
&\quad -\sum T(f_1^{r_1},\ldots,(b-b_Q)f_j^{r_j},\ldots,f_m^{r_m})(x)\\
&:= B_1 - B_2 - B_3 - B_4,
\end{aligned}
$$

where in the last sum each $r_i = 0$ or ∞ and in each term there is at least one $r_l = 0$ and $r_m = \infty$.

By Hölder's inequality, we have

$$
\begin{aligned}
\frac{1}{|Q|}\int_Q |B_1(z)|dz &\leqslant \left(\frac{1}{|Q|}\int_Q |b(z)-b_Q|^{s'}dz\right)^{1/s'}\left(\frac{1}{|Q|}\int_Q |T(\vec{f})(z)|^s dz\right)^{1/s}\\
&\leqslant C\|b\|_*(M(|T(\vec{f})|)^s)^{1/s}(x).
\end{aligned}
$$

Choose $1 < u, q$ and $q_i < \infty$ such that $uq_i = s_i$ and $\frac{1}{q} = \sum_{i=1}^m \frac{1}{q_i}$.

Thus applying Hölder's inequality and the boundedness of T (see [17]), we obtain that

$$
\begin{aligned}
\frac{1}{|Q|}\int_Q |B_2(z)|dz &\leqslant \left(\frac{1}{|Q|}\int_Q |T(f_1^0,\ldots,(b-b_Q)f_j^0,\ldots,f_m^0)(z)|^q dz\right)^{1/q}\\
&\leqslant |Q|^{-\frac{1}{q}}\left(\prod_{i\neq j}\|f_i^0\|_{q_i}\right)\|(b-b_Q)f_j^0\|_{q_j}\\
&\leqslant |Q|^{-\sum_{i=1}^m \frac{1}{q_i}}\left(\prod_{i\neq j}\|f_i\chi_{4\sqrt{n}Q}\|_{q_i}\right)\|(b-b_Q)\chi_{4\sqrt{n}Q}\|_{q_j u'}\|f_j\chi_{4\sqrt{n}Q}\|_{q_j u}\\
&\leqslant C\|b\|_*\prod_{i=1}^m (M(|f_i|)^{s_i})^{1/s_i}(x).
\end{aligned}
$$

Now we denote Q by $Q = Q(x_0, l(Q))$. If $y_i \notin 4\sqrt{n}Q$, then

$$
|x - x_0| \leqslant \frac{1}{2}\max_{1\leqslant i\leqslant n}|x - y_i|
$$

for any $x \in Q$. By the smoothness estimates of kernel K, we conclude that

$$
\begin{aligned}
|B_3(x) - B_3(x_0)| &\leqslant \int_{(\mathbb{R}^n)^m} |K(x,\vec{y}) - K(x_0,\vec{y})||b(y_j)-b_Q|\prod_{i=1}^m |f_i^\infty(y_i)|d\vec{y}\\
&\leqslant \int_{(\mathbb{R}^n)^m} \frac{A|x-x_0|^\varepsilon}{(\sum_{i=1}^m |x-y_i|)^{nm+\varepsilon}}|b(y_j)-b_Q|\prod_{i=1}^m |f_i^\infty(y_i)|d\vec{y}\\
&\leqslant C\int_{\mathbb{R}^n\setminus 4\sqrt{n}Q} \frac{l(Q)^{\frac{\varepsilon}{m}}|(b-b_Q)f_j(y_j)|}{|x-y_j|^{n+\frac{\varepsilon}{m}}}dy_j\\
&\quad\times\prod_{i\neq j}\int_{\mathbb{R}^n\setminus 4\sqrt{n}Q}\frac{l(Q)^{\frac{\varepsilon}{m}}|f_i(y_i)|}{|x-y_i|^{n+\frac{\varepsilon}{m}}}dy_i\\
&\leqslant C\|b\|_*\prod_{i=1}^m (M(|f_i|)^{s_i})^{1/s_i}(x).
\end{aligned}
$$

