AN INTRINSIC SQUARE FUNCTION ON WEIGHTED
HERZ SPACES WITH VARIABLE EXPONENT

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Abstract. We define new generalized Herz spaces having weight and variable exponent, that is, weighted Herz spaces with variable exponent. We prove the boundedness of an intrinsic square function on those spaces under proper assumptions on each exponent and weight.

1. Introduction

The boundedness of the Hardy–Littlewood maximal operator M on function spaces is very important in real analysis because it realizes boundedness of many other operators, for example, singular integrals, fractional integrals, and commutators involving BMO functions. Muckenhoupt [21] has established the theory on weights called the Muckenhoupt A_p theory in the study of weighted function spaces and greatly developed real analysis.

On the other hand, the theory on function spaces with variable exponent has been rapidly developed after the work [20] where Kováčik and Rákosník have clarified fundamental properties of Lebesgue spaces with variable exponent. On spaces with variable exponent, the boundedness of the operator M is also a notable problem. As sufficient conditions for the boundedness the log-Hölder continuous conditions have been established and well known now ([2, 3, 7, 8, 15]).

Recently a generalization of the Muceknhoupt weights in terms of variable exponent has been studied. Diening and Hästö [9] have initially defined the new class of weights $A_{p(\cdot)}$ and proved the equivalence between the $A_{p(\cdot)}$ condition and the boundedness of M on weighted Lebesgue spaces with variable exponent. The equivalence has been independently proved by Cruz-Uribe, Fiorenza and Neugebauer [4]. The first author and his collaborators have studied the relation between $A_{p(\cdot)}$ and the wavelet theory ([14, 16]).

An intrinsic square function is one of the remarkable operators in modern real analysis. Many researchers have studied characterizations of general function spaces via intrinsic functions ([12, 25, 26, 27, 28, 29, 30, 31]). In particular we focus on the work [27] by Wang where the boundedness of some intrinsic functions including S_β

on weighted Herz spaces has been proved under proper assumptions on every exponent and weight. Our aim in this paper is to extend the boundedness of the intrinsic square function S_β to the variable exponent case. We will define weighted Herz spaces with variable exponent having variable integral exponent $p(\cdot)$ and weight w, and prove the boundedness of S_β on those spaces based on fundamental facts on general Banach function spaces.

Throughout this paper we will use the following notation.

1. The symbol C always denotes a positive constant independent of main parameters. We remark that the value of C may be different from one occurrence to another.

2. Given a measurable set $S \subset \mathbb{R}^n$, we denote the Lebesgue measure of S by $|S|$. In addition, χ_S means the characteristic function of S.

3. A ball is always an open ball in $\mathbb{R}^n$, that is, a ball B is a set given by

$$B := \{y \in \mathbb{R}^n : |x - y| < r\},$$

using a point $x \in \mathbb{R}^n$ and a positive number r.

2. Preliminaries

2.1. Lebesgue spaces with variable exponent

We first define Lebesgue space with variable exponent.

DEFINITION 1. Let $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ be a measurable function. The Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ with variable exponent $p(\cdot)$ is the set of all complex-valued measurable functions f defined on $\mathbb{R}^n$ satisfying

$$\rho_p(f) := \int_{\mathbb{R}^n} |f(x)|^{p(x)} dx < \infty.$$

It is known (cf. [15, 20]) that the Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ becomes a Banach space equipped with a norm given by

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} := \inf \left\{ \lambda > 0 : \rho_p \left(\frac{f}{\lambda} \right) \leqslant 1 \right\}.$$

The measurable function $p(\cdot)$ is called a variable exponent in variable exponent analysis. In order to state variable exponent spaces deeply we define some notations on variable exponents.

DEFINITION 2.

1. Given a measurable function $r(\cdot) : \mathbb{R}^n \to (0, \infty)$, we write

$$r_+ := \|r(\cdot)\|_{L^\infty(\mathbb{R}^n)}, \quad r_- := \left\{ \left(\frac{1}{r(\cdot)} \right)_+ \right\}^{-1}.$$

2. The set $\mathscr{P}(\mathbb{R}^n)$ consists of all variable exponents $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ satisfying $1 < p_- \leqslant p_+ < \infty$.

3. A measurable function $r(\cdot) : \mathbb{R}^n \to (0, \infty)$ is said to be globally log-Hölder continuous if it satisfies the following two inequalities

$$|r(x) - r(y)| \leqslant \frac{C}{-\log(|x-y|)} \quad (|x-y| \leqslant 1/2),$$

$$|r(x) - r_\infty| \leqslant \frac{C}{\log(e + |x|)} \quad (x \in \mathbb{R}^n)$$

for some real constant r_∞. The set $LH(\mathbb{R}^n)$ consists of all globally log-Hölder continuous functions.

The globally log-Hölder continuous conditions are famous because they ensure the boundedness of the Hardy–Littlewood maximal operator M, defined by

$$Mf(x) := \sup_{B : \text{ball}, x \in B} \int_B |f(y)|\, dy,$$

on Lebesgue spaces with variable exponent. Hence we often consider those conditions as standard assumptions in the study of function spaces with variable exponents (cf. [2, 3, 7, 8, 15]).

2.2. Weighted Banach function spaces

We define Banach function space and state fundamental properties of it based on the book [1] by Bennett and Sharpley. For further informations on the theory of Banach function space including the proof of Lemma 1 below we refer to the book. We additionally show some properties of Banach function spaces in terms of boundedness of the Hardy–Littlewood maximal operator. We will also consider the weighted case based on the paper [18] by Karlovich and Spitkovsky.

