Journal of Mathematical Inequalities
Volume 11, Number 3 (2017), 763–780 doi:10.7153/jmi-2017-11-60

ON ROUGH GENERALIZED PARAMETRIC MARCINKIEWICZ INTEGRALS

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(*Communicated by S. Li*)

Abstract. We obtain certain sharp L^p bounds for the generalized parametric Marcinkiewicz integrals $\mathscr{M}^{(\lambda)}_{\Omega,h,\rho}$. The singular kernels are allowed to be rough on the unit sphere as well as in the radial direction. By the virtue of these estimates along with an extrapolation argument we obtain some new and improved results on generalized parametric Marcinkiewicz integrals. Our conditions on Ω and h are known to be the weakest conditions in their respective classes. One of our main results answers a question posed by Fan and Wu.

1. Introduction

Throughout this paper, let $\mathbf{R}^n$, $n \geqslant 2$, be the n-dimensional Euclidean space and $\mathbf{S}^{n-1}$ be the unit sphere in $\mathbf{R}^n$ equipped with the normalized Lebesgue surface measure $d\sigma$. Also, we let ξ' denote $\xi/|\xi|$ for $\xi \in \mathbf{R}^n \setminus \{0\}$ and p' denote the exponent conjugate to p, that is $1/p + 1/p' = 1$.

Let h be a measurable function on $\mathbf{R}_+$ and Ω be an integrable function Ω on $\mathbf{S}^{n-1}$ satisfying

$$\int_{\mathbf{S}^{n-1}} \Omega(y')d\sigma(y') = 0. \tag{1.1}$$

The generalized parametric Marcinkiewicz integral operator $\mathscr{M}^{(\lambda)}_{\Omega,h}$ is given by

$$\mathscr{M}^{(\lambda)}_{\Omega,h,\rho} f(x) = \left(\int_0^\infty \left| \frac{1}{t^\rho} \int_{|u|\leqslant t} f(x-u) \frac{\Omega(u')}{|u|^{n-\rho}} h(|u|) du \right|^\lambda \frac{dt}{t} \right)^{1/\lambda},$$

where $\lambda > 1$, $\rho = \alpha + i\beta$ $(\alpha, \beta \in \mathbf{R}$ with $\alpha > 0)$ and $f \in \mathscr{S}(\mathbf{R}^n)$, the space of Schwartz functions.

If $h \equiv 1$, $\rho = 1$ and $\lambda = 2$, $\mathscr{M}^{(2)}_{\Omega,1,1}$ is the classical Marcinkiewicz integral operator, which was first introduced by E. Stein in [25], as an extension of the notion of Marcinkiewicz function from one dimension to higher dimensions. In [25], Stein proved that if $\Omega \in \mathrm{Lip}_\alpha(\mathbf{S}^{n-1})$ $(0 < \alpha \leqslant 1)$, then $\mathscr{M}^{(2)}_{\Omega,1,1}$ is of type (p,p) for $1 < p \leqslant 2$

Mathematics subject classification (2010): Primary 42B20, Secondary 42B25, 42B35, 42B99.
Keywords and phrases: Marcinkiewicz integrals, Triebel-Lizorkin spaces, rough kernels, Orlicz spaces, Block spaces, extrapolation, L^p boundedness.

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Paper JMI-11-60

and of weak type $(1,1)$. In [7], Benedek, Calderón, and Panzone proved that $\mathscr{M}^{(2)}_{\Omega,1,1}$ is of type (p,p) for $p\in(1,\infty)$ if $\Omega\in C^1\left(\mathbf{S}^{n-1}\right)$. In [15], Hörmander proved that the parametric Marcinkiewicz operator $\mathscr{M}^{(2)}_{\Omega,1,\rho}$ is of type (p,p) for $p\in(1,\infty)$ if $\rho>0$ and $\Omega\in \mathrm{Lip}_{\alpha}(\mathbf{S}^{n-1})$ $(0<\alpha\leqslant 1)$. Later on, the study of $\mathscr{M}^{(2)}_{\Omega,h,\rho}$ and some of its extensions has attracted the attention of many authors. Readers may consult [29], [11], [2], [1], [6], [3], [4], [21], [22], among a large number of references for their development and applications.

On the other hand, the study of the generalized Marcinkiewicz integral operator $\mathscr{M}^{(\lambda)}_{\Omega,h,1}$ was first introduced in [9] and later it has attracted the attention of many authors (see for example, [19], [13], [5], [21], among others). Let us now recall the following results which will be relevant to our current study.

(1) If $h\equiv 1$, $\rho=1$ and $\Omega\in L(\log L)^{1/2}\left(\mathbf{S}^{n-1}\right)$, then $\mathscr{M}^{(2)}_{\Omega,1,1}$ is bounded on $L^p(\mathbf{R}^n)$ for $1<p<\infty$. Moreover, the exponent $1/2$ is the best possible (see [29] for $p=2$ and [6] for $1<p<\infty$).

(2) If $h\equiv 1$ and $\Omega\in B_q^{(0,-\frac{1}{2})}(\mathbf{S}^{n-1})$, then $\mathscr{M}^{(2)}_{\Omega,1,1}$ is bounded on $L^p(\mathbf{R}^n)$ for $1<p<\infty$. Moreover, the exponent $-1/2$ is the best possible (see [2]).

(3) If $1<\lambda<\infty$, $h\equiv 1$ and $\Omega\in L^q(\mathbf{S}^{n-1})$ for some $q>1$,

$$\left\|\mathscr{M}^{(\lambda)}_{\Omega,h,1}(f)\right\|_{L^p(\mathbf{R}^n)}\leqslant C\|f\|_{\dot{F}_p^{0,\lambda}(\mathbf{R}^n)} \tag{1.2}$$

for $1<p<\infty$ (see [9]).

(4) If $1<\lambda<\infty$, $h\in\Delta_{\max\{\lambda',2\}}(\mathbf{R}_+)$ and $\Omega\in L(\log L)(\mathbf{S}^{n-1})$, then (1.2) holds for $1<p<\infty$ (see [19]).

(5) If $h\equiv 1$ and $\Omega\in L(\log L)^{1/\lambda}(\mathbf{S}^{n-1})$ for $\lambda\geqslant 2$, then (1.2) holds for $1<p<\infty$. However, if $1<\lambda<2$ and $\Omega\in L(\log L)^{1/\lambda+\varepsilon}(\mathbf{S}^{n-1})$ for any $\varepsilon>0$, then (1.2) holds for $1<p<\infty$ (see [13]).

(6) If $1<\lambda<\infty, 1<p<\infty$, $\gamma>\frac{1}{2}\max\{\tilde{p},\tilde{q}\}$ with $\tilde{\eta}=\max\{\eta,\eta'\}$, $h\in\Delta_{\max\{2,\gamma\}}(\mathbf{R}_+)$ and $\Omega\in L(\log L)(\mathbf{S}^{n-1})$, then (1.2) holds (see [21]).

In view of the above results, the following questions are very natural:

QUESTION 1. Determine whether the ε in the condition $L(\log L)^{1/\lambda+\varepsilon}(\mathbf{S}^{n-1})$ in (5) can be removed?

In fact, this question was formally raised by the authors in [13]. Also, in view of results in (4) and (6) above (see [19] and [21]), we notice in one hand that the condition $\Omega\in L(\log L)(\mathbf{S}^{n-1})$ falls short of the natural condition $L(\log L)^{1/\lambda}(\mathbf{S}^{n-1})$, while on the other hand the conditions on h which is $h\in\Delta_{\max\{\lambda',2\}}(\mathbf{R}_+)$ in [19] and $h\in\Delta_{\max\{\gamma,2\}}(\mathbf{R}_+)$ in [21] are too restrictive. So the second question is the following:

QUESTION 2. Determine whether the inequality (1.2) holds if $h\in\Delta_{\gamma}(\mathbf{R}_+)$ for some $\gamma>1$ and $\Omega\in L(\log L)^{1/\lambda}(\mathbf{S}^{n-1})$?

