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THE UPPER BOUNDS FOR MULTIPLICATIVE SUM ZAGREB INDEX OF SOME GRAPH OPERATIONS

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Abstract. Let G be a simple graph with vertex set V(G) and edge set E(G). In [7], Eliasi et al. introduced the multiplicative sum Zagreb index of a graph G which is denoted by $\Pi_1^*(G)$ and is defined by

$$\Pi_1^*(G) = \prod_{uv \in V(G)} (d_G(u) + d_G(v)).$$

In this paper, we present some upper bounds for the multiplicative sum Zagreb indices of the join, rooted product, corona product, tensor product, Cartesian product, strong product, hierarchical product, lexicographic product of graphs.

1. Introduction

Molecular descriptors have found a wide application in QSPR/QSAR studies [13]. Among them, topological indices have a prominent place. The Zagreb indices are among the oldest degree-based topological invariants, were introduced by Gutman et al [9]. For details on theory and applications see in [12, 19].

Let G be a simple graph with the edge set $E(G)$ and vertex set $V(G)$. Also let n and m, respectively, be the number of vertices and edges of G. The first and second Zagreb indices of G are denoted by $M_1(G)$ and $M_2(G)$, respectively and defined as:

$$M_1(G) = \sum_{u \in V(G)} d_G^2(u)$$

and

$$M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v),$$

where $d_G(u)$ denotes the degree of the vertex u of G. The first Zagreb index can also be expressed as the following;

$$M_1(G) = \sum_{uv \in E(G)} (d_G(u) + d_G(v)).$$

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The multiplicative Zagreb indices were introduced by Todeschini et al. in 2010 [14]. The first and second multiplicative Zagreb indices of G are denoted by $\Pi_1(G)$ and $\Pi_2(G)$, respectively, and defined as;

$$\Pi_1(G) = \prod_{u \in V(G)} d_G^2(u)$$

and

$$\Pi_2(G) = \prod_{uv \in E(G)} d_G(u) d_G(v).$$

In 2012, Eliasi et al. [7] introduced the multiplicative sum Zagreb index of a graph G. The multiplicative sum Zagreb index is defined by

$$\Pi_1^*(G) = \prod_{uv \in V(G)} \left(d_G(u) + d_G(v) \right).$$

We refer the reader to [8, 11, 16, 17] for mathematical properties of the multiplicative Zagreb indices. Some more properties and applications of graph products can be seen in [1, 2, 10]. For other undefined notations and terminology from graph theory, the readers are referred to [5, 15].

Many graphs of general and in particular of chemical interest arise from simple graphs via various graph operations sometimes known as graph products. Hence, it is important to understand how certain invariants of such composite graphs related to the corresponding invariants of their components.

Yeh et al. [18] examined Wiener index of composite graphs. In [6], some upper bounds for the multiplicative Zagreb index of various graph operations were obtained. Azari [3] presented sharp lower bounds on the Narumi–Katayama index of several graph operations. Also, Azari et al. [4] gave some lower bounds for the multiplicative sum Zagreb index of graph operations.

In this paper, we obtain upper bounds for the multiplicative sum Zagreb index of some graph operations.

2. Main results

In this section, we give some upper bounds for the multiplicative sum Zagreb index of various graph operations such as join, rooted product, corona product, tensor product, etc.

We first recall the arithmetic-geometric mean inequality which will be used in this paper.

LEMMA 1. *Let $x_1, x_2, \ldots, x_n$ be nonnegative numbers. Then*

$$\frac{x_1 + x_2 + \ldots + x_n}{n} \geqslant \sqrt[n]{x_1 x_2 \ldots x_n},$$

with equality if and only if $x_1 = x_2 = \ldots = x_n$.

2.1. Join

Let G_1 and G_2 be vertex-disjoint graphs. Then the join, G_1+G_2, of G_1 and G_2 is the supergraph of $G_1 \cup G_2$ in which each vertex of G_1 is adjacent to every vertex of G_2. The join of two graphs is also known as their sum. The degree of a vertex u of G_1+G_2 is defined by:

$$d_{G_1+G_2}(u) = \begin{cases} d_{G_1}(u)+n_2, \text{ if } u \in V(G_1) \\ d_{G_2}(u)+n_1, \text{ if } u \in V(G_2) \end{cases}.$$

THEOREM 1. *Let G_1 and G_2 be vertex-disjoint graphs. The multiplicative sum Zagreb index of G_1+G_2 satisfies the following inequality:*

$$\Pi_1^*(G) \leqslant \left[\frac{M_1(G_1)+2m_1n_2}{m_1}\right]^{m_1} \left[\frac{M_1(G_2)+2m_2n_1}{m_2}\right]^{m_2} \times \left[\frac{2m_1n_2+2m_2n_1+n_1n_2(n_1+n_2)}{n_1n_2}\right]^{n_1n_2}, \quad (1)$$

with equality if and only if G_1 and G_2 are regular graphs.