DEFINITION 3. Let $\mathscr{M}$ be the set of all complex-valued measurable functions defined on $\mathbb{R}^n$, and X a linear subspace of $\mathscr{M}$.

1. The space X is said to be a Banach function space if there exists a functional $\|\cdot\|_X : \mathscr{M} \to [0, \infty]$ satisfying the following properties: Let $f, g, f_j \in \mathscr{M}$ $(j = 1, 2, \cdots)$, then

   (a) $f \in X$ holds if and only if $\|f\|_X < \infty$.

   (b) Norm property:

      i. Positivity: $\|f\|_X \geqslant 0$.
      ii. Strict positivity: $\|f\|_X = 0$ holds if and only if $f(x) = 0$ for almost every $x \in \mathbb{R}^n$.
      iii. Homogeneity: $\|\lambda f\|_X = |\lambda| \cdot \|f\|_X$ holds for all complex numbers λ.

    iv. Triangle inequality: $\|f+g\|_X \leqslant \|f\|_X + \|g\|_X$.

(c) Symmetry: $\|f\|_X = \|\,|f|\,\|_X$.

(d) Lattice property: If $0 \leqslant g(x) \leqslant f(x)$ for almost every $x \in \mathbb{R}^n$, then $\|g\|_X \leqslant \|f\|_X$.

(e) Fatou property: If $0 \leqslant f_j(x) \leqslant f_{j+1}(x)$ for all j and $f_j(x) \to f(x)$ as $j \to \infty$ for almost every $x \in \mathbb{R}^n$, then $\lim\limits_{j \to \infty} \|f_j\|_X = \|f\|_X$.

(f) For every measurable set $F \subset \mathbb{R}^n$ such that $|F| < \infty$, $\|\chi_F\|_X$ is finite. Additionally there exists a constant $C_F > 0$ depending only on F such that for all $h \in X$,

$$\int_F |h(x)|\,dx \leqslant C_F \|h\|_X.$$

2. Suppose that X is a Banach function space equipped with a norm $\|\cdot\|_X$. The associated space X' is defined by

$$X' := \{f \in \mathcal{M} \,:\, \|f\|_{X'} < \infty\},$$

where

$$\|f\|_{X'} := \sup_g \left\{ \left| \int_{\mathbb{R}^n} f(x)g(x)\,dx \right| \,:\, \|g\|_X \leqslant 1 \right\}.$$

LEMMA 1. *Let X be a Banach function space. Then the following hold:*

1. *The associated space X' is also a Banach function space.*

2. *(The Lorentz–Luxemberg theorem.) $(X')' = X$ holds, in particular, the norms $\|\cdot\|_{(X')'}$ and $\|\cdot\|_X$ are equivalent.*

3. *(The generalized Hölder inequality.) If $f \in X$ and $g \in X'$, then we have*

$$\int_{\mathbb{R}^n} |f(x)g(x)|\,dx \leqslant \|f\|_X \|g\|_{X'}.$$

    Kováčik and Rákosník [20] have proved that the generalized Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ with variable exponent $p(\cdot)$ is a Banach function space and the associate space is $L^{p'(\cdot)}(\mathbb{R}^n)$ with norm equivalence, where $p'(\cdot)$ is the conjugate exponent given by $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$.

    If we assume some conditions for boundedness of the Hardy–Littlewood maximal operator M on X, then the norm $\|\cdot\|_X$ has properties similar to the Muckenhoupt weights.

    LEMMA 2. *Let X be a Banach function space. Suppose that the Hardy–Littlewood maximal operator M is weakly bounded on X, that is,*

$$\|\chi_{\{Mf>\lambda\}}\|_X \leqslant C\lambda^{-1}\|f\|_X \tag{1}$$

is true for all $f \in X$ and $\lambda > 0$. Then we have

$$\sup_{B:\mathrm{ball}} \frac{1}{|B|} \|\chi_B\|_X \|\chi_B\|_{X'} < \infty. \tag{2}$$

The proof of Lemma 2 is found in the first author's paper [14, Lemmas 2.4 and 2.5] and [17, Lemmas G' and H].

REMARK 1. If M is bounded on X, that is,

$$\|Mf\|_X \leqslant C\|f\|_X$$

holds for all $f \in X$, then we can easily check that (1) holds. On the other hand, if M is bounded on the associate space X', then Lemma 1 shows that (2) is true.

Below we define weighted Banach function space and give some properties of it. Let X be a Banach function space. The set $X_{\text{loc}}(\mathbb{R}^n)$ consists of all measurable function f such that $f\chi_E \in X$ for any compact set E with $|E| < \infty$. Given a function W such that $0 < W(x) < \infty$ for almost every $x \in \mathbb{R}^n$, $W \in X_{\text{loc}}(\mathbb{R}^n)$ and $W^{-1} \in (X')_{\text{loc}}(\mathbb{R}^n)$, we define the weighted Banach function space

$$X(\mathbb{R}^n, W) := \{f \in \mathscr{M} \,:\, fW \in X\}.$$

Then the following hold.

LEMMA 3.

1. *The weighted Banach function space $X(\mathbb{R}^n, W)$ is a Banach function space equipped the norm*
$$\|f\|_{X(\mathbb{R}^n, W)} := \|fW\|_X.$$

2. *The associate space of $X(\mathbb{R}^n, W)$ is also a Banach function space and equals to $X'(\mathbb{R}^n, W^{-1})$.*

The properties above naturally arise from those of usual Banach function spaces and the proof is found in [18].

