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INEQUALITIES WITH APPLICATIONS INVOLVING k–BETA RANDOM VARIABLE

SHAHID MUBEEN, SANA IQBAL AND ABDUR REHMAN

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Abstract. In this paper, we introduce some properties of beta k-distribution defined in [1]. We present some inequalities involving beta k-distribution via some classical inequalities, like Chebyshev's inequality for synchronous (asynchronous) mappings and Holder's inequality. Also, we discuss the inequalities for harmonic mean, variance and coefficient of variation of β_k random variable involving the parameter $k>0$. If $k=1$, we get the classical results.

1. Introduction

A process which generates raw data is called an experiment and an experiment which gives different results under similar conditions, even though it is repeated a large number of times, is termed as a random experiment. A variable whose values are determined by the outcomes of a random experiment is called a random variable or simply a variate. The random variables are usually denoted by capital letters X,Y and Z while the values associated to them by corresponding small letters x,y and z. The random variables are classified into two classes namely discrete and continuous random variables.

A random variable that can assume only a finite or countably infinite number of values, is known as discrete random variable while a variable which can assume each and every value within some interval is called a continuous random variable. The distribution function of a random variable X is denoted by $F(x)$. A random variable X may also be defined as continuous if its distribution function $F(x)$ is continuous and differentiable everywhere except at isolated points in the given range. Let the derivative of $F(x)$ be denoted by $f(x)$ i.e., $f(x)=\frac{d}{dx}F(x)$. Since $F(x)$ is a non-decreasing function of x, so

$$f(x)\geqslant 0 \quad \text{and} \quad F(x)=\int_{-\infty}^{x} f(x)dx, \text{for all}\, x.$$

Here, the function $f(x)$ is called the probability density function $p.d.f$ or simply a density function of the random variable X. A probability density function has the properties

$$f(x)\geqslant 0, \text{for all}\, x \quad \text{and} \quad F(x)=\int_{-\infty}^{\infty} f(x)dx=1.$$

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A moment designates the power to which the deviations are raised before averaging them. In statistics, we have three kinds of moments as:

(i) Moments about any value $x = A$ is the rth power of the deviation of variable from A and is called the rth moment of the distribution about A.

(ii) Moments about $x = 0$ is the rth power of the deviation of variable from 0 and is called the rth moment of the distribution about 0.

(iii) Moments about mean i.e., $x = \overline{x}$ for sample and $x = \mu$ for population, is the rth power of the deviation of variable from mean and is called the rth moment of the distribution about mean. These moments are also called central moments or mean moments and are used to describe the set of data.

NOTE. The moments about any number $x = A$ and about $x = 0$ are denoted by μ_r' while about mean position, by μ_r and $\mu_0 = \mu_0' = 1$.

A link between the moments about arbitrary mean and actual mean of the data can be established in the following results.

$$\mu_r = \binom{r}{0}\mu_r' - \binom{r}{1}\mu_{r-1}'\mu_1' + \binom{r}{2}\mu_{r-2}'\mu_1'^2 - \binom{r}{3}\mu_{r-3}'\mu_1'^3 + \dots . \tag{1.1}$$

Putting $r = 0,1,2,3,4,\dots$ in the relation (1.1), we get

$$\begin{aligned}
\mu_0 &= \mu_0' = 1,\ \mu_1 = \mu_1' - \mu_0'\mu_1' = \mu_1' - \mu_1' = 0, \\
\mu_2 &= \mu_2' - (\mu_1')^2 = \sigma^2 = (\text{variance}),\ \mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3, \\
\mu_4 &= \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4 \dots .
\end{aligned}$$

Conversely, we have

$$\mu_r' = \binom{r}{0}\mu_r + \binom{r}{1}\mu_{r-1}\mu_1' + \binom{r}{2}\mu_{r-2}\mu_1'^2 + \binom{r}{3}\mu_{r-3}\mu_1'^3 + \cdots . \tag{1.2}$$

For $r = 0,1$, we get the same results as above . So, putting $r = 2,3,4,\cdots$ in the relation (1.2), we get

$$\begin{aligned}
\mu_2' &= \mu_2 + 2\mu_1\mu_1' + \mu_1'^2\mu_0 = \mu_2, \\
\mu_3' &= \mu_3 + 3\mu_2\mu_1' + 3\mu_1'^2\mu_1 + \mu_1'^3, \\
&\vdots
\end{aligned}$$

REMARKS. From the above discussion, we observe that the first moment about the mean position is always zero while the second moment is equal to the variance.

If a random variable X assumes all the values from a to b, then for a continuous distribution, the rth moment about the arbitrary number A and mean μ respectively, are given by

$$\mu_r' = \int_a^b (x-A)^r f(x)dx \tag{1.3}$$

and

$$\mu_r = \int_a^b (x-\mu)^r f(x)dx. \tag{1.4}$$

In a random experiment with n outcomes, suppose a variable X assumes the values $x_1,\ldots,x_n$ with corresponding probabilities $p_1,\ldots,p_n$, then the paring (x_i,p_i), $i=1,2,\ldots$ is called probability distribution and $\Sigma p_i = 1$ (in case of discrete distributions). Also, if $f(x)$ is a continuous probability distribution function defined on an interval $[a,b]$, then $\int_a^b f(x)dx = 1$. The expected value of the variate is defined as the first moment of the probability distribution about $x=0$ i.e.,

$$\mu_1' = E(X) = \int_a^b xf(x)dx \tag{1.5}$$

and the rth moment about mean of the probability distribution is defined as $E(X-\mu)^r$, where μ is the mean of the distribution.