One of our main purposes in this paper is to answer the above questions in the affirmative. In fact, we shall prove even more. To be able to state our results, we need to recall the following definition. For $1\leqslant\gamma\leqslant\infty$, let $\Delta_{\gamma}(\mathbf{R}_+)$ denote the collection of all measurable functions $h : [0, \infty) \longrightarrow \mathbf{C}$ satisfying

$$\|h\|_{\Delta_\gamma} = \sup_{k\in\mathbf{Z}} \left(\int_{2^k}^{2^{k+1}} |h(t)|^\gamma dt/t \right)^{1/\gamma} < \infty$$

and $\mathscr{L}_\gamma(\mathbf{R}_+)$ denote the collection of all measurable functions $h : [0, \infty) \longrightarrow \mathbf{C}$ satisfying

$$L_\gamma(h) = \sup_{k\in\mathbf{Z}} \left(\int_{2^k}^{2^{k+1}} |h(t)| (\log(2+|h(t)|))^\gamma dt/t \right) < \infty.$$

Also, we let $\mathscr{N}_\gamma(\mathbf{R}_+)$ denote the class of all measurable functions h on $\mathbf{R}_+$ such that

$$N_\gamma(h) = \sum_{m=1} m^\gamma 2^m d_m(h) < \infty,$$

where $d_m(h) = \sup_{k\in\mathbf{Z}} 2^{-k} |E(k,m)|$ with $E(k,m) = \left\{ t \in (2^k, 2^{k+1}] : 2^{m-1} < |h(t)| \leqslant 2^m \right\}$ for $m \geqslant 2$ and $E(k,1) = \left\{ t \in (2^k, 2^{k+1}] : |h(t)| \leqslant 2 \right\}$.
We remark that $\Delta_\gamma(\mathbf{R}_+) \subset \mathscr{N}_\alpha(\mathbf{R}_+) \subset \mathscr{L}_\alpha(\mathbf{R}_+)$ for any $\gamma \geqslant 1, \alpha > 0$ and for a given $\alpha > 1$, $\mathscr{L}_{\gamma+\alpha}(\mathbf{R}_+) \subset \mathscr{N}_\gamma(\mathbf{R}_+)$ for any $\gamma > 0$.

The statement of our main results of this paper are the following:

THEOREM 1.1. *Suppose that Ω satisfies (1.1), $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $q \in (1,2]$ and $h \in \Delta_\gamma(\mathbf{R}_+)$ for some $\gamma \in (1,2]$. Then*

$$\left\| \mathscr{M}^{(\lambda)}_{\Omega,h,\rho} f \right\|_{L^p(\mathbf{R}^n)} \leqslant C_p (q-1)^{-\frac{1}{\lambda}} (\gamma-1)^{-\frac{1}{\lambda}} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|h\|_{\Delta_\gamma} \|f\|_{\dot{F}^{0,\lambda}_p(\mathbf{R}^n)} \ \text{if } \lambda \leqslant p < \infty \tag{1.3}$$

and

$$\left\| \mathscr{M}^{(\lambda)}_{\Omega,h,\rho} f \right\|_{L^p(\mathbf{R}^n)} \leqslant C_p (q-1)^{-1} (\gamma-1)^{-1} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|h\|_{\Delta_\gamma} \|f\|_{\dot{F}^{0,\lambda}_p(\mathbf{R}^n)} \ \text{if } 1 < p < \lambda, \tag{1.4}$$

where C_p is a positive constant independent of γ, q, Ω and h.

We notice in Theorem 1.1 that the exponent -1 is not sharp in the case $1 < p < \lambda$. However if $h \in \Delta_\gamma(\mathbf{R}_+)$ for some $\gamma > 2$ we have the following sharper result:

THEOREM 1.2. *Suppose that Ω satisfies (1.1), $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $q \in (1,2]$ and $h \in \Delta_\gamma(\mathbf{R}_+)$ for some $\gamma > 2$. Then*

$$\left\| \mathscr{M}^{(\lambda)}_{\Omega,h,\rho} f \right\|_{L^p(\mathbf{R}^n)} \leqslant C (q-1)^{-1/\lambda} \|h\|_{\Delta_\gamma} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|f\|_{\dot{F}^{0,\lambda}_p(\mathbf{R}^n)}$$

for $1 < p < \lambda$ if $2 < \gamma < \infty$ and $\lambda' \geqslant \gamma$, and

$$\left\| \mathscr{M}^{(\lambda)}_{\Omega,h,\rho} f \right\|_{L^p(\mathbf{R}^n)} \leqslant C (q-1)^{-1/\lambda} \|h\|_{\Delta_\gamma} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|f\|_{\dot{F}^{0,\lambda}_p(\mathbf{R}^n)}$$

for $\gamma' < p < \infty$ if $2 < \gamma \leqslant \infty$ and $\lambda' < \gamma$, where C_p is a positive constant independent of γ, q, Ω and h.

By the estimates in Theorems 1.1–1.2 and applying extrapolation we obtain the following results:

THEOREM 1.3. *Suppose that* Ω *satisfies (1.1).*

(a) If $\Omega \in L(\log L)^{1/\lambda}\left(\mathbf{S}^{n-1}\right)$ *and* $h \in \mathcal{N}_{1/\lambda}(\mathbf{R}_+)$, *then*

$$\left\|\mathscr{M}_{\Omega,h,\rho}^{(\lambda)} f\right\|_{L^p(\mathbf{R}^n)} \leqslant C_p \left(1+\|\Omega\|_{L(\log L)^{1/\lambda}\left(\mathbf{S}^{n-1}\right)}\right)\left(1+N_{1/\lambda}(h)\right)\|f\|_{\dot{F}_p^{0,\lambda}(\mathbf{R}^n)}$$

for $\lambda \leqslant p < \infty$;

(b) If $\Omega \in L(\log L)\left(\mathbf{S}^{n-1}\right)$ *and* $h \in \mathcal{N}_1(\mathbf{R}_+)$, *then*

$$\left\|\mathscr{M}_{\Omega,h,\rho}^{(\lambda)} f\right\|_{L^p(\mathbf{R}^n)} \leqslant C_p \left(1+\|\Omega\|_{L(\log L)\left(\mathbf{S}^{n-1}\right)}\right)\left(1+N_1(h)\right)\|f\|_{\dot{F}_p^{0,\lambda}(\mathbf{R}^n)},$$

for $1 < p < \lambda$, *where the constant* C_p *is independent of* Ω *and* h.

THEOREM 1.4. *Suppose that* Ω *satisfies (1.1).*

(a) If $\Omega \in B_q^{(0,\frac{1}{\lambda}-1)}\left(\mathbf{S}^{n-1}\right)$ *for some* $q > 1$ *and* $h \in \mathcal{N}_{1/\lambda}(\mathbf{R}_+)$, *then*

$$\left\|\mathscr{M}_{\Omega,h,\rho}^{(\lambda)} f\right\|_{L^p(\mathbf{R}^n)} \leqslant C_p \left(1+\|\Omega\|_{B_q^{(0,\frac{1}{\lambda}-1)}\left(\mathbf{S}^{n-1}\right)}\right)\left(1+N_{1/\lambda}(h)\right)\|f\|_{\dot{F}_p^{0,\lambda}(\mathbf{R}^n)}$$

for $\lambda \leqslant p < \infty$;

(b) If $\Omega \in B_q^{(0,0)}\left(\mathbf{S}^{n-1}\right)$ *for some* $q > 1$ *and* $h \in \mathcal{N}_1(\mathbf{R}_+)$, *then*

$$\left\|\mathscr{M}_{\Omega,h,\rho}^{(\lambda)} f\right\|_{L^p(\mathbf{R}^n)} \leqslant C_p \left(1+\|\Omega\|_{B_q^{(0,0)}\left(\mathbf{S}^{n-1}\right)}\right)\left(1+N_1(h)\right)\|f\|_{\dot{F}_p^{0,\lambda}(\mathbf{R}^n)}$$

for $1 < p < \lambda$, *where the constant* C_p *is independent of* Ω *and* h.

THEOREM 1.5. *Suppose that* Ω *satisfies (1.1) and* $h \in \Delta_\gamma(\mathbf{R}_+)$ *for some* $\gamma > 2$.

(a) If $\Omega \in L(\log L)^{1/\lambda}\left(\mathbf{S}^{n-1}\right)$, *then, for some positive constant* C_p *that is independent of* Ω *and* h, *the following inequality holds*

$$\left\|\mathscr{M}_{\Omega,h,\rho}^{(\lambda)} f\right\|_{L^p(\mathbf{R}^n)} \leqslant C_p \left(1+\|\Omega\|_{L(\log L)^{1/\lambda}\left(\mathbf{S}^{n-1}\right)}\right)\|h\|_{\Delta_\gamma}\|f\|_{\dot{F}_p^{0,\lambda}(\mathbf{R}^n)}$$

for $1 < p < \lambda$ *if* $2 < \gamma < \infty$ *and* $\lambda' \geqslant \gamma$, *and for* $\gamma' < p < \infty$ *if* $2 < \gamma \leqslant \infty$ *and* $\lambda' < \gamma$.

(b) If $\Omega \in B_q^{(0,\frac{1}{\lambda}-1)}\left(\mathbf{S}^{n-1}\right)$, *then, for some positive constant* C_p *that is independent of* Ω *and* h, *the following inequality holds*

$$\left\|\mathscr{M}_{\Omega,h,\rho}^{(\lambda)} f\right\|_{L^p(\mathbf{R}^n)} \leqslant C_p \left(1+\|\Omega\|_{L(\log L)^{1/\lambda}\left(\mathbf{S}^{n-1}\right)}\right)\|h\|_{\Delta_\gamma}\|f\|_{\dot{F}_p^{0,\lambda}(\mathbf{R}^n)}$$

for $1 < p < \lambda$ *if* $2 < \gamma < \infty$ *and* $\lambda' \geqslant \gamma$, *and for* $\gamma' < p < \infty$ *if* $2 < \gamma \leqslant \infty$ *and* $\lambda' < \gamma$.