2.3. Muckenhoupt weights with variable exponent

A locally integrable and positive function defined on $\mathbb{R}^n$ is called a weight. We define fundamental classes of weights known as the Muckenhoupt classes in terms of variable exponent.

DEFINITION 4. Suppose $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$. A weight w is said to be an $A_{p(\cdot)}$ weight if

$$\sup_{B:\text{ball}} \frac{1}{|B|} \|w^{1/p(\cdot)}\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|w^{-1/p(\cdot)}\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} < \infty.$$

The set $A_{p(\cdot)}$ consists of all $A_{p(\cdot)}$ weights.

REMARK 2. Our symbol $A_{p(\cdot)}$ differs from that in the papers [4, 9]. If $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ equals to a constant $p \in (1, \infty)$, then the definition above is equivalent to the well known Muckenhoupt A_p weights ([21]).

We shall give the definitions of the Muckenhoupt classes A_p with $p = 1, \infty$.

DEFINITION 5.

1. A weight w is said to be a Muckenhoupt A_1 weight if $Mw(x) \leqslant Cw(x)$ holds for almost every $x \in \mathbb{R}^n$. The set A_1 consists of all Muckenhoupt A_1 weights. For every $w \in A_1$, the finite value

$$[w]_{A_1} := \sup_{B:\text{ball}} \left\{ \frac{1}{|B|} \int_B w(x)\,dx \cdot \|w^{-1}\|_{L^\infty(B)} \right\}$$

is said to be a Muckenhoupt A_1 constant.

2. A weight belonging to the set

$$A_\infty := \bigcup_{1 < p < \infty} A_p$$

is said to be a Muckenhoupt A_∞ weight.

REMARK 3.

1. We note that if $w \in A_1$, then

$$\frac{1}{|B|} \int_B w(x)\,dx \leqslant [w]_{A_1} \inf_{x \in B} w(x)$$

holds for all balls B.

2. It is known that the monotone property $A_p \subset A_q \subset A_\infty$ holds for every constants $1 \leqslant p < q < \infty$.

We will use a classical result on the Muckenhoupt weights. Below we write

$$w(S) := \int_S w(x)\,dx$$

for a measurable set S and a weight w.

LEMMA 4. (Chapter 7 in [10]) *If $w \in A_1$, then there exist positive constants $\delta < 1$ and C depending only on n and $[w]_{A_1}$ such that for all balls B and all measurable sets $E \subset B$,*

$$\frac{w(E)}{w(B)} \leqslant C \left(\frac{|E|}{|B|} \right)^\delta.$$

Diening and Hästö [9] have pointed out that Definition 4 does not directly imply the monotone property of the Muckenhoupt class $A_{p(\cdot)}$. In order to obtain the property they have generalized the Muckenhoupt class as follows:

DEFINITION 6. (Diening and Hästö [9]) Suppose $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$. A weight w is said to be an $\tilde{A}_{p(\cdot)}$ weight if

$$\sup_{B:\text{ball}} |B|^{-p_B} \|w\chi_B\|_{L^1(\mathbb{R}^n)} \|w^{-1}\chi_B\|_{L^{p'(\cdot)/p(\cdot)}(\mathbb{R}^n)} < \infty,$$

where p_B is the harmonic average of $p(\cdot)$ over B, namely,

$$p_B := \left(\frac{1}{|B|} \int_B \frac{1}{p(x)} dx \right)^{-1}.$$

The set $\tilde{A}_{p(\cdot)}$ consists of all $\tilde{A}_{p(\cdot)}$ weights.

Based on the definition $\tilde{A}_{p(\cdot)}$ Diening and Hästö [9, Lemma 3.1] have proved the next monotone property.

THEOREM 1. *Suppose* $p(\cdot), q(\cdot) \in \mathscr{P}(\mathbb{R}^n) \cap LH(\mathbb{R}^n)$ *and* $p(\cdot) \leqslant q(\cdot)$. *Then we have*

$$A_1 \subset A_{p_-} \subset \tilde{A}_{p(\cdot)} \subset \tilde{A}_{q(\cdot)} \subset A_{q_+} \subset A_\infty.$$

Before we state the relation between the generalized Muckenhoupt condition and boundedness of the Hardy–Littlewood maximal operator, we define explicitly weighted Lebesgue spaces with variable exponent.

DEFINITION 7. Let $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ and w be a weight. The weighted Lebesgue space with variable exponent $L^{p(\cdot)}(w)$ is defined by

$$L^{p(\cdot)}(w) := L^{p(\cdot)}(\mathbb{R}^n, w^{1/p(\cdot)}).$$

Namely the space $L^{p(\cdot)}(w)$ is a Banach function space equipped with the norm

$$\|f\|_{L^{p(\cdot)}(w)} := \|fw^{1/p(\cdot)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

THEOREM 2. *Suppose* $p(\cdot) \in \mathscr{P}(\mathbb{R}^n) \cap LH(\mathbb{R}^n)$. *Then the following three conditions are equivalent:*

(A) $w \in A_{p(\cdot)}$.

(B) $w \in \tilde{A}_{p(\cdot)}$.

(C) The Hardy–Littlewood maximal operator is bounded on the weighted variable Lebesgue space $L^{p(\cdot)}(w)$.