Now we consider B_4. Here we write $(4\sqrt{n}Q)^l \times (\mathbb{R}^n \backslash 4\sqrt{n}Q)^{m-l} = R_l$. Without loss of generality, we may assume that $r_1 = \ldots = r_l = 0$ and $r_{l+1} = \ldots = r_m = \infty$. When $1 \leqslant j \leqslant l$, by the size estimate of K, we conclude that

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q |T(f_1^0,\ldots,(b-b_Q)f_j^0,\ldots,f_l^0,f_{l+1}^\infty,\ldots,f_m^\infty)|(z)dz \\
&\leqslant \frac{1}{|Q|}\int_Q\int_{R_l} |K(z,\vec{y})f_1^0(y_1)\ldots(b(y_j)-b_Q)f_j^0(y_j)\ldots f_{l+1}^\infty(y_{l+1})\ldots f_m^\infty(y_m)|d\vec{y}dx \\
&\leqslant \frac{1}{|Q|}\int_Q\int_{R_l} \frac{A|f_1^0(y_1)|\ldots|b(y_j)-b_Q||f_j^0(y_j)|\ldots|f_{l+1}^\infty(y_{l+1})|\ldots|f_m^\infty(y_m)|}{(\sum_{i=1}^m |z-y_i|)^{nm}}d\vec{y}dx \\
&\leqslant \frac{C}{|Q|}\int_Q\Bigg(\frac{1}{|4\sqrt{n}Q|}\int_{4\sqrt{n}Q}|b(y_j)-b_Q||f_j(y_j)|dy_j\prod_{\substack{i=1\\ i\neq j}}^{l}\frac{1}{|4\sqrt{n}Q|}\int_{4\sqrt{n}Q}|f_i(y_i)|dy_i \\
&\quad\times(|Q|^{\frac{l}{m-l}})^{m-l}\prod_{k=l+1}^m\int_{\mathbb{R}\backslash 4\sqrt{n}Q}\frac{|f_k(y_k)|}{|z-y_k|^{n+\frac{nl}{m-l}}}dy_k\Bigg)dz \\
&\leqslant C\|b\|_*\prod_{i=1}^m(M(|f_i|)^{s_i})^{1/s_i}(x).
\end{aligned}$$

Similarly, when $l+1 \leqslant j \leqslant m$, we also can get

$$\frac{1}{|Q|}\int_Q |T(f_1^0,\ldots,(b-b_Q)f_j^0,\ldots,f_l^0,f_{l+1}^\infty,\ldots,f_m^\infty)|(z)dz \leqslant C\|b\|_*\prod_{i=1}^m(M(|f_i|)^{s_i})^{1/s_i}(x).$$

Observe that $M_\delta^\sharp(T(\vec{f}))(x) \leqslant C\prod_{i=1}^m Mf_i(x)$ for $0<\delta<1/m$ and all $x\in\mathbb{R}^n$, where each f_i is smooth and with compactly supported (cf. [33]) and $\|Mf\|_{p(\cdot)} \leqslant C\|M^\sharp f\|_{p(\cdot)}$ when $p(\cdot)\in LH\cap\mathscr{P}^0$ (see [7]).

Then applying Lemma 2.1, we can obtain

$$\begin{aligned}
\|T(\vec{f})\|_{p(\cdot)} &\leqslant \|M_\delta(T(\vec{f}))\|_{p(\cdot)} \leqslant \|M_\delta^\sharp(T(\vec{f}))\|_{p(\cdot)} \\
&\leqslant C\|\prod_{i=1}^m Mf_i\|_{p(\cdot)} \leqslant C\prod_{i=1}^m\|f_i\|_{p_i(\cdot)}.
\end{aligned}$$