Proof. Let $G = G_1+G_2$. By definition of the multiplicative sum Zagreb index, we have:

$$\begin{aligned}\Pi_1^*(G) &= \prod_{uv\in E(G)} [d_G(u)+d_G(v)] \\ &= \prod_{uv\in E(G_1)} [d_{G_1}(u)+d_{G_1}(v)+2n_2] \prod_{uv\in E(G_2)} [d_{G_2}(u)+d_{G_2}(v)+2n_1] \\ &\quad \times \prod_{\substack{u\in V(G_1)\\ v\in V(G_2)}} [d_{G_1}(u)+d_{G_2}(v)+n_1+n_2].\end{aligned}$$

Using Lemma 1, we get

$$\begin{aligned}\Pi_1^*(G) &\leqslant \left[\frac{\sum_{uv\in E(G_1)}[d_{G_1}(u)+d_{G_1}(v)+2n_2]}{m_1}\right]^{m_1} \left[\frac{\sum_{uv\in E(G_2)}[d_{G_2}(u)+d_{G_2}(v)+2n_1]}{m_2}\right]^{m_2} \\ &\quad \times \left[\frac{\sum_{\substack{u\in V(G_1)\\ v\in V(G_2)}}[d_{G_1}(u)+d_{G_2}(v)+n_1+n_2]}{n_1n_2}\right]^{n_1n_2} \\ &= \left[\frac{M_1(G_1)+2m_1n_2}{m_1}\right]^{m_1} \left[\frac{M_1(G_2)+2m_2n_1}{m_2}\right]^{m_2} \\ &\quad \times \left[\frac{2m_1n_2+2m_2n_1+n_1n_2(n_1+n_2)}{n_1n_2}\right]^{n_1n_2}.\end{aligned}$$

Now, suppose that the equality holds in (1). Then, for every $uv \in E(G_1)$ and $xy \in E(G_1)$,

$$d_{G_1}(u)+d_{G_1}(v)+2n_2 = d_{G_1}(x)+d_{G_1}(y)+2n_2$$

and for every $uv \in E(G_2)$ and $xy \in E(G_2)$,

$$d_{G_2}(u)+d_{G_2}(v)+2n_1 = d_{G_2}(x)+d_{G_2}(y)+2n_1.$$

Thus one can easily see that the equality holds in (1) if and only if both G_1 and G_2 are regular graphs. □

Let $G = G_1+G_2+\ldots+G_k$. Thus $\overline{n_i} = n-n_i \quad (1 \leqslant i \leqslant k)$. Also we have

$$d_G(u) = \{d_{G_i}(u)+\overline{n_i}, u \in V(G_i)\}.$$

From Theorem 1, we have the following result.

COROLLARY 1. *Let $G_1, G_2, \ldots, G_k$ be vertex-disjoint graphs. If $G = G_1+G_2+\ldots+G_k$, then*

$$\Pi_1^*(G) \leqslant \prod_{i=1}^{k}\left[\frac{M_1(G_i)+2m_i\overline{n_i}}{m_i}\right]^{m_i}\prod_{1\leqslant i\leqslant j\leqslant k}\left[\frac{2m_in_j+2m_jn_i+n_in_j(\overline{n_i}+\overline{n_j})}{n_in_j}\right]^{n_in_j}.$$

EXAMPLE 1. Consider cycle graphs C_p and C_q. We thus have

$$\Pi_1^*(C_p+C_q) = 2^{p+q}(p+2)^q(q+2)^p(p+q+4)^{pq}.$$

2.2. Rooted product

Let H be a labeled graph on k vertices with the vertex set $V(H) = \{1,2,\ldots,k\}$ and let G be a sequence of k rooted graphs $G_1, G_2, \ldots, G_k$. The rooted product of H by G, denoted by $H(G) = H(G_1, G_2, \ldots, G_k)$ is the graph obtained by identifying the root vertex of G_i with the i-th vertex of H. We denote the root vertex of G_i, which is assumed to be non-isolated, by w_i and the degree of w_i in G_i by $d(w_i)$, $1 \leqslant i \leqslant k$.

THEOREM 2. *Let $H(G)$ be rooted product of H by G. Then*

$$\Pi_1^*[H(G)] \leqslant \left[\frac{M_1(H)+\sum\limits_{i\in V(H)} d_H(i).d(w_i)}{m_H}\right]^{m_H} \times \sum_{i=1}^{k}\left[\left[\frac{M_1(G_i)-d^2(w_i)-d_H(i).d(w_i)-\sum\limits_{u\sim w_i} d_{G_i}(u))}{m_{G_i}-d(w_i)}\right]^{m_{G_i}-d(w_i)} \times \left[\frac{d^2(w_i)+d_H(i).d(w_i)+\sum\limits_{x\sim w_i} d_{G_i}(x)}{d(w_i)}\right]^{d(w_i)}\right]$$

where $i \sim j$ denotes the vertex i is adjacent to the vertex j.