Cruz-Uribe, Fiorenza and Neugebauer [4] have proved $(A) \Leftrightarrow (C)$. On the other hand, Diening and Hästö [9] have proved $(B) \Leftrightarrow (C)$. By Theorem 2 we can identify $A_{p(\cdot)}$ and $\tilde{A}_{p(\cdot)}$, provided that $p(\cdot) \in \mathscr{P}(\mathbb{R}^n) \cap LH(\mathbb{R}^n)$. Moreover Theorem 1 gives us the following monotone property.

COROLLARY 1. *Suppose $p(\cdot), q(\cdot) \in \mathscr{P}(\mathbb{R}^n) \cap LH(\mathbb{R}^n)$ and $p(\cdot) \leqslant q(\cdot)$. Then we have*

$$A_1 \subset A_{p_-} \subset A_{p(\cdot)} \subset A_{q(\cdot)} \subset A_{q_+} \subset A_\infty.$$

REMARK 4. Based on Diening and Hästö [9, Proposition 3.1], we can construct weights belonging to $A_{p(\cdot)}$ as follows: Let $p(\cdot) \in \mathscr{P}(\mathbb{R}^n) \cap LH(\mathbb{R}^n)$ and $w_1, w_2 \in A_1$. Then we have that $w_1 w_2^{1-p(\cdot)} \in A_{p(\cdot)}$.

3. Main result

3.1. Definition of the intrinsic function and Herz spaces

We first define the intrinsic square function $S_\beta f(x)$.

DEFINITION 8. Given a point $x \in \mathbb{R}^n$, we define a set

$$\Gamma(x) := \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < t\},$$

where $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0, \infty)$. Let $0 < \beta \leqslant 1$ be a constant. The set $\mathscr{C}_\beta$ consists of all functions φ defined on $\mathbb{R}^n$ such that

1. $\operatorname{supp} \varphi \subset \{|x| \leqslant 1\}$,

2. $\int_{\mathbb{R}^n} \varphi(x) \, dx = 0$,

3. $|\varphi(x) - \varphi(x')| \leqslant |x - x'|^\beta$ for $x, x' \in \mathbb{R}^n$.

For every $(y,t) \in \mathbb{R}^{n+1}_+$ we write $\varphi_t(y) = t^{-n} \varphi\left(\frac{y}{t}\right)$. Then we define a maximal function for $f \in L^1_{\text{loc}}(\mathbb{R}^n)$,

$$A_\beta f(y,t) := \sup_{\varphi \in \mathscr{C}_\beta} |f * \varphi_t(y)| \quad \left((y,t) \in \mathbb{R}^{n+1}_+\right).$$

Using above, we define the intrinsic square function with order β by

$$S_\beta f(x) := \left(\int\!\!\int_{\Gamma(x)} A_\beta f(y,t)^2 \, \frac{dy \, dt}{t^{n+1}}\right)^{1/2}.$$

In order to define weighted Herz spaces with variable exponent, we use a local weighted Lebesgue spaces with variable exponent.

DEFINITION 9. Let $\Omega \subset \mathbb{R}^n$ be a measurable set, $p(\cdot) : \Omega \to [1, \infty)$ a measurable function and w a positive and locally integrable function defined on Ω. The set $L^{p(\cdot)}_{\text{loc}}(\Omega, w^{1/p(\cdot)})$ consists of all functions f satisfying the following condition: for all measurable subsets $E \subset \Omega$ there exists a constant $\lambda > 0$ such that

$$\int_E \left|\frac{f(x)}{\lambda}\right|^{p(x)} w(x) \, dx < \infty.$$

We additionally use the following notation.

1. For every integer k, we write $B_k := \{|x| \leqslant 2^k\}$, $D_k := B_k \setminus B_{k-1}$ and $\chi_k := \chi_{D_k}$.

2. For every non-negative integer m, we write $C_m := D_m$ if $m \geqslant 1$ and $C_0 := B_0$.

Now we are ready to define the Herz spaces.

DEFINITION 10. *Let* $\alpha \in \mathbb{R}$, $0 < q < \infty$, $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ *and* w *be a weight.*

1. *The homogeneous weighted Herz space* $\dot{K}_{p(\cdot)}^{\alpha,q}(w)$ *with variable exponent is defined by*

$$\dot{K}_{p(\cdot)}^{\alpha,q}(w) := \{f \in L_{\mathrm{loc}}^{p(\cdot)}(\mathbb{R}^n \setminus \{0\}, w^{1/p(\cdot)}) : \|f\|_{\dot{K}_{p(\cdot)}^{\alpha,q}(w)} < \infty\},$$

where

$$\|f\|_{\dot{K}_{p(\cdot)}^{\alpha,q}(w)} := \left(\sum_{k=-\infty}^{\infty} 2^{\alpha k q} \|f\chi_k\|_{L^{p(\cdot)}(w)}^q \right)^{1/q}.$$

2. *The non-homogeneous weighted Herz space* $K_{p(\cdot)}^{\alpha,q}(w)$ *with variable exponent is defined by*

$$K_{p(\cdot)}^{\alpha,q}(w) := \{f \in L_{\mathrm{loc}}^{p(\cdot)}(\mathbb{R}^n, w^{1/p(\cdot)}) : \|f\|_{K_{p(\cdot)}^{\alpha,q}(w)} < \infty\},$$

where

$$\|f\|_{K_{p(\cdot)}^{\alpha,q}(w)} := \left(\sum_{m=0}^{\infty} 2^{\alpha m q} \|f\chi_{C_m}\|_{L^{p(\cdot)}(w)}^q \right)^{1/q}.$$

3.2. Key lemmas

Lemmas 5 and 6 below have been proved by the first author [13, Proposition 2.4] in the case $X = L^{p(\cdot)}(\mathbb{R}^n)$. His proof of Lemma 6 is due to Diening's work [8]. Recently a self-contained proof based on the Rubio de Francia algorithm [22, 23, 24] has given by Cruz-Uribe, Hernández and Martell [5, Proof of Lemma 3.3]. Based on [13, 5] we will give the complete proofs of those lemmas.