Thus, for every $p_i(\cdot)\in LH\cap\mathscr{P}$ we have

$$\|T(\vec{f})\|_{p(\cdot)} \leqslant C\prod_{i=1}^m\|f_i\|_{p_i(\cdot)}.$$

Noting that $\frac{1}{s} = \sum_{i=1}^m\frac{1}{s_i} > \sum_{i=1}^m\frac{1}{p_i^-} = \frac{1}{p^-}$, we have $s<p^-$. Then we can obtain

$$\begin{aligned}
\|[b,T]_j(\vec{f})\|_{p(\cdot)} &\leqslant \|M[b,T]_j(\vec{f})\|_{p(\cdot)} \leqslant \|M^\sharp[b,T]_j(\vec{f})\|_{p(\cdot)} \\
&\leqslant C\|b\|_*\left(\left\|(M(|T(\vec{f})|)^s)^{1/s}\right\|_{p(\cdot)} + \left\|\prod_{i=1}^m(M|f_i|^{s_i})^{1/s_i}\right\|_{p(\cdot)}\right) \\
&\leqslant C\|b\|_*\prod_1^m\|f_i\|_{p_i(\cdot)},
\end{aligned}$$

where each f_i is smooth and with compactly supported and, hence, $[b,T]_j$ extends as a bounded operator from $L^{p_1(\cdot)} \times \ldots \times L^{p_m(\cdot)}$ into $L^{p(\cdot)}$.

Here we use the techniques applied by Chaffee [2] and Janson [22] to prove our proof of the necessity. Choose $(\vec{y'}) = (y'_1, y'_2, \ldots, y'_m) \in \mathbb{R}^{mn}$, $\delta > 0$ such that in the ball $B = B((\vec{y'}), \delta\sqrt{mn})$, $\frac{1}{K(\vec{y})}$ can be represented as a Fourier series which absolutely converges. That is

$$\frac{1}{K(\vec{y})} = \sum_l a_l e^{\mathbf{i}v_l\cdot(\vec{y})},$$

whenever $\vec{y} \in B$. For any cube $Q = Q(y_0, r)$, let $\tilde{y}_i = y_0 - \frac{ry'_i}{\delta}$, and $Q'_i = Q(\tilde{y}_i, r)$, where $i = 1, 2, \ldots, m$. It is easy to see that $\frac{\delta}{r}(x - y_1, x - y_2, \ldots, x - y_m) \in B$ for any $x \in Q$ and $y_i \in Q'_i$. Denoting by $t(x) = \operatorname{sgn}(b(x) - b_{Q'_j})$, we conclude that

$$\begin{aligned}
|b(x) - b_{Q'_j}| &= \frac{t(x)}{|Q'_j|}\int_{Q'_j}(b(x) - b(y_j))dy_j \\
&= r^{-mn}t(x)\int_{\prod_{i=1}^m Q'_i}(b(x) - b(y_j))d\vec{y} \\
&= \delta^{-mn}t(x)\int_{\prod_{i=1}^m Q'_i}(b(x) - b(y_j))\frac{K(x - y_1, x - y_2, \ldots, x - y_m)}{K(\frac{\delta(x-y_1)}{r}, \frac{\delta(x-y_2)}{r}, \ldots, \frac{\delta(x-y_m)}{r})}d\vec{y} \\
&= \sum_l a_l \delta^{-mn}t(x)\int_{\prod_{i=1}^m Q'_i} e^{\mathbf{i}\frac{\delta}{r}v_l\cdot(x-y_1, x-y_2, \ldots, x-y_m)} \\
&\quad \times (b(x) - b(y_j))K(x - y_1, x - y_2, \ldots, x - y_m)d\vec{y} \\
&= \sum_l a_l \delta^{-mn}t(x)\int_{(\mathbb{R}^n)^m}(b(x) - b(y_j))K(x - y_1, x - y_2, \ldots, x - y_m) \\
&\quad \times e^{i\frac{\delta}{r}v_l\cdot(x,x,\ldots,x)}\prod_{i=1}^m e^{-\mathbf{i}\frac{\delta}{r}v_l^i\cdot y_i}\chi_{Q'_i}d\vec{y} \\
&=: \sum_l a_l \delta^{-mn}\int_{(\mathbb{R}^n)^m}(b(x) - b(y_j))K(x - y_1, x - y_2, \ldots, x - y_m) \\
&\quad \times h_l(x)\prod_{i=1}^m f_i(y_i)d\vec{y} \\
&= \sum_l a_l \delta^{-mn}h_l(x)[b,T]_j(\vec{f})(x),
\end{aligned}$$