Proof. By definition of the multiplicative sum Zagreb index, we have:

$$\Pi_1^*[H(G)] = \prod_{ij\in E(H)} [d_H(i)+d_H(j)+d(w_i)+d(w_j)] \times \prod_{i=1}^{k} \left[\prod_{\substack{ij\in E(G_i)\\ i,j\neq w_i}} [d_{G_i}(i)+d_{G_i}(j)] \prod_{w_i j\in E(G_i)} [d_{G_i}(w_i)+d_{G_i}(j)+d_H(i)] \right].$$

From Lemma 1, we have

$$\begin{aligned}\Pi_1^*[H(G)] &\leqslant \left[\frac{\sum_{ij\in E(H)}[d_H(i)+d_H(j)+d(w_i)+d(w_j)]}{m_H}\right]^{m_H} \\ &\times \sum_{i=1}^{k} \left[\left[\frac{\sum_{\substack{ij\in E(G_i)\\ i,j\neq w_i}}[d_{G_i}(i)+d_{G_i}(j)]}{m_{G_i}-d(w_i)}\right]^{m_{G_i}-d(w_i)} \times \left[\frac{\sum_{w_i j\in E(G_i)}[d_{G_i}(w_i)+d_{G_i}(j)+d_H(i)]}{d(w_i)}\right]^{d(w_i)} \right] \\ &= \left[\frac{M_1(H)+\sum_{i\in V(H)} d_H(i).d(w_i)}{m_H}\right]^{m_H} \\ &\times \sum_{i=1}^{k} \left[\left[\frac{M_1(G_i)-d^2(w_i)-d_H(i).d(w_i)-\sum_{u\sim w_i} d_{G_i}(u)}{m_{G_i}-d(w_i)}\right]^{m_{G_i}-d(w_i)} \times \left[\frac{d^2(w_i)+d_H(i).d(w_i)+\sum_{x\sim w_i} d_{G_i}(x)}{d(w_i)}\right]^{d(w_i)} \right]. \quad \square\end{aligned}$$

2.3. Corona product

The corona product of graphs G_1 and G_2, by written $G_1 \circ G_2$, is the graph obtained by taking one copy of G_1 and n_1 copies of G_2 by joining the i-th vertex of G_1 to every vertex in i-th copy of G_2 for $1 \leqslant i \leqslant n_1$. The degree of a vertex u of $G_1 \circ G_2$ is defined by:

$$d_{G_1\circ G_2}(u) = \begin{cases} d_{G_1}(u)+n_2, & u\in V(G_1) \\ d_{G_2}(u)+1, & u\in V(G_2) \end{cases}$$

where $|V(G_1\circ G_2)| = n_1(1+n_2)$ and $|E(G_1\circ G_2)| = m_1+n_1m_2+n_2n_1$.

THEOREM 3. *Let G_1 and G_2 are two graphs. If $G = G_1\circ G_2$, then*

$$\Pi_1^*(G) \leqslant \left[\frac{M_1(G_1)+2m_1n_2}{m_1}\right]^{m_1} \left[\frac{M_1(G_2)+2m_2}{m_2}\right]^{n_1m_2} \times \left[\frac{2m_1n_2+2m_2n_1+n_1n_2(n_2+1)}{n_1n_2}\right]^{n_1n_2} \tag{2}$$

with equality if and only if G_1 and G_2 are regular graphs.

Proof. By the definition of Π_1^* index, we have

$$\Pi_1^*(G) = \prod_{uv \in E(G_1)} [d_{G_1}(u) + d_{G_1}(v) + 2n_2] \left[\prod_{uv \in E(G_2)} [d_{G_2}(u) + d_{G_2}(v) + 2] \right]^{n_1} \times \prod_{\substack{u \in V(G_1) \\ v \in V(G_2)}} [d_{G_1}(u) + d_{G_2}(v) + n_2 + 1].$$

By Lemma 1, we have

$$\Pi_1^*(G) \leqslant \left[\frac{\sum_{uv \in E(G_1)} [d_{G_1}(u) + d_{G_1}(v) + 2n_2]}{m_1} \right]^{m_1} \left[\frac{\sum_{uv \in E(G_2)} [d_{G_2}(u) + d_{G_2}(v) + 2]}{m_2} \right]^{n_1 m_2} \times \left[\frac{\sum_{u \in V(G_1)} \sum_{v \in V(G_2)} [d_{G_1}(u) + d_{G_2}(v) + n_2 + 1]}{n_1 n_2} \right]^{n_1 n_2}$$

and hence

$$\Pi_1^*(G) \leqslant \left[\frac{M_1(G_1) + 2m_1 n_2}{m_1} \right]^{m_1} \left[\frac{M_1(G_2) + 2m_2}{m_2} \right]^{n_1 m_2} \times \left[\frac{2m_1 n_2 + 2m_2 n_1 + n_1 n_2 (n_2 + 1)}{n_1 n_2} \right]^{n_1 n_2}.$$