LEMMA 5. *Let* X *be a Banach function space. Suppose that the Hardy–Littlewood maximal operator* M *is weakly bounded on* X. *Then we have that for all balls* $B \subset \mathbb{R}^n$ *and all measurable sets* $E \subset B$,

$$\frac{|E|}{|B|} \leqslant C \frac{\|\chi_E\|_X}{\|\chi_B\|_X}. \tag{3}$$

Proof. Take a ball B, a measurable set $E \subset B$ and a number $0 < \lambda < |E|/|B|$ arbitrarily. Then we see that $M(\chi_E)(x) > \lambda$ for almost every $x \in B$. Hence we have

$$\|\chi_B\|_X \leqslant \|\chi_{\{M(\chi_E) > \lambda\}}\|_X \leqslant C\lambda^{-1}\|\chi_E\|_X.$$

Therefore we get inequality (3) because $0 < \lambda < |E|/|B|$ is arbitrary. $\square$

LEMMA 6. *Let X be a Banach function space. Suppose that M is bounded on the associate space X'. Then there exists a constant $0 < \delta < 1$ such that for all balls $B \subset \mathbb{R}^n$ and all measurable sets $E \subset B$,*

$$\frac{\|\chi_E\|_X}{\|\chi_B\|_X} \leqslant C\left(\frac{|E|}{|B|}\right)^\delta. \tag{4}$$

Proof. Let $A := \|M\|_{X' \to X'}$ and define a function

$$Rg(x) := \sum_{k=0}^{\infty} \frac{M^k g(x)}{(2A)^k} \quad (g \in X'), \tag{5}$$

where

$$M^k g := \begin{cases} |g| & (k=0), \\ Mg & (k=1), \\ M(M^{k-1}g) & (k \geqslant 2). \end{cases}$$

For every $g \in X$, the function Rg satisfies the following properties:

1. $|g(x)| \leqslant Rg(x)$ for almost every $x \in \mathbb{R}^n$.

2. $\|Rg\|_{X'} \leqslant 2\|g\|_{X'}$, namely the operator R is bounded on X'.

3. $M(Rg)(x) \leqslant 2ARg(x)$, that is, Rg is a Muckenhoupt A_1 weight such that $[Rg]_{A_1} \leqslant 2A$.

Thus by applying Lemma 4 to Rg, we can take positive constants C and $\delta < 1$ so that for all balls B and all measurable sets $E \subset B$,

$$\frac{Rg(E)}{Rg(B)} \leqslant C\left(\frac{|E|}{|B|}\right)^\delta.$$

Now we fix $g \in X'$ with $\|g\|_{X'} \leqslant 1$ arbitrarily. By virtue of generalized Hölder's inequality we have

$$\int_{\mathbb{R}^n} |\chi_E(x)g(x)|\,dx \leqslant Rg(E)$$

$$\leqslant C\left(\frac{|E|}{|B|}\right)^\delta \cdot Rg(B)$$

$$\leqslant C\left(\frac{|E|}{|B|}\right)^\delta \cdot \|\chi_B\|_X \|Rg\|_{X'}$$

$$\leqslant C\left(\frac{|E|}{|B|}\right)^\delta \|\chi_B\|_X.$$

Therefore by the duality we get

$$\|\chi_E\|_X \leqslant C \sup_g \left\{ \left| \int_{\mathbb{R}^n} \chi_E(x)g(x)\,dx \right| : g \in X', \ \|g\|_{X'} \leqslant 1 \right\} \leqslant C \left(\frac{|E|}{|B|} \right)^{\delta} \|\chi_B\|_X.$$

This completes the proof of the lemma. □

Wilson [30] has proved the following boundedness of the square function on weighted Lebesgue spaces.

THEOREM 3. *Let* $0 < \beta \leqslant 1$, $1 < p < \infty$ *and* $w \in A_p$. *Then the square function* S_β *is bounded on the weighted Lebesgue space* $L^p(w)$.

The next extrapolation theorem on weighted Lebesgue spaces has recently proved by Cruz-Uribe and Wang [6, Theorem 2.6].

THEOREM 4. *Suppose that there exists a constant* $1 < p_0 < \infty$ *such that for every* $w_0 \in A_{p_0}$, *the inequality* $\|f\|_{L^{p_0}(w_0)} \leqslant C \|g\|_{L^{p_0}(w_0)}$ *holds for all* $f \in L^{p_0}(w_0)$ *and all measurable functions* g. *Let* $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ *and* w *be a weight. If the Hardy–Littlewood maximal operator* M *is bounded on* $L^{p(\cdot)}(w)$ *and on* $L^{p'(\cdot)}(w^{-\frac{1}{p(\cdot)-1}})$, *then we have the inequality* $\|f\|_{L^{p(\cdot)}(w)} \leqslant C \|g\|_{L^{p(\cdot)}(w)}$ *holds for all* $f \in L^{p(\cdot)}(w)$ *and all measurable functions* g.

REMARK 5. For general variable exponent $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, it is not proved that the assumption $w \in A_{p(\cdot)}$ implies the equivalence of the following two conditions

(a) M is bounded on $L^{p(\cdot)}(w)$.