NOTE. For discrete probability distribution, all the above results and notations are same, just replacing the integral sign by the summation sign $(\sum)$. The definitions given in the introduction are taken from [2–4].

2. β_k Function and beta k-distribution

The gamma k-function introduced by Diaz and Teruel [5] is

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}},\ k>0, x\in \mathbb{C}\setminus k\mathbb{Z}^- \tag{1.6}$$

which is the generalization of $\Gamma(x)$ and the integral form of Γ_k is given by

$$\Gamma_k(x) = \int_0^\infty t^{x-1}e^{-\frac{t^k}{k}}dt,\ Re(x)>0. \tag{1.7}$$

Also, the researchers [6–11] have worked on the generalized k-gamma function and discussed the following properties:

$$\Gamma_k(x+k) = x\Gamma_k(x), \tag{1.8}$$

$$\Gamma_k(k) = 1, \tag{1.9}$$

$$(x)_{n,k} = \frac{\Gamma_k(x+nk)}{\Gamma_k(x)} \tag{1.10}$$

where $(x)_{n,k}$, is the Pochhammer k-symbol and also have the representation $(x)_{n,k} = x(x+k)(x+2k)(x+3k)\cdots(x+(n-1)k)$. We obtain the usual Pochhammer's symbol $(\alpha)_n$ by taking $k=1$. The authors [6] defined the k-beta function as

$$\beta_k(x,y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)},\ Re(x)>0,\ Re(y)>0 \tag{1.11}$$

and the integral form of $\beta_k(x,y)$ is

$$\beta_k(x,y) = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}dt. \tag{1.12}$$

From the definition of $\beta_k(x,y)$ given in (1.11) and (1.12), we can easily prove that

$$\beta_k(x,y) = \frac{1}{k}\beta\left(\frac{x}{k},\frac{y}{k}\right). \tag{1.13}$$

Note that when $k \to 1$, $\beta_k(x,y) \to \beta(x,y)$ and $\Gamma_k \to \Gamma$.

For more details about the theory of special k-functions like, gamma, polygamma, beta, hypergeometric k-functions, solutions of k-hypergeometric differential equations, contegious functions relations, inequalities and integral representations with applications involving gamma and beta k-functions, gamma and beta probability k-distributions and so forth (see [12–17]).

DEFINITION 2.1. Let X be a continuous random variable, then it is said to have a beta k-distribution with two parameters m and n, if its probability density k-function $(pdkf)$ is defined by [1,18]

$$f_k(x) = \begin{cases} \frac{1}{k\beta_k(m,n)} x^{\frac{m}{k}-1}(1-x)^{\frac{n}{k}-1}, & 0 \leqslant x \leqslant 1;\ m,n,k>0 \\ 0, \quad \text{elsewhere.} \end{cases} \tag{1.14}$$

In the above distribution, the k-beta variable is referred to as $\beta_k(m,n)$ and its k-distribution function $F_k(x)$ is given by

$$F_k(x) = \begin{cases} 0, \quad x<0, \\ \int_0^1 \frac{1}{k\beta_k(m,n)} x^{\frac{m}{k}-1}(1-x)^{\frac{n}{k}-1}, & 0 \leqslant x \leqslant 1;\ m,n,k>0 \\ 0, \quad x>1. \end{cases} \tag{1.15}$$

REMARKS. We can call the above function an incomplete beta k-function because, if $k=1$, it is an incomplete beta function tabulated in [19].

PROPOSITION 2.2. *The beta k-distribution $\beta_k(m,n)$, $m,n,k>0$ satisfies the following properties:*

(i) *Beta k-distribution is a proper probability distribution i.e., area of $\beta_k(m,n)$ under the curve $f_k(z)$ is unity.*

(ii) *The mean of this distribution is $\frac{m}{m+n}$.*

(iii) *The variance of $\beta_k(m,n)$ in terms of k is $\frac{mnk}{(m+n)^2(m+n+k)}$.*

(iv) *The harmonic mean of $\beta_k(m,n)$ in terms of k is $\frac{m-k}{m+n-k}$.*

Proof. (i) By using the definition of beta k-distribution, we have

$$\int_0^z f_k(z)dz = \int_0^z \frac{1}{k\beta_k(m,n)} z^{\frac{m}{k}-1}(1-z)^{\frac{n}{k}-1}dz, \qquad 0 \leqslant z \leqslant 1;\ m,n>0.$$

By the relation (1.12), we get

$$\int_0^z f_k(z)dz = \int_0^1 \frac{1}{k\beta_k(m,n)} z^{\frac{m}{k}-1}(1-z)^{\frac{n}{k}-1}dz = \frac{\beta_k(m,n)}{\beta_k(m,n)} = 1.$$