Now suppose that equality holds in (2). Then all the inequalities in the above argument must be equalities. Thus we have

$$d_{G_1}(u_1) + d_{G_1}(v_1) + 2n_2 = d_{G_1}(u_2) + d_{G_1}(v_2) + 2n_2$$

for $u_1 v_1, u_2 v_2 \in E(G_1)$ and

$$d_{G_2}(u_1) + d_{G_2}(v_1) + 2 = d_{G_2}(u_2) + d_{G_2}(v_2) + 2$$

for $u_1 v_1, u_2 v_2 \in E(G_2)$ and

$$d_{G_1}(u_1) + d_{G_2}(v_1) + n_2 + 1 = d_{G_1}(u_2) + d_{G_2}(v_2) + n_2 + 1$$

for $u_1, u_2 \in V(G_1)$ and $v_1, v_2 \in V(G_2)$.

Hence we conclude that G_1 and G_2 are regular graphs. □

2.4. Tensor product

The tensor product $G_1 \otimes G_2$ of two simple graphs G_1 and G_2 is the graph with $V(G_1 \otimes G_2) = V_1 \times V_2$ and (u_1, u_2) and (v_1, v_2) are adjacent in $G_1 \otimes G_2$ if and only if u_1 is adjacent to v_1 in G_1 and u_2 is adjacent to v_2 in G_2. Also we know that $d_{G_1 \otimes G_2}(u) = d_{G_1}(u) d_{G_2}(u)$ and $m(G_1 \otimes G_2) = 2m_1 m_2$.

THEOREM 4. *Let $G = G_1 \otimes G_2$ be the tensor product of two graphs G_1 and G_2. Then*

$$\Pi_1^*(G) \leqslant \left[\frac{M_1(G_1)M_1(G_2)}{2m_1m_2}\right]^{2m_1m_2} \tag{3}$$

with equality if and only if either G_1 and G_2 are regular graphs or G_1 is star graph and G_2 is regular graph.

Proof. By the definition of the multiplicative sum Zagreb index, we have

$$\begin{aligned}\Pi_1^*(G) &= \prod_{(u_1,u_2)(v_1,v_2)\in E(G)} [d_G((u_1,u_2)) + d_G((v_1,v_2))] \\ &= \prod_{(u_1,u_2)(v_1,v_2)\in E(G)} [d_{G_1}(u_1)d_{G_2}(u_2) + d_{G_1}(v_1)d_{G_2}(v_2)].\end{aligned}$$

On the other hand, by Lemma 1

$$\begin{aligned}\Pi_1^*(G) &\leqslant \left[\frac{\sum_{(u_1,u_2)(v_1,v_2)\in E(G)}[d_{G_1}(u_1)d_{G_2}(u_2) + d_{G_1}(v_1)d_{G_2}(v_2)]}{2m_1m_2}\right]^{2m_1m_2} \\ &= \left[\frac{\sum_{u_1v_1\in E(G_1)}\sum_{u_2v_2\in E(G_2)}[d_{G_1}(u_1)d_{G_2}(u_2) + d_{G_1}(v_1)d_{G_2}(v_2)]}{2m_1m_2}\right]^{2m_1m_2} \\ &= \left[\frac{\sum_{u_1v_1\in E(G_1)} M_1(G_2)[d_{G_1}(u_1) + d_{G_1}(v_1)]}{2m_1m_2}\right]^{2m_1m_2} \\ &= \left[\frac{M_1(G_1)M_1(G_2)}{2m_1m_2}\right]^{2m_1m_2}.\end{aligned}$$

Now suppose that equality holds in (3). Then all the inequalities in the above argument must be equalities. Thus we have

$$d_{G_1}(u_1)d_{G_2}(u_2) + d_{G_1}(v_1)d_{G_2}(v_2) = d_{G_1}(a_1)d_{G_2}(a_2) + d_{G_1}(b_1)d_{G_2}(b_2)$$

for any $u_1v_1, a_1b_1 \in E(G_1)$ and for any $u_2v_2, a_2b_2 \in E(G_2)$. Hence the equality holds in (3) if and only if either G_1 and G_2 are regular graphs or G_1 is star graph and G_2 is regular graph. □

EXAMPLE 2. Consider cycle graphs C_p and C_q and the complete graph K_2. We thus have

$$\Pi_1^*(C_p \otimes C_q) = 2^{8pq} \text{ and } \Pi_1^*(K_2 \otimes C_p) = 2^{4p}.$$

2.5. Cartesian product

The Cartesian product of G_1 and G_2; denoted by $G_1\times G_2$; is the graph defined as follows. The vertex set of $G_1\times G_2$ is $V(G_1)\times V(G_2)$. The vertices (u_1,v_1) and (u_2,v_2) are adjacent if either $u_1 = u_2$ and v_1 is adjacent to v_2 in G_2; or $v_1 = v_2$ and u_1 is adjacent to u_2 in G_1. Also we know that $m(G_1\times G_2) = n_1m_2 + m_1n_2$ and $d_{G_1\times G_2}(u_1,u_2) = d_{G_1}(u_1) + d_{G_2}(u_2)$, respectively.