REMARKS.

1. We notice that by Theorem 1.5 (ii) if $h \in L^\infty(0,\infty)$ and $\Omega \in L(\log L)^{1/\lambda}\left(\mathbf{S}^{n-1}\right)$, then $\mathscr{M}_{\Omega,h,\rho}^{(\lambda)}$ is bounded on L^p for the full range $(1,\infty)$ and hence we get a complete answer to Question 1 which in turn answers a question posed in [13].

2. We notice that Theorems 1.3 and 1.5 improve greatly the results in [13], [19] and [21].

3. Theorems 1.3–1.5 generalize very much the main results in [2], [3], [4] and [6].

4. It is known that the conditions $\Omega \in B_q^{(0,-1/2)}\left(\mathbf{S}^{n-1}\right)$ $(q>1)$ and $\Omega \in L(\log L)^{1/2}\left(\mathbf{S}^{n-1}\right)$ are optimal for the L^2 boundedness of $\mathscr{M}_{\Omega,1,1}^{(2)}$ to hold in the sense that $1/2$ cannot be replaced by any smaller number. Also, the conditions imposed on h in Theorems 1.3–1.5 are the weakest known conditions.

Throughout the rest of the paper the letter C denotes a positive whose value may be different at each appearance.

2. Some definitions and lemmas

The class $L(\log L)^{\alpha}(\mathbf{S}^{n-1})$ (for $\alpha>0$) denotes the class of all measurable functions Ω on $\mathbf{S}^{n-1}$ which satisfy

$$\|\Omega\|_{L(\log L)^{\alpha}(\mathbf{S}^{n-1})}=\int_{\mathbf{S}^{n-1}}|\Omega(y)|\log^{\alpha}(2+|\Omega(y)|)d\sigma(y)<\infty.$$

Now we recall the definition of the block space $B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})$. This space was introduced by Jiang and Lu (see [20]) in their study of the mapping properties of homogeneous singular integral operators and it is defined as follows: A $q-$block on $\mathbf{S}^{n-1}$ is an L^q $(1<q\leqslant\infty)$ function $b(x)$ that satisfies (i) $\operatorname{supp}(b)\subset I$; (ii) $\|b\|_{L^q}\leqslant|I|^{-1/q'}$, where $|I|=\sigma(I)$, and $I=B(x_0',\theta_0)=\{x'\in\mathbf{S}^{n-1}:\left|x'-x_0'\right|<\theta_0\}$ is a cap on $\mathbf{S}^{n-1}$ for some $x_0'\in\mathbf{S}^{n-1}$ and $\theta_0\in(0,1]$. The block space $B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})$ is defined by

$$B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})=\left\{\Omega\in L^1(\mathbf{S}^{n-1}):\Omega=\sum_{\mu=1}^{\infty}\lambda_\mu b_\mu, M_q^{(0,\upsilon)}\left(\{\lambda_\mu\}\right)<\infty\right\},$$

where each λ_μ is a complex number; each b_μ is a $q-$block supported on a cap I_μ on $\mathbf{S}^{n-1}$, $\upsilon>-1$ and

$$M_q^{(0,\upsilon)}\left(\{\lambda_\mu\}\right)=\sum_{\mu=1}^{\infty}\left|\lambda_\mu\right|\left\{1+\log^{(\upsilon+1)}(\left|I_\mu\right|^{-1})\right\}.$$

Let $\|\Omega\|_{B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})}=N_q^{(0,\upsilon)}(\Omega)=\inf\{M_q^{(0,\upsilon)}\left(\{\lambda_\mu\}\right):\ \Omega=\sum_{\mu=1}^{\infty}\lambda_\mu b_\mu$ and each b_μ is a q-block function supported on a cap I_μ on $\mathbf{S}^{n-1}\}$. Then $\|\cdot\|_{B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})}$ is a norm on the space $B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})$ and $(B_q^{(0,\upsilon)}(\mathbf{S}^{n-1}),\|\cdot\|_{B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})})$ is a Banach space.

REMARK. For any $q>1$ and $0<\upsilon\leqslant 1$, the following inclusions hold and are proper:

$$
\begin{aligned}
L^q(\mathbf{S}^{n-1}) &\subset L(\log L)(\mathbf{S}^{n-1}) \subset H^1(\mathbf{S}^{n-1}) \subset L^1(\mathbf{S}^{n-1}),\\
\bigcup_{r>1} L^r(\mathbf{S}^{n-1}) &\subset B_q^{(0,\upsilon)}(\mathbf{S}^{n-1}) \text{ for any } -1<\upsilon \text{ and } q>1,\\
L(\log L)^{\beta}(\mathbf{S}^{n-1}) &\subset L(\log L)^{\alpha}(\mathbf{S}^{n-1}) \text{ if } 0<\alpha<\beta,\\
L(\log L)^{\alpha}(\mathbf{S}^{n-1}) &\subset H^1(\mathbf{S}^{n-1}) \text{ for all } \alpha\geqslant 1.
\end{aligned}
$$

Regarding the relationship between $L(\log L)^{\alpha}(\mathbf{S}^{n-1})$ and $H^1(\mathbf{S}^{n-1})$ for $0<\alpha<1$, it is known that neither one is contained in the other. Here, $H^1(\mathbf{S}^{n-1})$ is the Hardy space on the unit sphere in the sense of Coifman and Weiss [8]. The question with regard to the relationship between $B_q^{(0,\upsilon-1)}(\mathbf{S}^{n-1})$ and $L(\log^+ L)^{\upsilon}(\mathbf{S}^{n-1})$ (for $\upsilon>0$) remains open.

Now we recall the definition of the Triebel-Lizorkin spaces $\dot{F}_p^{\alpha,q}(\mathbf{R}^n)$. For $1<p,q<\infty$ and $\alpha\in\mathbf{R}$, the homogeneous Triebel-Lizorkin space $\dot{F}_p^{\alpha,q}(\mathbf{R}^n)$ is defined by the space of all tempered distributions $f\in\mathscr{S}'(\mathbf{R}^n)$ satisfying

$$
\dot{F}_p^{\alpha,q}(\mathbf{R}^n)=\left\{f\in\mathscr{S}'(\mathbf{R}^n):\|f\|_{\dot{F}_p^{\alpha,q}(\mathbf{R}^n)}=\left\|\left(\sum_{k\in\mathbf{Z}}2^{k\alpha q}\left|\Psi_k*f\right|^q\right)^{1/q}\right\|_{L^p(\mathbf{R}^n)}<\infty\right\},
$$

where $\mathscr{S}'(\mathbf{R}^n)$ denotes the tempered distribution class on $\mathbf{R}^n, \hat{\Psi}_k(\xi)=\Phi(2^{-k}\xi)$ for $k\in\mathbf{Z}$ and $\Phi\in C_0^{\infty}(\mathbf{R}^n)$ is a radial function satisfying the following conditions:

(i) $0\leqslant\Phi\leqslant 1$;

(ii) supp $\Phi\subset\left\{\xi:\frac{1}{2}\leqslant|\xi|\leqslant 2\right\}$;

(iii) $\Phi(\xi)\geqslant c>0$ if $\frac{3}{5}\leqslant|\xi|\leqslant\frac{5}{3}$;

(iv) $\sum_{j\in\mathbf{Z}}\Phi(2^{-j}\xi)=1$ $(\xi\neq 0)$.

It is well-known that $\mathscr{S}(\mathbf{R}^n)$ is dense in $\dot{F}_p^{\alpha,q}(\mathbf{R}^n)$ and that the following hold:

(1) $L^p(\mathbf{R}^n)=\dot{F}_p^{0,2}(\mathbf{R}^n)$;

(2) $\left(\dot{F}_p^{\alpha,q}(\mathbf{R}^n)\right)^*=\dot{F}_{p'}^{-\alpha,q'}(\mathbf{R}^n)$;

(3) $\dot{F}_p^{\alpha,q_1}(\mathbf{R}^n)\subset\dot{F}_p^{\alpha,q_2}(\mathbf{R}^n)$ if $q_1\leqslant q_2$.