(ii) The mean of the k-distribution, denoted by μ_k, is given by

$$\begin{aligned}\mu_k &= E_k(Z) = \int_0^z z f_k(z)dz \\ &= \int_0^z \frac{1}{k\beta_k(m,n)} z.z^{\frac{m}{k}-1}(1-z)^{\frac{n}{k}-1}dz, \quad 0 \leqslant z \leqslant 1;\ m,n > 0.\end{aligned}$$

Using the relations (1.11), (1.12) and (1.8), we have

$$\begin{aligned}\mu_k &= \int_0^1 \frac{1}{k\beta_k(m,n)} z^{\frac{m}{k}}(1-z)^{\frac{n}{k}-1}dz = \frac{\beta_k(m+k,n)}{\beta_k(m,n)} \\ &= \frac{\Gamma_k(m+k)\Gamma_k(n)\Gamma_k(m+n)}{\Gamma_k(m)\Gamma_k(n)\Gamma_k(m+n+k)} = \frac{m}{m+n}.\end{aligned}$$

(iii) The variance of $\beta_k(m,n)$ is given by,

$$\sigma_k^2 = E_k(Z^2) - (E_k(Z))^2 \tag{1.16}$$

and

$$\begin{aligned}E_k(Z^2) &= \int_0^1 \frac{1}{k\beta_k(m,n)} z^{\frac{m}{k}+1}(1-z)^{\frac{n}{k}-1}dz = \frac{\beta_k(m+2k,n)}{\beta_k(m,n)} \\ &= \frac{\Gamma_k(m+2k)\Gamma_k(n)\Gamma_k(m+n)}{\Gamma_k(m)\Gamma_k(n)\Gamma_k(m+n+2k)} = \frac{m(m+k)}{(m+n)(m+n+k)}.\end{aligned}$$

Thus, substituting the values of $E_k(Z^2)$ and $E_k(Z)$ in equation (1.16) along with some algebraic calculations we get the desired result.

(iv) Let X be a $\beta_k(m,n)$ variate, then we have the expected value of $\frac{1}{X}$ as

$$E_k\left(\frac{1}{X}\right) = \frac{1}{k\beta_k(m,n)} \int_0^1 \frac{1}{x} x^{\frac{m}{k}-1}(1-x)^{\frac{n}{k}-1}dx = \frac{1}{k\beta_k(m,n)} \int_0^1 x^{\frac{m}{k}-1-1}(1-x)^{\frac{n}{k}-1}dx$$

which implies that

$$E_k\left(\frac{1}{X}\right) = \frac{\beta_k(m-k,n)}{\beta_k(m,n)} = \frac{\Gamma_k(m-k)\Gamma_k(n)}{\Gamma_k(m+n-k)} \frac{\Gamma_k(m+n)}{\Gamma_k(m)\Gamma_k(n)} = \frac{m+n-k}{m-k}.$$

Now, harmonic mean in terms of k-symbol is given by

$$H.M = \frac{1}{E_k(1/X)} = \frac{m-k}{m+n-k}. \quad \square$$

THEOREM 2.3. *The rth moment of the beta k-distribution is given by*

$$\frac{(m)_{r,k}}{(m+n)_{r,k}}.$$

Where, $(m)_{r,k}$ shows the Pochhammer k-symbol.

Proof. The rth moment in terms of $k>0$, about the origin is

$$\mu'_{r,k}=E_k(X^r)=\frac{1}{k\beta_k(m,n)}\int_0^1 x^r.x^{\frac{m}{k}-1}(1-x)^{\frac{n}{k}-1}dx$$

$$\begin{aligned} E_k(X^r)&=\frac{1}{k\beta_k(m,n)}\int_0^1 x^{\frac{m}{k}+r-1}(1-x)^{\frac{n}{k}-1}dx=\frac{\beta_k(m+rk,n)}{\beta_k(m,n)} \qquad (1.17)\\ &=\frac{\Gamma_k(m+rk)\Gamma_k(m+n)}{\Gamma_k(m)\Gamma_k(m+rk+n)}. \end{aligned}$$

Using the relation $(x)_{n,k}=\frac{\Gamma_k(x+nk)}{\Gamma_k(x)}$, in the numerator as well as in the denominator, we get the desired result. $\square$

3. Applications to beta k-distribution via Chebychev's integral inequality

In this section, we prove some inequalities which involve beta k-distribution by using some natural inequalities [20]. The following result is well known in the literature as Chebychev's integral inequality for synchronous (asynchronous) functions. Here, we use this result to prove some k-analog inequalities [21] and some new inequalities.