THEOREM 5. *Let $G = G_1 \times G_2$ be the Cartesian product of two graphs G_1 and G_2. Then*

$$\Pi_1^*(G) \leqslant \frac{[4m_1m_2 + n_1M_1(G_2)]^{n_1m_2} . [4m_1m_2 + n_2M_1(G_1)]^{n_2m_1}}{n_1^{m_2} n_2^{m_1} m_2^{n_1m_2} m_1^{n_2m_1}} \tag{4}$$

with equality if and only if G_1 and G_2 are regular graphs.

Proof. Let $G = G_1 \times G_2$. Then

$$\begin{aligned}
\Pi_1^*(G) &= \prod_{(u_1,u_2)(v_1,v_2)\in E(G)} [d_G((u_1,u_2)) + d_G((v_1,v_2))] \\
&= \prod_{u_1\in V(G_1)} \prod_{u_2v_2\in E(G_2)} [2d_{G_1}(u_1) + d_{G_2}(u_2) + d_{G_2}(v_2)] \\
&\quad \times \prod_{u_2\in V(G_2)} \prod_{u_1v_1\in E(G_1)} [2d_{G_2}(u_2) + d_{G_1}(u_1) + d_{G_2}(v_1)].
\end{aligned}$$

Hence from the arithmetic geometric inequality, we have

$$\begin{aligned}
\Pi_1^*(G) &\leqslant \prod_{u_1\in V(G_1)} \left[\frac{\sum_{u_2v_2\in E(G_2)}[2d_{G_1}(u_1) + d_{G_2}(u_2) + d_{G_2}(v_2)]}{m_2}\right]^{m_2} \\
&\quad \times \prod_{u_2\in V(G_2)} \left[\frac{\sum_{u_1v_1\in E(G_1)}[2d_{G_2}(u_2) + d_{G_1}(u_1) + d_{G_2}(v_1)]}{m_1}\right]^{m_1} \\
&= \frac{1}{m_2^{n_1m_2}} \prod_{u_1\in V(G_1)} [2m_2d_{G_1}(u_1) + M_1(G_2)]^{m_2} \frac{1}{m_1^{n_2m_1}} \\
&\quad \times \prod_{u_2\in V(G_2)} [2m_1d_{G_2}(u_2) + M_1(G_1)]^{m_1}.
\end{aligned}$$

By applying Lemma 1, we get,

$$\begin{aligned}
\Pi_1^*(G) &\leqslant \frac{1}{m_2^{n_1m_2}} \frac{1}{m_1^{n_2m_1}} \left[\frac{\sum_{u_1\in V(G_1)}[2m_2d_{G_1}(u_1) + M_1(G_2)]}{n_1}\right]^{n_1m_2} \\
&\quad \times \left[\frac{\sum_{u_2\in V(G_2)}[2m_1d_{G_2}(u_2) + M_1(G_1)]}{n_2}\right]^{m_1n_2} \\
&= \frac{[4m_1m_2 + n_1M_1(G_2)]^{n_1m_2} [4m_1m_2 + n_2M_1(G_1)]^{n_2m_1}}{n_1^{m_2} n_2^{m_1} m_2^{n_1m_2} m_1^{n_2m_1}}.
\end{aligned}$$

Now suppose that equality holds in (4). Then all the inequalities in the above argument must be equalities. Thus we have

$$2d_{G_1}(u_1) + d_{G_2}(u_2) + d_{G_2}(v_2) = 2d_{G_1}(u_1) + d_{G_2}(a_2) + d_{G_2}(b_2)$$

for any $u_2v_2, a_2b_2 \in E(G_2)$ and

$$2d_{G_2}(u_2) + d_{G_1}(u_1) + d_{G_1}(v_1) = 2d_{G_2}(u_2) + d_{G_1}(a_1) + d_{G_1}(b_1)$$

for any $u_1v_1, a_1b_1 \in E(G_1)$ and

$$2m_2d_{G_1}(u_1)+M_1(G_2) = 2m_2d_{G_1}(v_1)+M_1(G_2)$$

for any $u_1, v_1 \in V(G_1)$ and

$$2m_1d_{G_2}(u_2)+M_1(G_1) = 2m_1d_{G_2}(v_2)+M_1(G_1)$$

for any $u_2, v_2 \in V(G_2)$. Hence the equality holds in (4) if and only if both G_1 and G_2 are regular graphs. □