where

$$f_i(y_i) = e^{-\mathbf{i}\frac{\delta}{r}v_l^i\cdot y_i}\chi_{Q'_i}(y_i),$$

$$h_l(x) = e^{\mathbf{i}\frac{\delta}{r}v_l\cdot(x,x,\ldots,x)}t(x).$$

Applying the boundedness of $[b,T]_j$ on variable Lebesgue spaces, we obtain that

$$\begin{aligned}\|(b-b_{Q'_j})\chi_Q\|_{p(\cdot)} &= \Big\|\sum_l a_l\delta^{-mn}h_l[b,T]_j(\vec{f})\Big\|_{p(\cdot)}\\ &\leqslant \sum_l a_l\delta^{-mn}\|[b,T]_j\|_{\mathscr{B}(\prod_{i=1}^m L^{p_i(\cdot)},L^{p(\cdot)})}\prod_{i=1}^m\|f_i\|_{p_i(\cdot)}\\ &\leqslant C\|[b,T]_j\|_{\mathscr{B}(\prod_{i=1}^m L^{p_i(\cdot)},L^{p(\cdot)})}\prod_{i=1}^m\|\chi_{Q'_i}\|_{p_i(\cdot)}.\end{aligned}$$

When $|Q|\leqslant 1$, we can choose a proper $\delta>0$ such that $\bigcap_{i=1}^m Q'_i\neq\emptyset$, then by Lemma 2.3, then there exist $z\in\bigcap_{i=1}^m Q'_i\neq\emptyset$ such that

$$\begin{aligned}\|Q\|_{p(\cdot)}^{-1}\|(b-b_{Q'_j})\chi_Q\|_{p(\cdot)} &\leqslant C\|[b,T]_j\|_{\mathscr{B}(\prod_{i=1}^m L^{p_i(\cdot)},L^{p(\cdot)})}\|Q\|_{p(\cdot)}^{-1}\prod_{i=1}^m\|\chi_{Q'_i}\|_{p_i(\cdot)}\\ &\leqslant C\|[b,T]_j\|_{\mathscr{B}(\prod_{i=1}^m L^{p_i(\cdot)},L^{p(\cdot)})}|Q|^{-\frac{1}{p(z)}+\sum_{i=1}^m\frac{1}{p_i(z)}}\\ &\leqslant C\|[b,T]_j\|_{\mathscr{B}(\prod_{i=1}^m L^{p_i(\cdot)},L^{p(\cdot)})}.\end{aligned}$$

When $|Q|>1$, similarly we can get

$$\|Q\|_{p(\cdot)}^{-1}\|(b-b_{Q'_j})\chi_Q\|_{p(\cdot)}\leqslant C\|[b,T]_j\|_{\mathscr{B}(\prod_{i=1}^m L^{p_i(\cdot)},L^{p(\cdot)})}.$$

Therefore, by Lemma 2.4 we have completed our proof. □

Next, we will prove Theorem 1.3.