Let $\{a_k:k\in\mathbf{Z}\}$ be a lacunary sequence of positive numbers in the sense that $\frac{a_{k+1}}{a_k}\geqslant a>1$ for each $k\in\mathbf{Z}$. A sequence $\{\Phi_k:k\in\mathbf{Z}\}$ of $C^{\infty}(\mathbf{R}^n)$ functions is said to be a partition of unity adapted to $\{a_k:k\in\mathbf{Z}\}$ if

$$
\begin{aligned}
&\mathrm{Supp}\widehat{\Phi_k}\subset\{\xi\in\mathbf{R}^n:a_{k-1}\leqslant|\xi|\leqslant a_{k+1}\}\,(k\in\mathbf{Z}),\\
&\sum_{k\in\mathbf{Z}}\widehat{\Phi_k}(\xi)=1\,(\xi\in\mathbf{R}^n\setminus\{0\}),
\end{aligned}
$$

and

$$
\left|\xi^{\alpha}\partial^{\beta}\widehat{\Phi_k}(\xi)\right|\leqslant C_\beta
$$

for any multi-index β. Let $\mathscr{P}$ be the set of all polynomials on $\mathbf{R}^n$. Let $1 < p,q < \infty$ and $\alpha \in \mathbf{R}$. For $f \in \mathscr{S}'(\mathbf{R}^n)/\mathscr{P}$, we define the norm $\|f\|_{\dot{F}_p^{\alpha,q}(\{\Phi_k\}_{k\in\mathbf{Z}},\mathbf{R}^n)}$ by

$$\|f\|_{\dot{F}_p^{\alpha,q}(\{\Phi_k\}_{k\in\mathbf{Z}},\mathbf{R}^n)} = \left\| \left(\sum_{k\in\mathbf{Z}} a_k^{\alpha q} |\Phi_k * f|^q \right)^{1/q} \right\|_{L^p(\mathbf{R}^n)}.$$

The following result is stated in Proposition 1 in [21] for $\alpha \neq 0$, but the proof of this part works also for $\alpha = 0$ as pointed in [27].

LEMMA 2.1. *Let $\alpha \in \mathbf{R}$ and $1 < p,q < \infty$. Let $\{a_k : k \in \mathbf{Z}\}$ be a lacunary sequence of positive numbers with $\frac{a_{k+1}}{a_k} \geqslant a > 1$ $(k \in \mathbf{Z})$. Then $\|f\|_{\dot{F}_p^{\alpha,q}(\{\Phi_k\}_{k\in\mathbf{Z}},\mathbf{R}^n)}$ is equivalent to the usual homogeneous Triebel-Lizorkin space norm $\|f\|_{\dot{F}_p^{\alpha,q}(\mathbf{R}^n)}$ if $\frac{a_{k+1}}{a_k} \leqslant d$ $(k \in \mathbf{Z})$ for some $d \geqslant a$.*

Let $\theta \geqslant 2$. For a suitable measurable function $h : \mathbf{R}_+ \longrightarrow C$ and $\Omega : \mathbf{S}^{n-1} \to \mathbf{R}$, we define the family of measures $\{\sigma_{t,\Omega,h} : t \in \mathbf{R}_+\}$ and the related maximal operators $\sigma^*_{\Omega,h}$ and $M_{\Omega,h,\theta}$ on $\mathbf{R}^n$ by

$$\int_{\mathbf{R}^n} f d\sigma_{t,\Omega,h} = \frac{1}{t^\rho} \int_{\frac{1}{2}t<|x|\leqslant t} \frac{\Omega(x)h(|x|)}{|x|^{n-\rho}} f(x)dx\,;$$
$$\sigma^*_{\Omega,h}(f) = \sup_{t\in\mathbf{R}_+} \left| \left|\sigma_{t,\Omega,h}\right| * f \right|,$$
$$M_{\Omega,h,\theta} f(x) = \sup_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^{(k+1)}} \left| \left|\sigma_{t,\Omega,h}\right| * f(x) \right| dt/t.$$

We shall need the following lemma from [4].

LEMMA 2.2. *Let $h \in \Delta_\gamma(\mathbf{R}_+)$ for some $1 < \gamma \leqslant 2$, $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $1 < q \leqslant 2$ and $\theta = 2^{q'\gamma'}$. Then for every $p, 1 < p \leqslant \infty$, there exists a positive constant C_p which is is independent of h, Ω, q and γ such that*

$$\left\|M_{\Omega,h,\theta}(f)\right\|_p \leqslant C_p(q-1)^{-1}(\gamma-1)^{-1} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|h\|_{\Delta_\gamma} \|f\|_p\,; \tag{2.1}$$
$$\left\|\sigma^*_{\Omega,h}(f)\right\|_p \leqslant C_p(q-1)^{-1}(\gamma-1)^{-1} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|h\|_{\Delta_\gamma} \|f\|_p \tag{2.2}$$

for every $f \in L^p(\mathbf{R}^n)$.

LEMMA 2.3. *Let $h \in \Delta_\gamma(\mathbf{R}_+)$ for some $1 < \gamma \leqslant 2$, $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $1 < q \leqslant 2$ and $\theta = 2^{q'\gamma'}$. Let λ be a real number with $\lambda > 1$. Then there exists a positive constant C_p which is independent of q, γ, Ω and h such that the following inequalities*

$$\left\| \left(\sum_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^{(k+1)}} \left|\sigma_{t,\Omega,h} * g_k\right|^\lambda dt/t \right)^{1/\lambda} \right\|_{L^p(\mathbf{R}^n)}$$
$$\leqslant C_p(q-1)^{-\frac{1}{\lambda}}(\gamma-1)^{-\frac{1}{\lambda}} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|h\|_{\Delta_\gamma} \left\| \left(\sum_{k\in\mathbf{Z}} |g_k|^\lambda \right)^{1/\lambda} \right\|_{L^p(\mathbf{R}^n)} \quad \text{for } \lambda \leqslant p < \infty; \tag{2.3}$$

and

$$\left\| \left(\sum_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^{(k+1)}} \left|\sigma_{t,\Omega,h} * g_k\right|^{\lambda} dt/t \right)^{1/\lambda} \right\|_{L^p(\mathbf{R}^n)} \leqslant C_p(q-1)^{-1}(\gamma-1)^{-1} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|h\|_{\Delta_\gamma} \left\| \left(\sum_{k\in\mathbf{Z}} |g_k|^{\lambda} \right)^{1/\lambda} \right\|_{L^p(\mathbf{R}^n)} \quad \text{for } 1<p<\lambda \tag{2.4}$$

hold for arbitrary functions $\{g_k(\cdot)\}_{k\in\mathbf{Z}}$ *on* $\mathbf{R}^n$.

Proof. Let us first consider the case $p \geqslant \lambda$. By duality there exists a nonnegative function b in $L^{(p/\lambda)'}(\mathbf{R}^n)$ with $\|b\|_{(p/\lambda)'} \leqslant 1$ such that

$$\left\| \left(\sum_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^{(k+1)}} \left|\sigma_{t,\Omega,h} * g_k\right|^{\lambda} dt/t \right)^{1/\lambda} \right\|_{L^p(\mathbf{R}^n)}^{\lambda} = \int_{\mathbf{R}^n} \sum_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^{(k+1)}} \left|\sigma_{t,\Omega,h} * g_k(x)\right|^{\lambda} b(x) \frac{dt}{t} dx. \tag{2.5}$$

By Hölder's inequality we get

$$\left|\sigma_{t,\Omega,h} * g_k(x)\right|^{\lambda} \leqslant C \|h\|_{\Delta_1}^{(\lambda/\lambda')} \left(\|\Omega\|_{L^1(\mathbf{S}^{n-1})}\right)^{(\lambda/\lambda')} \times \left(\int_{\frac{1}{2}t}^{t} \int_{\mathbf{S}^{n-1}} |g_k(x-sy)|^{\lambda} |\Omega(y)| |h(s)| d\sigma(y) ds/s \right).$$

Therefore, by a change of variable we have

$$\left\| \left(\sum_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^{(k+1)}} \left|\sigma_{t,\Omega,h} * g_k\right|^{\lambda} dt/t \right)^{1/\lambda} \right\|_{L^p(\mathbf{R}^n)}^{\lambda} \leqslant C \|h\|_{\Delta_1}^{(\lambda/\lambda')} \left(\|\Omega\|_{L^1(\mathbf{S}^{n-1})}\right)^{(\lambda/\lambda')} \int_{\mathbf{R}^n} \left(\sum_{k\in\mathbf{Z}} |g_k(x)|^{\lambda} \right) M_{|\Omega|,|h|,\theta} \tilde{b}(-x) dx, \tag{2.6}$$

where $\tilde{b}(x) = b(-x)$. Thus, by Lemma 2.2 , (2.6) and Hölder's inequality we get (2.3) for $\lambda < p < \infty$. Now if $p = \lambda$, we have

$$\left\| \left(\sum_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^{(k+1)}} \left|\sigma_{t,\Omega,h} * g_k\right|^{\lambda} dt/t \right)^{1/\lambda} \right\|_{L^p(\mathbf{R}^n)}^{\lambda} = \sum_{k\in\mathbf{Z}} \int_{\mathbf{R}^n} \int_{\theta^k}^{\theta^{(k+1)}} \left|\sigma_{t,\Omega,h} * g_k(x)\right|^{\lambda} \frac{dt}{t} dx.$$