LEMMA 3.1. *Let $f,g,h: I\subseteq\mathbb{R}\to\mathbb{R}$ be such that $h(x)\geqslant 0$ for all $x\in I$ and h, hfg, hf and hg are integrable on I. If f,g are synchronous (asynchronous) on I, i.e.,*

$$(f(x)-f(y))(g(x)-g(y))\geqslant(\leqslant)=0 \, for\, all x,\, y\in I.$$

Then, we have the inequality (see [22])

$$\int_I h(x)dx\int_I h(x)f(x)g(x)dx\geqslant(\leqslant)\int_I h(x)f(x)dx\int_I h(x)g(x)dx. \qquad (1.18)$$

This lemma can be proved by using Korkine's identity [23]

$$\begin{aligned}&\int_I h(x)dx\int_I h(x)f(x)g(x)dx-\int_I h(x)f(x)dx\int_I h(x)g(x)dx\\ &=\frac{1}{2}\int_I\int_I h(x)h(y)(f(x)-f(y))(g(x)-g(y))dxdy.\end{aligned}$$

DEFINITION 3.2. Two positive real numbers a and b are said to be similarly (oppositely) unitary if

$$(a-1)(b-1)\geqslant(\leqslant)0. \qquad (1.19)$$

THEOREM 3.3. *Let the random variables X and Y be such that $X \sim \beta_k(p,q)$ and $Y \sim \beta_k(m,n)$, $p,q,m,n > 0$. Further, let the random variables U and V be such that $U \sim \beta_k(p,n)$ and $V \sim \beta_k(m,q)$. If*

$$(p-m)(q-n) \leqslant (\geqslant) 0 \tag{1.20}$$

then, we have the inequality

$$\frac{E_k(X)^r E_k(Y)^r}{E_k(U)^r E_k(V)^r} \geqslant (\leqslant) \frac{\beta_k(p,n)\beta_k(m,q)}{\beta_k(p,q)\beta_k(m,n)}, k > 0,\ r = 1,2,\ldots.$$

Proof. For $k > 0$, consider the mappings $f, g, h : [0,1] \to [0,\infty)$ given by

$$f(x) = x^{\frac{p-m}{k}}, \quad g(x) = (1-x)^{\frac{q-n}{k}} \quad \text{and} \quad h(x) = x^{\frac{r+m}{k}-1}(1-x)^{\frac{n}{k}-1}.$$

Now, differentiation of f and g gives

$$f'(x) = \frac{(p-m)}{k} x^{\frac{p-m}{k}-1}, \quad g'(x) = \frac{(n-q)}{k}(1-x)^{\frac{q-n}{k}-1}, \quad x \in (0,1).$$

As $k > 0$, so using the relations (1.19) and (1.20), we see that the mappings f and g are synchronous (asynchronous) having the same (opposite) monotonicity on [0, 1] and h is non negative on [0, 1]. Thus, using Chebychev's integral inequality (1.18) for the functions f, g, and h defined above, we have

$$\begin{aligned}&\int_0^1 x^{\frac{m}{k}+r-1}(1-x)^{\frac{n}{k}-1}dx. \int_0^1 x^{\frac{m}{k}+r-1}(1-x)^{\frac{n}{k}-1} x^{\frac{p-m}{k}}(1-x)^{\frac{q-n}{k}}dx \\ &\quad \geqslant (\leqslant) \int_0^1 x^{\frac{m}{k}+r-1}(1-x)^{\frac{n}{k}-1} x^{\frac{p-m}{k}}dx. \int_0^1 x^{\frac{m}{k}+r-1}(1-x)^{\frac{n}{k}-1}(1-x)^{\frac{q-n}{k}}dx.\end{aligned}$$

This implies

$$\begin{aligned}&\frac{1}{k}\int_0^1 x^{\frac{m}{k}+r-1}(1-x)^{\frac{n}{k}-1}dx. \frac{1}{k}\int_0^1 x^{\frac{p}{k}+r-1}(1-x)^{\frac{q}{k}-1}dx \\ &\quad \geqslant (\leqslant) \frac{1}{k}\int_0^1 x^{\frac{p}{k}+r-1}(1-x)^{\frac{n}{k}-1}dx. \frac{1}{k}\int_0^1 x^{\frac{m}{k}+r-1}(1-x)^{\frac{q}{k}-1}dx.\end{aligned} \tag{1.21}$$

Also, from the moment generating function given in the theorem (2.3), using the relation (1.17), we observe that

$$E_k(X)^r \beta_k(p,q) = \frac{1}{k}\int_0^1 x^{\frac{p}{k}+r-1}(1-x)^{\frac{q}{k}-1}dx, \tag{1.22}$$

$$E_k(Y)^r \beta_k(m,n) = \frac{1}{k}\int_0^1 x^{\frac{m}{k}+r-1}(1-x)^{\frac{n}{k}-1}dx, \tag{1.23}$$

$$E_k(U)^r \beta_k(p,n) = \frac{1}{k}\int_0^1 x^{\frac{p}{k}+r-1}(1-x)^{\frac{n}{k}-1}dx \tag{1.24}$$

and

$$E_k(V)^r \beta_k(m,q) = \frac{1}{k}\int_0^1 x^{\frac{m}{k}+r-1}(1-x)^{\frac{q}{k}-1}dx. \tag{1.25}$$

Applying the relation (1.22) to (1.25) in (1.21) and rearranging the terms, we get the required proof. □