(b) M is bounded on $L^{p'(\cdot)}(w^{-\frac{1}{p(\cdot)-1}})$.

If we additionally suppose that $p(\cdot) \in LH(\mathbb{R}^n)$, then (a) is immediately true. We note that $p(\cdot) \in LH(\mathbb{R}^n) \cap \mathscr{P}(\mathbb{R}^n)$ implies $p'(\cdot) \in LH(\mathbb{R}^n) \cap \mathscr{P}(\mathbb{R}^n)$. Thus (b) is also true.

Combing the two theorems above, we have the following boundedness of the intrinsic square function on weighted Lebesgue spaces with variable exponent.

COROLLARY 2. *Let* $0 < \beta \leqslant 1$, $p(\cdot) \in LH(\mathbb{R}^n) \cap \mathscr{P}(\mathbb{R}^n)$ *and* $w \in A_{p(\cdot)}$. *Then the intrinsic square function* S_β *is bounded on* $L^{p(\cdot)}(w)$.

In order to prove the main theorem we will additionally use the following Hardy type inequality.

LEMMA 7. *Let* $\{g_k\}_{k=-\infty}^{\infty}$ *be a positive sequence,* $0 < q < \infty$ *and* $\delta > 0$. *We denote*

$$G_j := \sum_{k=-\infty}^{\infty} 2^{-|k-j|\delta} g_k$$

for each $j \in \mathbb{Z}$. Then the inequality

$$\left(\sum_{j=-\infty}^{\infty} G_j^q \right)^{1/q} = \left(\sum_{j=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} 2^{-|k-j|\delta} g_k \right)^q \right)^{1/q} \leqslant C \left(\sum_{k=-\infty}^{\infty} g_k^q \right)^{1/q}$$

holds.

The proof of Lemma 7 is found in [11, Lemma A.2.1]. We remark that Kempka and Vybíral [19, Lemma 8] have proved the Hardy type inequality for general function spaces case.

3.3. Statement of the main result

THEOREM 5. *Let $0 < \beta \leqslant 1$, $p(\cdot) \in LH(\mathbb{R}^n) \cap \mathscr{P}(\mathbb{R}^n)$, $0 < q < \infty$, $1/p_- < r < 1$, $w \in A_{rp(\cdot)}$ and $-n\delta < \alpha < n(1-r)$, where $0 < \delta < 1$ is a constant satisfying*

$$\frac{\|\chi_{B_k}\|_{L^{p(\cdot)}(w)}}{\|\chi_{B_l}\|_{L^{p(\cdot)}(w)}} \leqslant C 2^{\delta n(k-l)}$$

for all $k, l \in \mathbb{Z}$ with $k \leqslant l$. Then the intrinsic square function S_β is bounded on $\dot{K}_{p(\cdot)}^{\alpha,q}(w)$ and on $K_{p(\cdot)}^{\alpha,q}(w)$.

Proof. We prove the boundedness on the homogeneous space $\dot{K}_{p(\cdot)}^{\alpha,q}(w)$. The proof similar to below is valid for the non-homogeneous space $K_{p(\cdot)}^{\alpha,q}(w)$. We decompose $f \in \dot{K}_{p(\cdot)}^{\alpha,q}(w)$ as $f = f\chi_{B_{k+1} \setminus B_{k-2}} + f\chi_{B_{k-2}} + f\chi_{\mathbb{R}^n \setminus B_{k+1}}$. Thus we obtain

$$\|S_\beta f\|_{\dot{K}_{p(\cdot)}^{\alpha,q}(w)} \leqslant C \Bigg\{ \left(\sum_{k=-\infty}^{\infty} 2^{\alpha kq} \|S_\beta(f\chi_{B_{k+1} \setminus B_{k-2}})\chi_k\|_{L^{p(\cdot)}(w)}^q \right)^{1/q}$$

$$+ \left(\sum_{k=-\infty}^{\infty} 2^{\alpha kq} \|S_\beta(f\chi_{B_{k-2}})\chi_k\|_{L^{p(\cdot)}(w)}^q \right)^{1/q}$$

$$+ \left(\sum_{k=-\infty}^{\infty} 2^{\alpha kq} \|S_\beta(f\chi_{\mathbb{R}^n \setminus B_{k+1}})\chi_k\|_{L^{p(\cdot)}(w)}^q \right)^{1/q} \Bigg\}$$

$$=: C(T_1 + T_2 + T_3).$$

For each $i = 1, 2, 3$ we start the estimate T_i.

We first consider T_1. Using the boundedness of S_β on $L^{p(\cdot)}(w)$ we get

$$T_1 \leqslant \left(\sum_{k=-\infty}^{\infty} 2^{\alpha kq} \|S_\beta(f\chi_{B_{k+1} \setminus B_{k-2}})\|_{L^{p(\cdot)}(w)}^q \right)^{1/q}$$

$$\leqslant C \left(\sum_{k=-\infty}^{\infty} 2^{\alpha kq} \|f\chi_{B_{k+1} \setminus B_{k-2}}\|_{L^{p(\cdot)}(w)}^q \right)^{1/q}$$

$$\leqslant C \|f\|_{\dot{K}_{p(\cdot)}^{\alpha,q}(w)}.$$

Next we estimate T_2. By the definition of the operator S_β we have

$$T_2 = \left(\sum_{k=-\infty}^{\infty} 2^{\alpha k q} \left\| S_\beta \left(\sum_{l=-\infty}^{k-2} f\chi_l \right) \chi_k \right\|_{L^{p(\cdot)}(w)}^q \right)^{1/q}$$

$$\leqslant \left(\sum_{k=-\infty}^{\infty} 2^{\alpha k q} \left(\sum_{l=-\infty}^{k-2} \left\| S_\beta (f\chi_l) \chi_k \right\|_{L^{p(\cdot)}(w)} \right)^q \right)^{1/q}. \tag{6}$$