By Hölder's inequality we have

$$\begin{aligned}
&\left\| \left(\sum_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^{(k+1)}} \left|\sigma_{t,\Omega,h} * g_k\right|^{\lambda} dt/t \right)^{1/\lambda} \right\|_{L^p(\mathbf{R}^n)}^{\lambda} \\
&\leqslant C\,\|h\|_{\Delta_1}^{(\lambda/\lambda')} \left(\|\Omega\|_{L^1(\mathbf{S}^{n-1})}\right)^{(\lambda/\lambda')} \\
&\quad \times \sum_{k\in\mathbf{Z}} \int_{\mathbf{R}^n} \int_{\theta^k}^{\theta^{(k+1)}} \left(\int_{\frac{1}{2}t}^{t} \int_{\mathbf{S}^{n-1}} |g_k(x-sy)|^{\lambda}\, |\Omega(y)|\, |h(s)|\, d\sigma(y) ds/s \right) dt/t dx \\
&\leqslant C(q-1)^{-1}(\gamma-1)^{-1}\,\|h\|_{\Delta_1}^{(\lambda/\lambda'+1)} \left(\|\Omega\|_{L^1(\mathbf{S}^{n-1})}\right)^{(\lambda/\lambda'+1)} \int_{\mathbf{R}^n} \left(\sum_{k\in\mathbf{Z}} |g_k(x)|^{\lambda} \right)
\end{aligned}$$

which in turns implies (2.3) for the case $p=\lambda$. Let us now prove (2.4). By duality, there exist functions $f=f_k(x,t)$ defined on $\mathbf{R}^n\times\mathbf{R}_+$ with $\left\|\left\|\|f_k\|_{L^{\lambda'}([\theta^k,\theta^{k+1}],dt/t)}\right\|_{l^{\lambda'}}\right\|_{L^{p'}} \leqslant 1$ such that

$$\begin{aligned}
&\left\| \left(\sum_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^{(k+1)}} \left|\sigma_{t,\Omega,h} * g_k\right|^{\lambda} dt/t \right)^{1/\lambda} \right\|_{p} \\
&= \int_{\mathbf{R}^n} \sum_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^{(k+1)}} \left(\sigma_{t,\Omega,h} * g_k(x)\right) f_k(x,t) \frac{dt}{t} dx \\
&\leqslant C_p(q-1)^{-\frac{1}{\lambda}}(\gamma-1)^{-\frac{1}{\lambda}} \left\| \left(\sum_{k\in\mathbf{Z}} |g_k|^{\lambda} \right)^{1/\lambda} \right\|_{p} \left\| (H(f))^{1/\lambda'} \right\|_{p'},
\end{aligned} \tag{2.7}$$

where

$$Hf(x) = \sum_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^{(k+1)}} \left|\sigma_{t,\Omega,h} * f_k(x,t)\right|^{\lambda'} dt/t.$$

Now, since $p'>\lambda'$, there exists a function $F\in L^{(p'/\lambda')'}(\mathbf{R}^n)$ such that

$$\|H(f)\|_{p'/\lambda'} = \sum_{k\in\mathbf{Z}} \int_{\mathbf{R}^n} \int_{\theta^k}^{\theta^{(k+1)}} \left|f_k(x,t) * \sigma_{t,\Omega,h}\right|^{\lambda'} \frac{dt}{t} F(x) dx. \tag{2.8}$$

By a similar argument as above, the choice of $f_k(x,t)$ and (2.2) we have

$$\begin{aligned}\|H(f)\|_{p'/\lambda'} &\leqslant C\|h\|_{\Delta_1}^{(\lambda'/\lambda)}\left(\|\Omega\|_{L^1(\mathbf{S}^{n-1})}\right)^{(\lambda'/\lambda)}\\ &\quad\times\int_{\mathbf{R}^n}\sigma^*_{|\Omega|,|h|}(\tilde{F})(-x)\left(\sum_{k\in\mathbf{Z}}\int_{\theta^k}^{\theta^{(k+1)}}|f_k(x,t)|^{\lambda'}dt/t\right)dx\\ &\leqslant C\|h\|_{\Delta_\gamma}^{(\lambda'/\lambda)}\left(\|\Omega\|_{L^1(\mathbf{S}^{n-1})}\right)^{(\lambda'/\lambda)}\left\|\left(\sum_{k\in\mathbf{Z}}\int_{\theta^k}^{\theta^{(k+1)}}|f_k(\cdot,t)|^{\lambda'}dt/t\right)\right\|_{p'/\lambda'}\\ &\quad\times\left\|\sigma^*_{|\Omega|,|h|}(F)\right\|_{(p'/\lambda')'}\\ &\leqslant C(q-1)^{-1}(\gamma-1)^{-1}\|h\|_{\Delta_\gamma}^{(1+\lambda'/\lambda)}\left(\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\right)^{(1+\lambda'/\lambda)}\|F\|_{(p'/\lambda')'}\end{aligned}$$

which when combined with (2.7) yields (2.4). The proof of Lemma 2.3 is complete. □

LEMMA 2.4. *Let $h\in\Delta_\gamma(\mathbf{R}_+)$ for some $2\leqslant\gamma<\infty$, $\Omega\in L^q(\mathbf{S}^{n-1})$ for some $1<q\leqslant 2$ and $\phi=2^{q'}$. Let λ be a real number such that $\lambda'\geqslant\gamma$. Then there exists a positive constant C_p which is independent of q,γ,Ω and h such that the following inequalities*

$$\begin{aligned}&\left\|\left(\sum_{k\in\mathbf{Z}}\int_{\phi^k}^{\phi^{(k+1)}}\left|\sigma_{t,\Omega,h}*g_k\right|^\lambda dt/t\right)^{1/\lambda}\right\|_{L^p(\mathbf{R}^n)}\\ &\leqslant C_p(q-1)^{-\frac{1}{\lambda}}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\left\|\left(\sum_{k\in\mathbf{Z}}|g_k|^\lambda\right)^{1/\lambda}\right\|_{L^p(\mathbf{R}^n)}\end{aligned}\tag{2.9}$$

holds for $1<p<2$ and for arbitrary functions $\{g_k(\cdot)\}_{k\in\mathbf{Z}}$ on $\mathbf{R}^n$.

Proof. As in the proof of Lemma 2.3, by duality, there exist functions $f=f_k(x,t)$ defined on $\mathbf{R}^n\times\mathbf{R}_+$ with $\left\|\left\|\|f_k\|_{L^{\lambda'}([\phi^k,\phi^{k+1}],dt/t)}\right\|_{l^{\lambda'}}\right\|_{L^{p'}}\leqslant 1$ such that

$$\begin{aligned}&\left\|\left(\sum_{k\in\mathbf{Z}}\int_{\phi^k}^{\phi^{(k+1)}}\left|\sigma_{t,\Omega,h}*g_k\right|^\lambda dt/t\right)^{1/\lambda}\right\|_p\\ &=\int_{\mathbf{R}^n}\sum_{k\in\mathbf{Z}}\int_{\phi^k}^{\phi^{(k+1)}}\left(\sigma_{t,\Omega,h}*g_k(x)\right)f_k(x,t)\frac{dt}{t}dx\\ &\leqslant C_p(q-1)^{-\frac{1}{\lambda}}\left\|\left(\sum_{k\in\mathbf{Z}}|g_k|^\lambda\right)^{1/\lambda}\right\|_p\left\|(H(f))^{1/\lambda'}\right\|_{p'},\end{aligned}\tag{2.10}$$

where

$$Hf(x)=\sum_{k\in\mathbf{Z}}\int_{\phi^k}^{\phi^{(k+1)}}\left|\sigma_{t,\Omega,h}*f_k(x,t)\right|^{\lambda'}dt/t.$$

Now, since $p' > \lambda'$, there exists a function $F \in L^{(p'/\lambda')'}(\mathbf{R}^n)$ such that

$$\|H(f)\|_{p'/\lambda'} = \sum_{k\in\mathbf{Z}} \int_{\mathbf{R}^n} \int_{\phi^k}^{\phi^{(k+1)}} \left| f_k(x,t) * \sigma_{t,\Omega,h} \right|^{\lambda'} \frac{dt}{t} F(x) dx. \tag{2.11}$$

By Hölder's inequality we get

$$\begin{aligned}\left|\sigma_{t,\Omega,h} * f_k(x,t)\right|^{\gamma'} \leqslant C &\left(\int_{t/2}^{t} |h(s)|^{\lambda} \frac{ds}{s}\right)^{\frac{\lambda'}{\lambda}} \left(\|\Omega\|_{L^1(\mathbf{S}^{n-1})}\right)^{\frac{\lambda'}{\lambda}} \\ &\times \left(\int_{\phi^k}^{\phi^{(k+1)}} \int_{\mathbf{S}^{n-1}} |\Omega(y)| \, |f_k(x-sy,t)|^{\lambda'} d\sigma(y) \frac{ds}{s}\right).\end{aligned} \tag{2.12}$$