2.6. Strong product

The strong product of G_1 and G_2; denoted by $G_1 \boxtimes G_2$; is the graph defined as follows. The vertex set of $G_1 \boxtimes G_2$ is $V(G_1) \times V(G_2)$. The vertices (u_1, v_1) and (u_2, v_2) are adjacent if either $u_1 = u_2$ and v_1 is adjacent to v_2 in G_2; or $v_1 = v_2$ and u_1 is adjacent to u_2 in G_1; or u_1 is adjacent to u_2 in G_1 and v_1 is adjacent to v_2 in G_2. Also we know that $m(G_1 \boxtimes G_2) = n_1m_2 + m_1n_2 + 2m_1m_2$ and $d_{G_1 \boxtimes G_2}(u_1, u_2) = d_{G_1}(u_1) + d_{G_2}(u_2) + d_{G_1}(u_1)d_{G_2}(u_2)$.

THEOREM 6. *Let $G = G_1 \boxtimes G_2$ be the strong product of two graphs G_1 and G_2. Then*

$$\Pi_1^*(G) \leqslant \frac{[4m_1m_2 + (2m_1+n_1)M_1(G_2)]^{n_1m_2}\,[4m_1m_2 + (2m_2+n_2)M_1(G_1)]^{n_2m_1}}{4^{m_1m_2}m_1^{n_2m_1+2m_1m_2}m_2^{n_1m_2+2m_1m_2}n_1^{n_1m_2}n_2^{n_2m_1}} \times [2m_2M_1(G_1) + 2m_1M_1(G_2) + M_1(G_1)M_1(G_2)]^{2m_1m_2} \quad (5)$$

with equality if and only if G_1 and G_2 are regular graphs.

Proof. From definition of strong product and multiplicative sum Zagreb index, we have

$$\begin{aligned}
\Pi_1^*(G) &= \prod_{(u_1,u_2)(v_1,v_2)\in E(G)} [d_G((u_1,u_2)) + d_G((v_1,v_2))] \\
&= \prod_{(u_1,u_2)(v_1,v_2)\in E(G)} [d_{G_1}(u_1) + d_{G_2}(u_2) + d_{G_1}(u_1)d_{G_2}(u_2) + d_{G_1}(v_1) + d_{G_2}(v_2) \\
&\qquad + d_{G_1}(v_1)d_{G_2}(v_2)] \\
&= \prod_{u_1\in V(G_1)} \prod_{u_2v_2\in E(G_2)} [2d_{G_1}(u_1) + d_{G_1}(u_1)(d_{G_2}(u_2) + d_{G_2}(v_2)) + d_{G_2}(u_2) \\
&\qquad + d_{G_2}(v_2)] \\
&\quad \times \prod_{u_2\in V(G_2)} \prod_{u_1v_1\in E(G_1)} [2d_{G_2}(u_2) + d_{G_2}(u_2)(d_{G_1}(u_1) + d_{G_1}(v_1)) + d_{G_1}(u_1) \\
&\qquad + d_{G_1}(v_1)] \\
&\quad \times \prod_{u_1v_1\in E(G_1)} \prod_{u_2v_2\in E(G_2)} [d_{G_1}(u_1) + d_{G_1}(v_1) + d_{G_2}(u_2) + d_{G_2}(v_2) \\
&\qquad + d_{G_1}(u_1)d_{G_2}(u_2) + d_{G_1}(v_1)d_{G_2}(v_2)].
\end{aligned}$$

By using Lemma 1, we have

$$\Pi_1^*(G) \leqslant \prod_{u_1\in V(G_1)} \left[\frac{\sum_{u_2v_2\in E(G_2)}[2d_{G_1}(u_1)+d_{G_1}(u_1)(d_{G_2}(u_2)+d_{G_2}(v_2)) + d_{G_2}(u_2)+d_{G_2}(v_2)]}{m_2}\right]^{m_2}$$

$$\times \prod_{u_2\in V(G_2)} \left[\frac{\sum_{u_1v_1\in E(G_1)}[2d_{G_2}(u_2)+d_{G_2}(u_2)(d_{G_1}(u_1)+d_{G_1}(v_1)) + d_{G_1}(u_1)+d_{G_1}(v_1)]}{m_1}\right]^{m_1}$$

$$\times \prod_{u_1v_1\in E(G_1)} \left[\frac{\sum_{u_2v_2\in E(G_2)}[d_{G_1}(u_1)+d_{G_1}(v_1)+d_{G_2}(u_2)+d_{G_2}(v_2) + d_{G_1}(u_1)d_{G_2}(u_2)+d_{G_1}(v_1)d_{G_2}(v_2)]}{2m_2}\right]^{2m_2}$$

$$= \prod_{u_1\in V(G_1)} \left[\frac{2m_2d_{G_1}(u_1)+M_1(G_2)d_{G_1}(u_1)+M_1(G_2)}{m_2}\right]^{m_2}$$

$$\times \prod_{u_2\in V(G_2)} \left[\frac{2m_1d_{G_2}(u_2)+M_1(G_1)d_{G_2}(u_2)+M_1(G_1)}{m_1}\right]^{m_1}$$

$$\times \prod_{u_1v_1\in E(G_1)} \left[\frac{2m_2(d_{G_1}(u_1)+d_{G_1}(v_1))+2M_1(G_2)d_{G_1}(u_1) + M_1(G_2)(d_{G_1}(u_1)+d_{G_1}(v_1))}{2m_2}\right]^{2m_2}.$$