Now we take $k \in \mathbb{Z}$, $l \leqslant k-2$, $x \in D_k$ and $(y,t) \in \Gamma(x)$. For every $\varphi \in \mathscr{C}_\beta$ we have

$$|(f\chi_l) * \varphi_t(y)| = \left| \int_{D_l} \varphi_t(y) f(z) \, dz \right| \leqslant C t^{-n} \int_{\{z \in D_l : |y-z| < t\}} |f(z)| \, dz.$$

Using a point $z \in D_l$ with $|y - z| < t$ we obtain

$$t = \frac{1}{2}(t+t) > \frac{1}{2}(|x-y| + |y-z|) \geqslant \frac{1}{2}|x-z| \geqslant \frac{1}{2}(|x| - |z|)$$

$$\geqslant \frac{1}{2}(|x| - 2^l) \geqslant \frac{1}{2}(|x| - 2^{k-2}) \geqslant \frac{1}{2}(|x| - 2^{-1}|x|) = \frac{|x|}{4}.$$

Hence we get

$$|S_\beta(f\chi_l)(x)| = \left(\int\int_{\Gamma(x)} \left(\sup_{\varphi \in \mathscr{C}_\beta} |(f\chi_l) * \varphi_t(y)|^2 \frac{dy\,dt}{t^{n+1}} \right)^2 \right)^{1/2}$$

$$\leqslant C \left(\int_{\frac{|x|}{4}}^{\infty} \int_{\{y : |x-y| < t\}} \left(\frac{1}{t^n} \int_{\{z \in D_l : |y-z| < t\}} |f(z)| \, dz \right)^2 \frac{dy\,dt}{t^{n+1}} \right)^{1/2}$$

$$\leqslant C \left(\int_{D_l} |f(z)| \, dz \right) \left(\int_{\frac{|x|}{4}}^{\infty} \left(\int_{\{y : |x-y| < t\}} dy \right) \frac{dt}{t^{3n+1}} \right)^{1/2}$$

$$= C \left(\int_{D_l} |f(z)| \, dz \right) \left(\int_{\frac{|x|}{4}}^{\infty} \frac{dt}{t^{2n+1}} \right)^{1/2}$$

$$= C \left(\int_{D_l} |f(z)| \, dz \right) |x|^{-n}.$$

Applying the generalized Hölder inequality and Lemma 2, we have

$$|S_\beta(f\chi_l)(x)| \leqslant C|x|^{-n} \left\| f w^{1/p(\cdot)} \chi_l \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \cdot \left\| w^{-1/p(\cdot)} \chi_l \right\|_{L^{p'(\cdot)}(\mathbb{R}^n)}$$

$$\leqslant C|x|^{-n} \left\| f w^{1/p(\cdot)} \chi_l \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \cdot \left\| w^{-1/p(\cdot)} \chi_{B_l} \right\|_{L^{p'(\cdot)}(\mathbb{R}^n)}$$

$$= C|x|^{-n} \left\| f w^{1/p(\cdot)} \chi_l \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \cdot \left\| \chi_{B_l} \right\|_{(L^{p(\cdot)}(w))'}$$

$$\leqslant C|x|^{-n} \left\| f w^{1/p(\cdot)} \chi_l \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \cdot |B_l| \left\| \chi_{B_l} \right\|_{L^{p(\cdot)}(w)}^{-1}.$$

We note that $x \in D_k$ implies $|x| > 2^{k-1}$, that is $|x|^{-n} < C|B_k|^{-1}$. Thus we get

$$\left| S_\beta (f\chi_l)(x) \right| \leqslant C \cdot \frac{|B_l|}{|B_k|} \|f\chi_l\|_{L^{p(\cdot)}(w)} \|\chi_{B_l}\|_{L^{p(\cdot)}(w)}^{-1}. \tag{7}$$

Combing (6) and (7), we obtain

$$T_2 \leqslant C \left(\sum_{k=-\infty}^{\infty} 2^{\alpha kq} \left(\sum_{l=-\infty}^{k-2} \frac{|B_l|}{|B_k|} \frac{\|\chi_k\|_{L^{p(\cdot)}(w)}}{\|\chi_{B_l}\|_{L^{p(\cdot)}(w)}} \|f\chi_l\|_{L^{p(\cdot)}(w)} \right)^q \right)^{1/q}$$

$$\leqslant C \left(\sum_{k=-\infty}^{\infty} 2^{\alpha kq} \left(\sum_{l=-\infty}^{k-2} \frac{|B_l|}{|B_k|} \frac{\|\chi_{B_k}\|_{L^{p(\cdot)}(w)}}{\|\chi_{B_l}\|_{L^{p(\cdot)}(w)}} \|f\chi_l\|_{L^{p(\cdot)}(w)} \right)^q \right)^{1/q}.$$