Now, since $\gamma \geqslant 2$ and $\lambda' \geqslant \gamma$ we have $\lambda \leqslant \gamma' \leqslant \gamma$. Thus by Hölder's inequality we have $\left(\int_{t/2}^{t} |h(s)|^{\lambda} \frac{ds}{s}\right)^{\frac{\lambda'}{\lambda}} \leqslant C \|h\|_{\Delta_\gamma}^{\lambda'}$ and hence by (2.11)–(2.12) and Hölder's inequality we get

$$\begin{aligned}\|H(f)\|_{p'/\lambda'} \leqslant C \|h\|_{\Delta_\gamma}^{(\lambda'/\gamma)} &\left(\|\Omega\|_{L^1(\mathbf{S}^{n-1})}\right)^{(\lambda'/\lambda)} \\ &\times \left\| \left(\sum_{k\in\mathbf{Z}} \int_{\phi^k}^{\phi^{(k+1)}} |f_k(\cdot,t)|^{\lambda'} dt/t \right) \right\|_{p'/\lambda'} \left\| \sigma^*_{|\Omega|,1}(F) \right\|_{(p'/\lambda')'}.\end{aligned} \tag{2.13}$$

Now, by switching to polar coordinates we have

$$\begin{aligned}\left|\sigma_{t,\Omega,1}\right| * |f|(x) &\leqslant \int_{\mathbf{S}^{n-1}} |\Omega(y)| \int_{\frac{1}{2}t}^{t} |f(x-sy)| \frac{ds}{s} d\sigma(y) \\ &\leqslant C \int_{\mathbf{S}^{n-1}} |\Omega(y)| M_y f(x) d\sigma(y),\end{aligned}$$

where

$$M_y f(x) = \sup_{\rho\in\mathbf{R}} \frac{1}{\rho} \int_0^{\rho} |f(x-sy)| ds$$

is the Hardy-Littlewood maximal function of f in the direction of y. Since M_y is bounded on $L^p(\mathbf{R}^n)$, $1 < p < \infty$ with bound independent of y, we get

$$\begin{aligned}\left\|\sigma^*_{\Omega,1}(f)\right\|_p &\leqslant C \int_{\mathbf{S}^{n-1}} |\Omega(y)| \left\|M_y(f)\right\|_p d\sigma(y) \\ &\leqslant C \|\Omega\|_{L^1(\mathbf{S}^{n-1})} \|f\|_p \text{ for } 1 < p < \infty.\end{aligned} \tag{2.14}$$

Thus by (2.13) and (2.14) we obtain

$$\|H(f)\|_{p'/\lambda'} \leqslant C \|h\|_{\Delta_\gamma}^{(1+\lambda'/\gamma)} \left(\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\right)^{(1+\lambda'/\lambda)} \|F\|_{(p'/\lambda')'} \tag{2.15}$$

which when combined by (2.10) implies get (2.9). □

LEMMA 2.5. *Let $h \in \Delta_\gamma(\mathbf{R}_+)$ for some $\gamma \geqslant 2$, $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $1 < q \leqslant 2$ and $\phi = 2^{q'}$. Let λ be a real number such that $\lambda' < \gamma$. Then for any p satisfying $\gamma' < p < \infty$ and $f \in L^p(\mathbf{R}^n)$, there exists a positive constant C_p which is independent of $q, \gamma, \lambda, \Omega$ and h such that the inequality*

$$\left\| \left(\sum_{k\in\mathbf{Z}} \int_{\phi^k}^{\phi^{(k+1)}} \left|\sigma_{t,\Omega,h} * g_k\right|^\lambda dt/t \right)^{1/\lambda} \right\|_{L^p(\mathbf{R}^n)} \leqslant C_p (q-1)^{-\frac{1}{\lambda}} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|h\|_{\Delta_\gamma} \left\| \left(\sum_{k\in\mathbf{Z}} |g_k|^\lambda \right)^{1/\lambda} \right\|_{L^p(\mathbf{R}^n)} \tag{2.16}$$

holds for arbitrary functions $\{g_k(\cdot)\}_{k\in\mathbf{Z}}$ on $\mathbf{R}^n$.

Proof. We follow a similar argument as in the proof of Lemma 2.3 in [14]. By a change of variable, we have

$$\left(\sum_{k\in\mathbf{Z}} \int_{\phi^k}^{\phi^{(k+1)}} \left|\sigma_{t,\Omega,h} * g_k\right|^\lambda \frac{dt}{t} \right)^{1/\lambda} \leqslant \left(\sum_{k\in\mathbf{Z}} \int_1^{\phi} \left|\sigma_{\phi^k t,\Omega,h} * g_k\right|^\lambda \frac{dt}{t} \right)^{1/\lambda}. \tag{2.17}$$

By Hölder's inequality we get

$$\left|\sigma_{\phi^k t,\Omega,h} * g_k(x)\right|^{\gamma'} \leqslant C \|h\|_{\Delta_\gamma}^{\gamma'} \left(\|\Omega\|_{L^1(\mathbf{S}^{n-1})} \right)^{\frac{\gamma'}{\gamma}} \left(\int_{\frac{1}{2}\phi^k t}^{\phi^k t} \int_{\mathbf{S}^{n-1}} |\Omega(y)| \, |g_k(x-sy)|^{\gamma'} d\sigma(y) \frac{ds}{s} \right). \tag{2.18}$$

Let $\gamma' < p < \infty$ and let $d = p/\gamma'$. By duality, there is a nonnegative function $f \in L^{d'}(\mathbf{R}^n)$ satisfying $\|f\|_{L^{d'}(\mathbf{R}^n)} \leqslant 1$ such that

$$\left\| \left(\sum_{k\in\mathbf{Z}} \int_1^{\phi} \left|\sigma_{\phi^k t,\Omega,h} * g_k\right|^{\gamma'} \frac{dt}{t} \right)^{1/\gamma'} \right\|_{L^p(\mathbf{R}^n)}^{\gamma'} = \int_{\mathbf{R}^n} \sum_{k\in\mathbf{Z}} \int_1^{\phi} \left|\sigma_{\phi^k t,\Omega,h} * g_k(x)\right|^{\gamma'} \frac{dt}{t} f(x) dx. \tag{2.19}$$

Therefore, by (2.18) and a change of variable we get

$$\left\| \left(\sum_{k\in\mathbf{Z}} \int_1^{\phi} \left|\sigma_{\theta^k t,\Omega,h} * g_k\right|^{\gamma'} \frac{dt}{t} \right)^{1/\gamma'} \right\|_{L^p(\mathbf{R}^n)}^{\gamma'} \leqslant C \|h\|_{\Delta_\gamma}^{\gamma'} \left(\|\Omega\|_{L^1(\mathbf{S}^{n-1})} \right)^{\frac{\gamma'}{\gamma}} \int_{\mathbf{R}^n} \sum_{k\in\mathbf{Z}} |g_k(x)|^{\gamma'} \sigma_{\Omega,1}^* \tilde{f}(-x) dx, \tag{2.20}$$

where $\tilde{f}(x) = f(-x)$. By Hölder's inequality, we obtain

$$\left\| \left(\sum_{k\in\mathbf{Z}} \int_1^{\phi} \left| \sigma_{\phi^k t,\Omega,h} * g_k \right|^{\gamma'} \frac{dt}{t} \right)^{1/\gamma'} \right\|_{L^p(\mathbf{R}^n)}^{\gamma'}$$
$$\leqslant C(q-1)^{-1} \|h\|_{\Delta_\gamma}^{\gamma'} \left(\|\Omega\|_{L^1(\mathbf{S}^{n-1})} \right)^{\frac{\gamma'}{\gamma}} \left\| \left(\sum_{k\in\mathbf{Z}} |g_k|^{\gamma'} \right)^{1/\gamma'} \right\|_{L^p(\mathbf{R}^n)}^{\gamma'} \left\| \sigma_{\Omega,1}^{*} \tilde{f} \right\|_{L^{d'}(\mathbf{R}^n)}.$$