COROLLARY 3.4. *For $q = p,\ n = m > 0$, the condition $(p-m)(q-n) \leqslant (\geqslant) 0$ will become $(p-m)^2 \geqslant 0$ and the theorem (3.3) becomes*

$$\frac{E_k(X)^r E_k(Y)^r}{E_k(U)^r E_k(V)^r} \leqslant \frac{\beta_k(p,m)\beta_k(m,p)}{\beta_k(p,p)\beta_k(m,m)},\ k > 0,\ r = 1,2,\cdots$$

and using the relation (1.11), we obtain

$$\frac{E_k(X)^r E_k(Y)^r}{E_k(U)^r E_k(V)^r} \leqslant \frac{\Gamma_k(2p)\Gamma_k(2m)}{\Gamma_k^2(p+m)},\ k > 0,\ r = 1,2,\cdots.$$

THEOREM 3.5. *Let the random variables X and Y be such that $X \sim \beta_k(p,q)$ and $Y \sim \beta_k(p,n)$. Then, for $p,q,m,n > 0$, we have the inequality for beta k-distribution*

$$\frac{E_k(X)^r}{E_k(Y)^r} \geqslant (\leqslant) \frac{\Gamma_k(p+q)\Gamma_k(m+n)}{\Gamma_k(p+n)\Gamma_k(m+q)},\ k > 0,\ r = 1,2,\ldots$$

according as

$$(p-m)(q-n) \leqslant (\geqslant) 0.$$

Proof. For $k > 0$, consider the mappings $f, g, h : [0,1] \to [0,\infty)$ given by

$$f(x) = x^{\frac{p-m}{k}+r}, \quad g(x) = (1-x)^{\frac{q-n}{k}} \quad \text{and} \quad h(x) = x^{\frac{m}{k}-1}(1-x)^{\frac{n}{k}-1}.$$

Using these mappings in the Chebychev's inequality, we have

$$\int_0^1 x^{\frac{m}{k}-1}(1-x)^{\frac{n}{k}-1}dx.\int_0^1 x^{\frac{p}{k}+r-1}(1-x)^{\frac{q}{k}-1}dx$$
$$\geqslant (\leqslant) \int_0^1 x^{\frac{p}{k}+r-1}(1-x)^{\frac{n}{k}-1}dx.\int_0^1 x^{\frac{m}{k}-1}(1-x)^{\frac{q}{k}-1}dx. \tag{1.26}$$

By the definitions of expected values of beta k-distribution given in the relations (1.22) to (1.25) and k-beta function, inequality (1.26) gives

$$k\beta_k(m,n).k\beta_k(p,q)E_k(X)^r \geqslant (\leqslant) kE_k(Y)^r\beta_k(p,n).k\beta_k(m,q).$$

As $k > 0$, so, dividing by k^2 and rearranging the terms, we get

$$\frac{E_k(X)^r}{E_k(Y)^r} \geqslant (\leqslant) \frac{\beta_k(p,n)\beta_k(m,q)}{\beta_k(m,n)\beta_k(p,q)},\ r = 1,2,\ldots. \tag{1.27}$$

Applying the relation (1.11), we get the desired proof. □

COROLLARY 3.6. *For $q = p,\ n = m > 0$, the theorem (3.5) becomes*

$$\frac{E_k(X)^r}{E_k(Y)^r} \leqslant \frac{\Gamma_k(2p)\Gamma_k(2m)}{\Gamma_k^2(p+m)},\ k > 0,\ r = 1,2,\ldots.$$

The following theorem gives an inequality for the harmonic means of the beta distributed random variables and beta functions in terms of the parameter $k > 0$.

THEOREM 3.7. *Let the random variables X and Y be such that $X \sim \beta_k(p,q)$ and $Y \sim \beta_k(p,n)$. Denote the harmonic mean of the random variables X and Y, in terms of k, respectively by $H_k(X) = \frac{1}{E_k(1/X)}$ and $H_k(Y) = \frac{1}{E_k(1/Y)}$. Then, for $p,q,m,n > 0$, we have the inequality for beta k-distribution*

$$\frac{H_k(Y)}{H_k(X)} \geqslant (\leqslant) \frac{\beta_k(p,n)\beta_k(m,q)}{\beta_k(m,n)\beta_k(p,q)}, \ k > 0, \ r = 1,2,\dots$$

according as

$$(p-m)(q-n) \leqslant (\geqslant) 0.$$

Proof. For $k > 0$, choose the mappings defined by

$$f(x) = x^{\frac{p-m}{k}-r}, \quad g(x) = (1-x)^{\frac{q-n}{k}} \quad \text{and} \quad h(x) = x^{\frac{m}{k}-1}(1-x)^{\frac{n}{k}-1}.$$

Using these mappings in the inequality (1.18), we get

$$\int_0^1 x^{\frac{m}{k}-1}(1-x)^{\frac{n}{k}-1}dx. \int_0^1 x^{\frac{p}{k}-r-1}(1-x)^{\frac{q}{k}-1}dx$$
$$\geqslant (\leqslant) \int_0^1 x^{\frac{p}{k}-r-1}(1-x)^{\frac{n}{k}-1}dx. \int_0^1 x^{\frac{m}{k}-1}(1-x)^{\frac{q}{k}-1}dx. \tag{1.28}$$