Again by using Lemma 1, we have

$$\Pi_1^*(G) \leqslant \frac{[4m_1m_2+(2m_1+n_1)M_1(G_2)]^{n_1m_2}\,[4m_1m_2+(2m_2+n_2)M_1(G_1)]^{n_2m_1}}{4^{m_1m_2}m_1^{n_2m_1+2m_1m_2}m_2^{n_1m_2+2m_1m_2}n_1^{n_1m_2}n_2^{n_2m_1}} \times [2m_2M_1(G_1)+2m_1M_1(G_2)+M_1(G_1)M_1(G_2)]^{2m_1m_2}.$$

The proof is similar to the proof of Theorem 5. □

2.7. Hierarchical product

Let $G=(V(G),E(G))$ and $H=(V(H),E(H))$ be two graphs and U be a non-empty subset of $V(H)$. Let $\Gamma = G\sqcap H(U)$ be the hierarchical product of G and H corresponding to U. Then $V(\Gamma)=V(G)\times V(H)$ and $(a,x)(b,y)\in E(\Gamma)$ if and only if $a=b, xy\in E(H)$ or $ab\in E(G), x=y\in U$. It is clear that if $U=V(H)$, then $G\sqcap H(U) = G\times H$, the Cartesian product of G and H.

LEMMA 2. (see [1]) *The degree of a vertex* $x=(x_N, x_{N-1},\ldots,x_2,x_1)$ *in the generalized hierarchical product* $H_N = G_N \sqcap \ldots \sqcap G_2(U_2)\sqcap G_1(U_1)$ *is*

$$\delta(x) = \delta(x_1)+\chi_{U_1}(x_1)\delta(x_2)+\ldots+[\chi_{U_1}(x_1)\ldots\chi_{U_{N-1}}(x_{N-1})]\delta(x_N),$$

where δ and χ_{U_i} denote, respectively, the degree and the characteristic function on the set U_i which is 1 on U_i and 0 outside U_i.

THEOREM 7. *Let $\Gamma = G \sqcap H(U)$. Then*

$$\Pi_1^*(\Gamma) \leqslant \frac{\left[n_1 M_1(G_2) + 2m_1 \sum_{x \in U} d(x)\right]^{n_1 m_2} \left[|U| M_1(G_2) + 2m_1 \sum_{x \in U} d(x)\right]^{m_1 |U|}}{(n_1 m_2)^{n_1 m_2} (m_1 |U|)^{m_1 |U|}}.$$

Proof. From definition of hierarchical product of G and H corresponding U, we have

$$\begin{aligned}\Pi_1^*(\Gamma) = & \prod_{u_1 \in V(G)} \prod_{u_2 v_2 \in E(H)} \left(d_{G_2}(u_2) + d_{G_2}(v_2) + d_{G_1}(u_1)\left[\chi_U(u_2) + \chi_U(v_2)\right]\right) \\ & \times \prod_{u_2 \in U} \prod_{u_1 v_1 \in E(G)} \left(2 d_{G_2}(u_2) + d_{G_1}(u_1) + d_{G_1}(v_1)\right).\end{aligned}$$

By using Lemma 1, we get

$$\begin{aligned}\Pi_1^*(\Gamma) & \leqslant \prod_{u_1 \in V(G_1)} \left[\frac{M_1(G_2) + dG_1(u_1) \sum_{x \in U} d(x)}{m_2}\right]^{m_2} \prod_{u_2 \in U} \left[\frac{2m_1 d_{G_2}(u_2) + M_1(G_1)}{m_1}\right]^{m_1} \\ & \leqslant \frac{\left[n_1 M_1(G_2) + 2m_1 \sum_{x \in U} d(x)\right]^{n_1 m_2} \left[|U| M_1(G_2) + 2m_1 \sum_{x \in U} d(x)\right]^{m_1 |U|}}{(n_1 m_2)^{n_1 m_2} (m_1 |U|)^{m_1 |U|}}. \quad \square\end{aligned}$$

2.8. Lexicographic product

The lexicographic product $G = G_1[G_2]$ of graphs G_1 and G_2 is a graph such that the vertex set of $G_1[G_2]$ is the cartesian product $V(G_1) \times V(G_2)$; and any two vertices (u,v) and (x,y) are adjacent in $G_1[G_2]$ if and only if either u is adjacent with x in G_1 or $u = x$ and v is adjacent with y in G_2. Also we know that $m(G_1[G_2]) = n_1 m_2 + n_2^2 m_1$ and $d_{G_1[G_2]}(u_1, u_2) = n_2 d_{G_1}(u_1) + d_{G_2}(u_2)$.