By the last inequality and (2.14) we get

$$\left\| \left(\sum_{k\in\mathbf{Z}} \int_1^{\phi} \left| \sigma_{\phi^k t,\Omega,h} * g_k \right|^{\gamma'} \frac{dt}{t} \right)^{1/\gamma'} \right\|_{L^p(\mathbf{R}^n)}$$
$$\leqslant C(q-1)^{-1/\gamma'} \|h\|_{\Delta_\gamma} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \left\| \left(\sum_{k\in\mathbf{Z}} |g_k|^{\gamma'} \right)^{1/\gamma'} \right\|_{L^p(\mathbf{R}^n)} \tag{2.21}$$

for any $\gamma' < p < \infty$. Now, define the linear operator T on any function $g = g_k(x)$ by

$$T(g_k(x)) = \sigma_{\phi^k t,\Omega,h} * g_k(x).$$

Then by (2.21) we have

$$\left\| \left\| \|T(g)\|_{L^{\gamma'}([1,\phi],\frac{dt}{t})} \right\|_{l^{\gamma'}(\mathbf{Z})} \right\|_{L^p(\mathbf{R}^n)}$$
$$\leqslant C(q-1)^{-1/\gamma'} \|h\|_{\Delta_\gamma} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \left\| \|g\|_{l^{\gamma'}(\mathbf{Z})} \right\|_{L^p(\mathbf{R}^n)} \tag{2.22}$$

for $\gamma' < p < \infty$. On the other hand, by Hölder's inequality and since M_y is bounded on $L^p(\mathbf{R}^n)$, $1 < p < \infty$ with bound independent of y, we get

$$\left\| \sigma_{\Omega,h}^{*}(f) \right\| \leqslant C \|h\|_{\Delta_\gamma} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|f\|_{L^p(\mathbf{R}^n)} \text{ for } \gamma' < p < \infty. \tag{2.23}$$

By (2.23) we have

$$\left\| \sup_{k\in\mathbf{Z}} \sup_{t\in[1,\phi]} \left| \sigma_{\phi^k t,\Omega,h} * g_k \right| \right\|_{L^p(\mathbf{R}^n)}$$
$$\leqslant \left\| \sigma_{\Omega,h}^{*}(\sup_{k\in\mathbf{Z}} |g_k|) \right\|_{L^p(\mathbf{R}^n)}$$
$$\leqslant C_p \|h\|_{\Delta_\gamma} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \left\| \sup_{k\in\mathbf{Z}} |g_k| \right\|_{L^p(\mathbf{R}^n)}. \tag{2.24}$$

and hence we have

$$\begin{aligned}
&\left\|\left\|\|T(g)\|_{L^\infty([1,\phi],\frac{dt}{t})}\right\|_{l^\infty(\mathbf{Z})}\right\|_{L^p(\mathbf{R}^n)} \\
&= \left\|\left\|\|T(g)\|_{L^\infty([1,\phi],dt)}\right\|_{l^\infty(\mathbf{Z})}\right\|_{L^p(\mathbf{R}^n)} \\
&\leqslant C\|h\|_{\Delta_\gamma}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\left\|\|g\|_{l^\infty(\mathbf{Z})}\right\|_{L^p(\mathbf{R}^n)}. \qquad (2.25)
\end{aligned}$$

Therefore, we can interpolate (2.22) and (2.25) (see [16], p. 481) for the vector-valued interpolation) to get (2.16). The lemma is proved. □

3. Proof of main results

Proof of Theorem 1.1. Assume $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $q \in (1,2]$ and $h \in \Delta_\gamma(R_+)$ for some $\gamma \in (1,2]$. By Minkowski's inequality we have

$$\begin{aligned}
\mathscr{M}^{(\lambda)}_{\Omega,h,\rho} f(x) &= \left(\int_0^\infty \left|2^{-k\rho}\sum_{k=0}^{\infty}\sigma_{t2^{-k},\Omega,h} * f(x)\right|^\lambda dt/t\right)^{1/\lambda} \\
&\leqslant \sum_{k=0}^{\infty} 2^{-k\alpha}\left(\left|\sigma_{t2^{-k},\Omega,h} * f(x)\right|^\lambda \frac{dt}{t}\right)^{1/\lambda} = \left(\frac{1}{1-2^{-\alpha}}\right)\mathscr{S}^{(\lambda)}_{\Omega,h} f(x), \quad (3.1)
\end{aligned}$$

where

$$\mathscr{S}^{(\lambda)}_{\Omega,h} f(x) = \left(\int_0^\infty \left|\frac{1}{t}\int_{\frac{1}{2}t<|u|\leqslant t} f(x-u)\frac{\Omega(u')h(|u|)}{|u|^{n-1}}du\right|^\lambda dt/t\right)^{1/\lambda}.$$

Then

$$\mathscr{S}^{(\lambda)}_{\Omega,h} f(x) = \left(\int_0^\infty \left|\sigma_{t,\Omega,h} * f\right|^\lambda \frac{dt}{t}\right)^{\frac{1}{\lambda}}.$$

Let $\theta = 2^{\gamma' q'}$ and let $\{\Phi_j\}_{-\infty}^{\infty}$ be a smooth partition of unity in $(0, \infty)$ adapted to the intervals $I_j = [\theta^{-(j+1)}, \theta^{-(j-1)}]$. More precisely, we require the following:

$$\Phi_j \in C^\infty,\ 0 \leqslant \Phi_j \leqslant 1,\ \sum_j \Phi_j(t) = 1;$$

$$\operatorname{supp}\Phi_j \subseteq I_j;$$

$$\left|\frac{d^s\Phi_j(t)}{dt^s}\right| \leqslant \frac{C}{t^s}$$

where C can be chosen to be independent of θ. Let $\widehat{\Psi_k}(\xi) = \Phi_k(|\xi|)$. Decompose

$$f * \sigma_{t,\Omega,h}(x) = \sum_{j\in\mathbf{Z}}\sum_{k\in\mathbf{Z}}(\Psi_{k+j} * \sigma_{t,\Omega,h} * f)(x)\chi_{[\theta^k,\theta^{(k+1)})}(t) := \sum_{j\in\mathbf{Z}} F_j(x,t)$$

and define

$$\mathscr{S}^{(\lambda)}_{\Omega,h,j}f(x)=\left(\int_0^\infty \left|F_j(x,t)\right|^\lambda \frac{dt}{t}\right)^{\frac{1}{\lambda}}.$$

Then

$$\mathscr{S}^{(\lambda)}_{\Omega,h}(f)\leqslant \sum_{j\in\mathbf{Z}}\mathscr{S}^{(\lambda)}_{\Omega,h,j}(f) \tag{3.2}$$

holds for $f\in\mathscr{S}(\mathbf{R}^n)$.

By (3.1) and (3.2) we notice that (1.3)–(1.4) are proved if we show that

$$\begin{aligned}&\left\|\mathscr{S}^{(\lambda)}_{\Omega,h,j}(f)\right\|_{L^p(\mathbf{R}^n)}\\ &\leqslant C2^{-\eta|j|}(q-1)^{-1/\lambda}(\gamma-1)^{-1/\lambda}\left\|h\right\|_{\Delta_\gamma}\left\|\Omega\right\|_{L^q(\mathbf{S}^{n-1})}\left\|f\right\|_{\dot{F}^{0,\lambda}_p(\mathbf{R}^n)}\end{aligned} \tag{3.3}$$

for $\lambda\leqslant p<\infty$; and

$$\begin{aligned}&\left\|\mathscr{S}^{(\lambda)}_{\Omega,h,j}(f)\right\|_{L^p(\mathbf{R}^n)}\\ &\leqslant C2^{-\eta|j|}(q-1)^{-1}(\gamma-1)^{-1}\left\|h\right\|_{\Delta_\gamma}\left\|\Omega\right\|_{L^q(\mathbf{S}^{n-1})}\left\|f\right\|_{\dot{F}^{0,\lambda}_p(\mathbf{R}^n)}\end{aligned} \tag{3.4}$$

for $1<p<\lambda$ for some positive constants C and η. Before starting proving (3.3)–(3.4) we need to get some necessary estimates.

First, by definition,

$$\hat{\sigma}_{t,\Omega,h}(\xi)=\frac{1}{t^\rho}\int_{\frac{1}{2}t}^{t}\int_{\mathbf{S}^{n-1}}e^{-is\xi\cdot x}\Omega(x)\frac{h(s)}{s^{1-\rho}}d\sigma(x)\,ds$$

It is easy to see that

$$\left|\hat{\sigma}_{t,\Omega,h}(\xi)\right|\leqslant C\left\|h\right\|_{\Delta_\gamma}\left\|\Omega\right\|_{L^1(\mathbf{S}^{n-1})}\qquad \text{for } t\in\mathbf{R}_+, \tag{3.5}$$

which in turn implies

$$\int_{\theta^k}^{\theta^{(k+1)}}\left|\hat{\sigma}_{t,\Omega,h}(\xi)\right|^2\frac{dt}{t}\leqslant C(q-1)^{-1}(\gamma-1)^{-1}\left\|h\right\|^2_{\Delta_\gamma}\left\|\Omega\right\|^2_{L^q(\mathbf{S}^{n-1})} \tag{3.6}$$

for some positive constant independent of q and γ.