From the definition of expected values of beta k-distribution, we observe

$$E_k\left(\frac{1}{X}\right)^r = \frac{1}{k\beta_k(p,q)} \int_0^1 (\frac{1}{x})^r x^{\frac{p}{k}-1}(1-x)^{\frac{q}{k}-1}dx = \frac{1}{k\beta_k(p,q)} \int_0^1 x^{\frac{p}{k}-r-1}(1-x)^{\frac{q}{k}-1}dx$$

and

$$E_k\left(\frac{1}{Y}\right)^r = \frac{1}{k\beta_k(p,n)} \int_0^1 (\frac{1}{x})^r x^{\frac{p}{k}-1}(1-x)^{\frac{n}{k}-1}dx = \frac{1}{k\beta_k(p,n)} \int_0^1 x^{\frac{p}{k}-r-1}(1-x)^{\frac{n}{k}-1}dx.$$

Using these values in the inequality (1.28) we have

$$k\beta_k(m,n)E_k\left(\frac{1}{X}\right)^r .k\beta_k(p,q) \geqslant (\leqslant) \ k\beta_k(m,q).k\beta_k(p,n)E_k(\frac{1}{Y})^r$$

which is equivalent to the theorem (3.7). $\square$

COROLLARY 3.8. *For $q = p, \ n = m > 0$, the theorem (3.7) becomes*

$$\frac{H_k(Y)}{H_k(X)} \leqslant \frac{\beta_k(p,m)\beta_k(m,p)}{\beta_k(m,m)\beta_k(p,p)}, \ k > 0$$

and by the relation (1.11), we have

$$\frac{HM_k(Y)}{HM_k(X)} \leqslant \frac{\Gamma_k(2p)\Gamma_k(2m)}{\Gamma_k^2(p+n)}, \ k > 0.$$

The following theorem gives an inequality for the variance of the beta distributed random variables and beta functions in terms of the parameter $k > 0$.

THEOREM 3.9. *Let the random variables X and Y be such that $X \sim \beta_k(p,q)$ and $Y \sim \beta_k(p,n)$. Denote the variances of the r.v.X and r.v.Y, in terms of k, respectively by $\sigma_k^2(X) = \mu'_{2,k}(X) - (\mu'_{1,k}(X))^2$ and $\sigma_k^2(Y) = \mu'_{2,k}(Y) - (\mu'_{1,k}(Y))^2$. Then, for $p,q,m,n > 0$, we have the inequality for beta k-distribution*

$$\sigma_k^2(X)\beta_k(m,n)\beta_k(p,q) - \sigma_k^2(Y)\beta_k(m,q)\beta_k(p,n)$$
$$\geqslant (\leqslant) \frac{\beta_k(m,q)\beta_k^2(p+k,n)}{\beta_k(p,n)} - \frac{\beta_k(m,n)\beta_k^2(p+k,q)}{\beta_k(p,q)},\ k>0$$

according as

$$(p-m)(q-n) \leqslant (\geqslant) 0.$$

Proof. From the inequality (1.27), taking $r = 2$, we get

$$\frac{E_k(X)^2}{E_k(Y)^2} \geqslant (\leqslant) \frac{\beta_k(p,n)\beta_k(m,q)}{\beta_k(m,n)\beta_k(p,q)}.$$

Using the value of $\mu'_{2,k}(\cdot) = E_k(\cdot)^2$ in terms of $\sigma_k^2(\cdot)$, we have

$$[\sigma_k^2(X) + (\mu'_{1,k}(X))^2]\beta_k(m,n)\beta_k(p,q) \geqslant (\leqslant) [\sigma_k^2(Y) + (\mu'_{1,k}(Y))^2]\beta_k(m,q)\beta_k(p,n) \tag{1.29}$$

which implies that

$$\sigma_k^2(X)\beta_k(m,n)\beta_k(p,q) - \sigma_k^2(Y)\beta_k(m,q)\beta_k(p,n)$$
$$\geqslant (\leqslant) \left(\mu'_{1,k}(Y)\right)^2 \beta_k(m,q)\beta_k(p,n) - \left(\mu'_{1,k}(X)\right)^2 \beta_k(p,q)\beta_k(m,n).$$