Proof. By Theorem 1 in [16] for a maximal Calderón-Zygmund operator T_*, choose $0<\eta<p^-$ and all $x\in\mathbb{R}^n$, then we have that

$$T_*(\vec{f})(x)\leqslant C\bigg((M(|T(\vec{f})|^\eta)(x))^{1/\eta}+\prod_1^m Mf_i(x)\bigg).$$

Thus, by the fact that given $p(\cdot)\in\mathscr{P}^0$ and $p^+<\infty$, then for any $s>0$, $\||f|^s\|_{p(\cdot)}=\|f\|_{sp(\cdot)}^s$ and Lemma 2.1, we can obtain that

$$\begin{aligned}\|T_*(\vec{f})\|_{p(\cdot)} &\leqslant C\big(\|(M(|T(\vec{f})|^\eta))^{1/\eta}\|_{p(\cdot)}+\|\prod_1^m Mf_i\|_{p(\cdot)}\big)\\ &\leqslant C\big(\|(M(|T(\vec{f})|^\eta))\|_{p(\cdot)/\eta}^{1/\eta}+\prod_{i=1}^m\|Mf_i\|_{p_i(\cdot)}\big)\\ &\leqslant C\prod_{i=1}^m\|f_i\|_{p_i(\cdot)}.\end{aligned}$$

It concluded the proof. □

Also, the proof of Theorem 1.4 is given.

Proof. For the proof we need some known results. First, the weighted L^p boundedness of $[b,T]$ is obtained in [26].

PROPOSITION 3.1. *Let $1 < p_0 < \infty$. Suppose that T is defined as in Theorem 1.4 and satisfies $\|Tf\|_{L^{p_0}(w_0)} \leqslant C\|f\|_{L^{p_0}(w_0)}$ for $w \in A_s(1 < s < \infty)$. Then $[b,T]$ is bounded on $L^{p_0}(w_0)$.*

The next extrapolation theorem of Rubio de Francia below can be found in [8].

PROPOSITION 3.2. *Given a family $\mathscr{F}$, assume that for some p_0, $1 < p_0 < \infty$, and for every $w_0 \in A_{p_0}$,*

$$\|f\|_{L^{p_0}(w_0)} \leqslant C\|g\|_{L^{p_0}(w_0)}, \quad (f,g) \in \mathscr{F}.$$

Then for every $1 < p,q < \infty$, $w \in A_p$ and sequence $\{(f_j,g_j)\} \subset \mathscr{F}$,

$$\left\| \left(\sum_j |f_j|^q \right)^{1/q} \right\|_{L^p(w_0)} \leqslant C \left\| \left(\sum_j |g_j|^q \right)^{1/q} \right\|_{L^p(w_0)}, \quad (f,g) \in \mathscr{F}.$$

To prove Theorem 1.4, we take $s = p_0$. By Proposition 3.1, we have

$$\|[b,T]f\|_{L^{p_0}(w_0)} \leqslant C\|f\|_{L^{p_0}(w_0)}$$

for every $w \in A_{p_0}$. Observe that $p(\cdot) \in LH \bigcap \mathscr{P}$, by Lemma 2.5, for $([b,T]f, f) \in \mathscr{F}$ and $f \in L^{p(\cdot)}(w)$, we have

$$\|([b,T]f)w\|_{L^{p(\cdot)}} \leqslant C\|fw\|_{L^{p(\cdot)}}.$$

Furthermore, as an immediate consequence of Proposition 3.2, for every $1 < p,q < \infty$, $w_0 \in A_p$ and sequence $\{([b,T]f_j, f_j)\} \subset \mathscr{F}$,we get

$$\left\| \left(\sum_j |[b,T]f_j|^q \right)^{1/q} \right\|_{L^p(w_0)} \leqslant C \left\| \left(\sum_j |f_j|^q \right)^{1/q} \right\|_{L^p(w_0)}.$$

From this we immediately get vector-valued inequalities for it in the variable exponent setting. □

Finally, the proof of Theorem 1.5 is similar to that of Theorem 1.4.

Proof. In the same way as Theorem 1.4, Theorem 1.5 follows immediately from Lemma 2.5, Proposition 3.2 and Corollary 1.2 in [34]. Here we omit its proof. □

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