Second, by (1.1) we have

$$\left|\hat{\sigma}_{t,\Omega,h}(\xi)\right|\leqslant\int_{\frac{1}{2}t}^{t}\int_{\mathbf{S}^{n-1}}\left|e^{-is\xi\cdot x}-1\right|\left|\Omega(x)\right|\left|h(s)\right|\frac{ds}{s}\leqslant C\left|\xi t\right|\left\|h\right\|_{\Delta_\gamma}\left\|\Omega\right\|_{L^1(\mathbf{S}^{n-1})}$$

which easily implies

$$\int_{\theta^k}^{\theta^{(k+1)}}\left|\hat{\sigma}_{t,\Omega,h}(\xi)\right|^2\frac{dt}{t}\leqslant C(q-1)^{-1}(\gamma-1)^{-1}\theta^2\left\|h\right\|^2_{\Delta_\gamma}\left\|\Omega\right\|^2_{L^q(\mathbf{S}^{n-1})}\left|\theta^k\xi\right|^2.$$

By the last estimate and (3.6) we get

$$\int_{\theta^k}^{\theta^{(k+1)}} \left|\hat{\sigma}_{t,\Omega,h}(\xi)\right|^2 \frac{dt}{t} \leqslant C(q-1)^{-1}(\gamma-1)^{-1}\|h\|_{\Delta_\gamma}^2\|\Omega\|_{L^q(\mathbf{S}^{n-1})}^2\left|\theta^k\xi\right|^{\frac{2}{\gamma'q'}}. \tag{3.7}$$

Third, by the proof of Corollary 4.1 of [10],

$$\left|\hat{\sigma}_{t,\Omega,h}(\xi)\right| \leqslant C|t\xi|^{-\beta/2}\|h\|_{\Delta_\gamma}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}$$

for some positive constant C and β with $\beta q' < 1$ which in turn implies

$$\int_{\theta^k}^{\theta^{(k+1)}} \left|\hat{\sigma}_{t,\Omega,h}(\xi)\right|^2 \frac{dt}{t} \leqslant C(q-1)^{-1}(\gamma-1)^{-1}\left|\theta^k\xi\right|^{-\beta}\|h\|_{\Delta_\gamma}^2\|\Omega\|_{L^q(\mathbf{S}^{n-1})}^2.$$

Therefore, by combining the last estimate with the trivial estimate (3.6) we obtain

$$\int_{\theta^k}^{\theta^{(k+1)}} \left|\hat{\sigma}_{t,\Omega,h}(\xi)\right|^2 \frac{dt}{t} \leqslant C(q-1)^{-1}(\gamma-1)^{-1}\|h\|_{\Delta_\gamma}^2\|\Omega\|_{L^q(\mathbf{S}^{n-1})}^2\left|\theta^k\xi\right|^{-\frac{\beta}{\gamma'q'}}. \tag{3.8}$$

We are now ready to prove (3.3). First we start with the case $p=\lambda=2$. By Plancherel's theorem we have

$$\begin{aligned}
\left\|\mathscr{S}^{(2)}_{\Omega,h,j}(f)\right\|_{L^2(\mathbf{R}^n)}^2 &= \sum_{k\in\mathbf{Z}}\int_{\mathbf{R}^n}\int_{\theta^k}^{\theta^{(k+1)}}\left|\Psi_{k+j}*\sigma_{t,\Omega,h}*f(x)\right|^2\frac{dt}{t}dx \\
&\leqslant \sum_{k\in\mathbf{Z}}\int_{I_{j+k}}\left(\int_{\theta^k}^{\theta^{(k+1)}}\left|\hat{\sigma}_{t,\Omega,h}(\xi)\right|^2\frac{dt}{t}\right)\left|\hat{f}(\xi)\right|^2 d\xi \\
&\leqslant C(q-1)^{-1}(\gamma-1)^{-1}\|h\|_{\Delta_\gamma}^2\|\Omega\|_{L^q(\mathbf{S}^{n-1})}^2 \\
&\quad\times\sum_{k\in\mathbf{Z}}\int_{I_{j+k}}\min\left(\left|\theta^k\xi\right|^{\frac{2}{\gamma'q'}},\left|\theta^k\xi\right|^{-\frac{\beta}{\gamma'q'}}\right)\left|\hat{f}(\xi)\right|^2 d\xi \\
&\leqslant C(q-1)^{-1}(\gamma-1)^{-1}\|h\|_{\Delta_\gamma}^2\|\Omega\|_{L^q(\mathbf{S}^{n-1})}^2 2^{-\eta|j|}\sum_{k\in\mathbf{Z}}\int_{I_{j+k}}\left|\hat{f}(\xi)\right|^2 d\xi \\
&\leqslant C(q-1)^{-1}(\gamma-1)^{-1}\|h\|_{\Delta_\gamma}^2\|\Omega\|_{L^q(\mathbf{S}^{n-1})}^2 2^{-\eta|j|}\|f\|_{L^2(\mathbf{R}^n)}^2.
\end{aligned}$$

Therefore,

$$\left\|\mathscr{S}^{(2)}_{\Omega,h,j}(f)\right\|_{L^2(\mathbf{R}^n)} \leqslant C(q-1)^{-\frac{1}{2}}(\gamma-1)^{-\frac{1}{2}}\|h\|_{\Delta_\gamma}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}2^{-\frac{\eta}{2}|j|}\|f\|_{L^2(\mathbf{R}^n)} \tag{3.9}$$

and since $\|f\|_{F_2^{0,2}(\mathbf{R}^n)} = \|f\|_{L^2(\mathbf{R}^n)}$ we get (3.3) for the case $p=\lambda=2$.

Now Lemma 2.3 we have

$$\left\|\mathscr{S}^{(\lambda)}_{\Omega,h,j}(f)\right\|_{L^p(\mathbf{R}^n)} \leqslant C(q-1)^{-\frac{1}{\lambda}}(\gamma-1)^{-\frac{1}{\lambda}}\|h\|_{\Delta_\gamma}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\|f\|_{\dot{F}_p^{0,\lambda}(\mathbf{R}^n)} \tag{3.10}$$

for $\lambda \leqslant p < \infty$ and

$$\left\|\mathscr{S}^{(\lambda)}_{\Omega,h,j}(f)\right\|_{L^p(\mathbf{R}^n)} \leqslant C(q-1)^{-1}(\gamma-1)^{-1}\|h\|_{\Delta_\gamma}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\|f\|_{\dot{F}_p^{0,\lambda}(\mathbf{R}^n)} \tag{3.11}$$

for $p<\lambda$. By interpolating (3.9) with (3.10)–(3.11) we get (3.3)–(3.4) and hence the proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2. Assume $\Omega\in L^q(\mathbf{S}^{n-1})$ for some $q\in(1,2]$ and $h\in\Delta_\gamma(R_+)$ for some $\gamma>2$. We argue as in the proof of Theorem 1and in this case θ is replaced by ϕ. So Theorem 2 is proved if we show that

$$\left\|\mathscr{S}^{(\lambda)}_{\Omega,h,j}(f)\right\|_{L^p(\mathbf{R}^n)}\leqslant C2^{-\eta|j|}(q-1)^{-1/\lambda}\|h\|_{\Delta_\gamma}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\|f\|_{\dot{F}^{0,\lambda}_p(\mathbf{R}^n)} \tag{3.12}$$

for $1<p<\lambda$ if $2<\gamma<\infty$ and $\lambda'\geqslant\gamma$, and

$$\left\|\mathscr{S}^{(\lambda)}_{\Omega,h,j}(f)\right\|_{L^p(\mathbf{R}^n)}\leqslant C2^{-\alpha|j|}(q-1)^{-1/\lambda}\|h\|_{\Delta_\gamma}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\|f\|_{\dot{F}^{0,\lambda}_p(\mathbf{R}^n)} \tag{3.13}$$

for $\gamma'<p<\infty$ if $2<\gamma\leqslant\infty$ and $\lambda'<\gamma$. By Lemmas 2.4 and 2.5 we have

$$\left\|\mathscr{S}_{\Omega,\lambda,h,j}(f)\right\|_{L^p(\mathbf{R}^n)}\leqslant C(q-1)^{-1/\lambda}\|h\|_{\Delta_\gamma}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\|f\|_{\dot{F}^{0,\lambda}_p(\mathbf{R}^n)} \tag{3.14}$$

for $1<p<\lambda$ if $2<\gamma<\infty$ and $\lambda'\geqslant\gamma$, and

$$\left\|\mathscr{S}_{\Omega,\lambda,h,j}(f)\right\|_{L^p(\mathbf{R}^n)}\leqslant C(q-1)^{-1/\lambda}\|h\|_{\Delta_\gamma}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\|f\|_{\dot{F}^{0,\lambda}_p(\mathbf{R}^n)} \tag{3.15}$$

for $\gamma'<p<\infty$ if $2<\gamma\leqslant\infty$ and $\lambda'<\gamma$. As above, by interpolation between (3.9) and (3.14)–(3.15) we get (3.12)–(3.13). Theorem 1.2 is proved. □

Proof of Theorem 1.3 and Theorem 1.4. A proof of each of these theorems follows by Theorems 1.1 and 1.2 and an extrapolation argument. For more details, see [4] and [23]. □

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(Received April 2, 2016)

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