From the relation (1.17), taking $r = 1$, we find the value of $\mu'_{1,k} = \frac{\beta_k(m+k,n)}{\beta_k(m,n)}$ and the above expression (for the parameters p,q,n) becomes

$$\geqslant (\leqslant) \left(\frac{\beta_k(p+k,n)}{\beta_k(p,n)}\right)^2 \beta_k(m,q)\beta_k(p,n) - \left(\frac{\beta_k(p+k,q)}{\beta_k(p,q)}\right)^2 \beta_k(p,q)\beta_k(m,n)$$

and use of the relation (1.11) will provide the required result. □

THEOREM 3.10. *Denote the coefficients of variation of the r.v.X and r.v.Y, in terms of k, respectively by $CV_k(X) =$ and $CV_k(Y)$, where $CV_k(\cdot) = \frac{\sqrt{\sigma_k^2(\cdot)}}{\mu'_{1,k}(\cdot)}$. Then, for $p,q,m,n > 0$, we have the inequality*

$$\frac{CV_k^2(X)+1}{CV_k^2(Y)+1} \geqslant (\leqslant) \frac{(p+q)\Gamma_k(m+n)\Gamma_k(p+q+k)}{(p+n)\Gamma_k(m+q)\Gamma_k(p+n+k)},\ k>0$$

according as

$$(p-m)(q-n) \leqslant (\geqslant) 0.$$

Proof. From the inequality (1.29), we have

$$(\mu'_{1,k}(X))^2\left[\frac{\sigma_k^2(X)}{(\mu'_{1,k}(X))^2}+1\right]\beta_k(m,n)\beta_k(p,q)$$
$$\geqslant (\leqslant)\,(\mu'_{1,k}(Y))^2\left[\frac{\sigma_k^2(Y)+1}{(\mu'_{1,k}(Y))^2}\right]\beta_k(m,q)\beta_k(p,n)$$

which implies that

$$\left[\frac{CV_k^2(X)+1}{CV_k^2(Y)+1}\right]\geqslant (\leqslant)\,\frac{(\mu'_{1,k}(Y))^2\beta_k(m,q)\beta_k(p,n)}{(\mu'_{1,k}(X))^2\beta_k(m,n)\beta_k(p,q)}. \tag{1.30}$$

From the proposition (2.2), we see that the mean of beta k-distribution with parameters p, q is $\mu'_{1,k}(X)=E_k(X)=\frac{p}{p+q}$ and with parameters p, n is $\mu'_{1,k}(Y)=\frac{p}{p+n}$. Thus, inequality (1.30) along with the relation (1.11) becomes

$$\left[\frac{CV_k^2(X)+1}{CV_k^2(Y)+1}\right]\geqslant (\leqslant)\,\frac{(p+q)^2\Gamma_k(p+q)\Gamma_k(m+n)}{(p+n)^2\Gamma_k(p+n)\Gamma_k(m+q)}$$

and use of the relation (1.8) gives the desired proof. □

COROLLARY 3.11. *Using the values of mean and variance of beta distribution in terms of the parameter $k>0$, from the inequality (1.29), we have the inequality*

$$\Gamma_k(p+n+k)\Gamma_k(m+q)\geqslant (\leqslant)\Gamma_k(p+q+k)\Gamma_k(m+n) \tag{1.31}$$

and

$$\frac{(p+n)[q+p(p+q+k)]}{(p+q)[n+p(p+n+k)]}\geqslant (\leqslant)\frac{\Gamma_k(p+q+2k)\Gamma_k(m+n)}{\Gamma_k(p+n+2k)\Gamma_k(m+q)},\ k>0 \tag{1.32}$$

according as

$$(p-m)(q-n)\leqslant (\geqslant)0.$$

Proof. As proved in the proposition (2.2), the value of mean and variance with parameters p, q is $\mu'_{1,k}=\frac{p}{p+q}$ and $\sigma_k^2=\frac{pqk}{(p+q)^2(p+q+k)}$. Thus, using these values along with some algebraic calculations, inequality (1.29) gives

$$[qk+p(p+q+k)](p+n)^2(p+n+k)\Gamma_k(m+q)\Gamma_k(p+n)$$
$$\geqslant (\leqslant)[nk+p(p+n+k)](p+q)^2(p+q+k)\Gamma_k(m+n)\Gamma_k(p+q).$$

By successive use of the relation (1.8) on both sides of the above inequality, we get the required proof. □

4. Some results via Holder's integral inequality

In this section, we prove some results involving the β_k random variable via Hölder's integral inequality. The mapping β_k, for two parameters, is logarithmically convex on the interval $\in [0,\infty)^2$ proved in [24] which is k-analog result [22]. Now, we have the following theorem.

THEOREM 4.1. *Let $(p,q),(m,n) \in [0,\infty)^2$ and $a,b \geqslant 0$ with $a+b=1$. For $k>0$, define the k-distributed random variables X and Y such that $X \sim \beta_k(ap+bm, aq+bn)$ and $Y \sim \beta_k(p,q)$. Then, we have the inequality for beta k-distribution*

$$\frac{E_k(X)^{ar}}{[E_k(Y)^r]^a} \leqslant \frac{[\beta_k(p,q)]^a[\beta_k(m,n)]^b}{\beta_k(ap+bm,aq+bn)},\ k>0,\ r=1,2,\dots. \tag{1.34}$$