THEOREM 8. *Let $G = G_1[G_2]$ be the lexicographic product of two graphs G_1 and G_2. Then*

$$\Pi_1^*(G) \leqslant \frac{\left[4m_1 m_2 n_2 + n_1 M_1(G_2)\right]^{n_1 m_2} \left[4m_1 m_2 n_2 + n_2^3 M_1(G_1)\right]^{n_2^2 m_1}}{(m_2 n_1)^{n_1 m_2} (m_1 n_2^2)^{m_1 n_2^2}}. \tag{6}$$

Proof. By definition of the multiplicative sum Zagreb index, we have

$$\begin{aligned}\Pi_1^*(G) = & \prod_{u_1 \in V(G_1)} \prod_{u_2 v_2 \in E(G_2)} \left(2n_2 d_{G_1}(u_1) + d_{G_2}(u_2) + d_{G_2}(v_2)\right) \\ & \times \prod_{u_1 v_1 \in E(G_1)} \prod_{\{u_2, v_2\} \subset V(G_2)} \left[n_2\left(d_{G_1}(u_1) + d_{G_1}(v_1)\right) + d_{G_2}(u_2) + d_{G_2}(v_2)\right].\end{aligned}$$

By applying Lemma 1, we have

$$\Pi_1^*(G) \leqslant \prod_{u_1\in V(G_1)} \left[\frac{\sum_{u_2v_2\in E(G_2)}(2n_2d_{G_1}(u_1)+d_{G_2}(u_2)+d_{G_2}(v_2))}{m_2}\right]^{m_2}$$

$$\times \prod_{u_1v_1\in E(G_1)} \left[\frac{\sum_{\{u_2,v_2\}\subset V(G_2)}\left[n_2\left(d_{G_1}(u_1)+d_{G_1}(v_1)\right)+d_{G_2}(u_2)+d_{G_2}(v_2)\right]}{n_2^2}\right]^{n_2^2}$$

$$= \frac{1}{m_2^{n_1m_2}n_2^{2n_2^2m_1}} \prod_{u_1\in V(G_1)} \left[2n_2m_2d_{G_1}(u_1)+M_1(G_2)\right]^{m_2}$$

$$\times \prod_{u_1v_1\in E(G_1)} \left[n_2^3\left(d_{G_1}(u_1)+d_{G_1}(v_1)\right)+4m_2n_2\right]$$

$$\leqslant \frac{1}{m_2^{n_1m_2}n_2^{2n_2^2m_1}} \left[\frac{\sum_{u_1\in V(G_1)}\left[2n_2m_2d_{G_1}(u_1)+M_1(G_2)\right]}{n_1}\right]^{n_1m_2}$$

$$\times \left[\frac{\sum_{,u_1v_1\in E(G_1)}\left[n_2^3\left(d_{G_1}(u_1)+d_{G_1}(v_1)\right)+4m_2n_2\right]}{m_1}\right]^{n_2^2m_1}$$

$$= \frac{\left[4m_1m_2n_2+n_1M_1(G_2)\right]^{n_1m_2}\left[4m_1m_2n_2+n_2^3M_1(G_1)\right]^{n_2^2m_1}}{(m_2n_1)^{n_1m_2}(m_1n_2^2)^{m_1n_2^2}}.$$

Now suppose that equality holds in (6). Then all the inequalities in the above argument must be equalities. Thus we have

$$2n_2d_{G_1}(u_1)+d_{G_2}(u_2)+d_{G_2}(v_2) = 2n_2d_{G_1}(a_1)+d_{G_2}(a_2)+d_{G_2}(b_2)$$

for any $u_1,a_1\in V(G_1), u_2v_2, a_2b_2\in E(G_2)$ and

$$n_2(d_{G_1}(u_1)+d_{G_1}(v_1))+d_{G_2}(u_2)+d_{G_2}(v_2) = n_2(d_{G_1}(a_1)+d_{G_1}(b_1))+d_{G_2}(a_2)+d_{G_2}(b_2)$$

for any $u_1v_1, a_1b_1\in E(G_1), \{u_2,v_2\},\{a_2,b_2\}\subset V(G_2)$. Hence the equality holds in (6) if and only if both G_1 and G_2 are regular graphs. □

EXAMPLE 3. Consider cycle graphs C_p and C_q. We thus have

$$\Pi_1^*(C_p[C_q]) = (4p+4)^{pq(q+1)}.$$

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