Proof. For $k>0$, choose the mappings defined by

$$f(t)=[t^{\frac{p}{k}+r-1}(1-t)^{\frac{q}{k}-1}]^a, \quad g(t)=[t^{\frac{m}{k}-1}(1-t)^{\frac{n}{k}-1}]^b \quad \text{and} \quad h(t)=1,$$

for $p=\frac{1}{a}$, $q=\frac{1}{b}$, $(\frac{1}{p}+\frac{1}{q}=1$ and $p \geqslant 1)$. Using these mappings in the Holder's integral inequality

$$\int_I f(t)g(t)h(t) \leqslant \Big(\int_I \{f(t)\}^{\frac{1}{a}}h(t)dt\Big)^a \Big(\int_I \{g(t)\}^{\frac{1}{b}}h(t)dt\Big)^b, \tag{1.35}$$

we have

$$\begin{aligned}&\int_0^1 \Big(t^{\frac{ap}{k}+\frac{bm}{k}+ar-1}(1-t)^{\frac{aq}{k}+\frac{bn}{k}-1}\Big)dt \\ &\qquad \leqslant \Big(\int_0^1 t^{\frac{p}{k}+r-1}(1-t)^{\frac{q}{k}-1}dt\Big)^a \Big(\int_0^1 t^{\frac{m}{k}-1}(1-t)^{\frac{n}{k}-1}dt\Big)^b.\end{aligned} \tag{1.36}$$

From (1.17), we observe that

$$E_k(X)^{ar}\beta_k(ap+bm,aq+bn)=\frac{1}{k}\int_0^1 t^{\frac{ap+bm}{k}+ar-1}(1-t)^{\frac{aq+bn}{k}-1}dt$$

and the inequality (1.36) gives

$$kE_k(X)^{ar}\beta_k(ap+bm,aq+bn) \leqslant [kE_k(Y)^r\beta_k(p,q)]^a[k\beta_k(m,n)]^b,\ r=1,2,\dots$$

which is equivalent to the required result. □

COROLLARY 4.2. *Setting $r=1$ and $E_k(Y)=\frac{p}{p+q}$ in the theorem (4.1), we have*

$$E_k(X)^a \leqslant \Big(\frac{p}{p+q}\Big)^a \frac{[\beta_k(p,q)]^a[\beta_k(m,n)]^b}{\beta_k(ap+bm,aq+bn)},\ k>0.$$

Using the relation (1.11), we observe

$$E_k(X)^a \leqslant \Big(\frac{p\Gamma_k(p)\Gamma_k(q)}{(p+q)\Gamma_k(p+q)}\Big)^a \frac{[\beta_k(m,n)]^b}{\beta_k(ap+bm,aq+bn)}$$

and use of the property (1.8) and (1.11) gives

$$E_k(X)^a \leqslant \Big(\frac{\Gamma_k(p+k)\Gamma_k(q)}{\Gamma_k(p+q+k)}\Big)^a \frac{[\beta_k(m,n)]^b}{\beta_k(ap+bm,aq+bn)} = \frac{[\beta_k(p+q,q)]^a[\beta_k(m,n)]^b}{\beta_k(ap+bm,aq+bn)}.$$

THEOREM 4.3. *Let $(p,q),(m,n) \in [0,\infty)^2$ and $a,b \geqslant 0$ with $a+b=1$. Denote the variance of the k-beta distributed random variables $X \sim \beta_k(ap+bm,aq+bn)$ and $Y \sim \beta_k(p,q)$ by $\sigma_k^2(X)$ and $\sigma_k^2(Y)$ respectively. Then, we have the inequality for beta k-distribution*

$$[\sigma_k^2(X^a)+E_k^2(X^a)]\beta_k(ap+bm,aq+bn) \leqslant \left[\sigma_k^2(Y)+\left(\frac{p}{p+q}\right)^2\right][\beta_k(p,q)]^a[\beta_k(m,n)]^b, k>0. \tag{1.37}$$

Proof. Taking $r=2$ in the inequality (1.34), we get

$$[E_k(X^a)^2]\beta_k(ap+bm,aq+bn) \leqslant [E_k(Y)^2]^a[\beta_k(p,q)]^a][\beta_k(m,n)]^b], k>0.$$

From the relation (1.16), using the value of $E_k(X)$ in terms of variance and also $E_k(X) = \frac{p}{p+q}$ provide the desired proof. □

COROLLARY 4.4. *As proved in the Proposition (2.2), for the random variable $X \sim \beta_k(m,n)$, $E_k(X) = \frac{m}{m+n}$ and variance is, $\sigma_k^2 = \frac{mnk}{(m+n)^2(m+n+k)}$. Now, inequality (1.37) implies that*

$$\sigma_k^2(X^a)+E_k^2(X^a) \leqslant \frac{p(p+k)}{(p+q)(p+q+k)}\frac{[\beta_k(p,q)]^a][\beta_k(m,n)]^b}{\beta_k(ap+bm,aq+bn)}.$$

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Shahid Mubeen
Department of Mathematics
University of Sargodha
Sargodha, Pakistan
e-mail: smjhanda@gmail.com

Sana Iqbal
Department of Mathematics
University of Sargodha
Sargodha, Pakistan
e-mail: sna1310@gmail.com

Abdur Rehman
Department of Mathematics
University of Sargodha
Sargodha, Pakistan
e-mail: abdurrehman2007@hotmail.com