For every $k, l \in \mathbb{Z}$ such that $k \geqslant l+2$, we see that $B_l \subset B_k$. We also note that $rp(\cdot) \in LH(\mathbb{R}^n) \cap \mathscr{P}(\mathbb{R}^n)$. Thus by virtue of Lemma 5 with $X = L^{rp(\cdot)}(w)$ we have

$$\frac{\|\chi_{B_k}\|_{L^{p(\cdot)}(w)}}{\|\chi_{B_l}\|_{L^{p(\cdot)}(w)}} = \left(\frac{\|\chi_{B_k}\|_{L^{rp(\cdot)}(w)}}{\|\chi_{B_l}\|_{L^{rp(\cdot)}(w)}} \right)^r \leqslant C \left(\frac{|B_k|}{|B_l|} \right)^r = C 2^{(k-l)nr}.$$

Hence we have

$$T_2 \leqslant C \left(\sum_{k=-\infty}^{\infty} 2^{\alpha kq} \left(\sum_{l=-\infty}^{k-2} 2^{n(r-1)(k-l)} \|f\chi_l\|_{L^{p(\cdot)}(w)} \right)^q \right)^{1/q}. \tag{8}$$

We shall continue the estimate remarking the range of q carefully.

Now let us suppose that $0 < q \leqslant 1$. For general non-negative sequence $\{a_\lambda\}_{\lambda \in \Lambda}$, it is well known that the inequality

$$\left(\sum_{\lambda \in \Lambda} a_\lambda \right)^q \leqslant \sum_{\lambda \in \Lambda} a_\lambda{}^q \tag{9}$$

holds. Applying (9) to (8), we have

$$T_2 \leqslant C \left(\sum_{k=-\infty}^{\infty} 2^{\alpha kq} \sum_{l=-\infty}^{k-2} 2^{qn(r-1)(k-l)} \|f\chi_l\|_{L^{p(\cdot)}(w)}^q \right)^{1/q}$$

$$= C \left(\sum_{l=-\infty}^{\infty} 2^{\alpha lq} \|f\chi_l\|_{L^{p(\cdot)}(w)}^q \sum_{k=l+2}^{\infty} 2^{q(k-l)(\alpha-n(1-r))} \right)^{1/q}$$

$$= C \left(\sum_{l=-\infty}^{\infty} 2^{\alpha lq} \|f\chi_l\|_{L^{p(\cdot)}(w)}^q \right)^{1/q} = C \|f\|_{\dot{K}_{p(\cdot)}^{\alpha,q}(w)},$$

where we have used the condition that $\alpha < n(1-r)$.

We next consider the case $1 < q < \infty$. Because we have supposed that $\alpha < n(1 - r)$, we can take a constant $1 < s < \infty$ so that $\alpha < \frac{n}{s}(1 - r)$. Using the usual Hölder inequality, for every $k \in \mathbb{Z}$ we get

$$\left(\sum_{l=-\infty}^{k-2} 2^{n(r-1)(k-l)} \|f\chi_l\|_{L^{p(\cdot)}(w)} \right)^q$$

$$= \left(\sum_{l=-\infty}^{k-2} 2^{(l-k)\cdot\frac{n}{s}(1-r)} 2^{-\alpha l} \cdot 2^{(l-k)\cdot\frac{n}{s'}(1-r)} 2^{\alpha l} \|f\chi_l\|_{L^{p(\cdot)}(w)} \right)^q$$

$$\leqslant \left(\sum_{l=-\infty}^{k-2} 2^{((l-k)\cdot\frac{n}{s}(1-r)-\alpha l)q'} \right)^{q/q'} \left(\sum_{l=-\infty}^{k-2} 2^{(l-k)\cdot\frac{nq}{s'}(1-r)} 2^{\alpha l q} \|f\chi_l\|_{L^{p(\cdot)}(w)}^q \right)$$

$$= \left(\sum_{l=-\infty}^{k-2} 2^{(l(\frac{n}{s}(1-r)-\alpha)-k\cdot\frac{n}{s}(1-r))q'} \right)^{q/q'} \left(\sum_{l=-\infty}^{k-2} 2^{(l-k)\cdot\frac{nq}{s'}(1-r)} 2^{\alpha l q} \|f\chi_l\|_{L^{p(\cdot)}(w)}^q \right)$$

$$= C \cdot 2^{-k\alpha q} \left(\sum_{l=-\infty}^{k-2} 2^{(l-k)\cdot\frac{nq}{s'}(1-r)} 2^{\alpha l q} \|f\chi_l\|_{L^{p(\cdot)}(w)}^q \right).$$

Applying this estimate to (8) we get

$$T_2 \leqslant C \left(\sum_{k=-\infty}^{\infty} 2^{\alpha k q} \cdot 2^{-k\alpha q} \left(\sum_{l=-\infty}^{k-2} 2^{(l-k)\cdot\frac{nq}{s'}(1-r)} 2^{\alpha l q} \|f\chi_l\|_{L^{p(\cdot)}(w)}^q \right) \right)^{1/q}$$

$$= C \left(\sum_{l=-\infty}^{\infty} 2^{\alpha l q} \|f\chi_l\|_{L^{p(\cdot)}(w)}^q \sum_{k=l+2}^{\infty} 2^{(l-k)\cdot\frac{nq}{s'}(1-r)} \right)^{1/q}$$

$$= C \left(\sum_{l=-\infty}^{\infty} 2^{\alpha l q} \|f\chi_l\|_{L^{p(\cdot)}(w)}^q \right)^{1/q} = C \|f\|_{\dot{K}_{p(\cdot)}^{\alpha,q}(w)}.$$

Finally we estimate T_3. We note that

$$T_3 \leqslant \left(\sum_{k=-\infty}^{\infty} 2^{\alpha k q} \left(\sum_{l=k+2}^{\infty} \|S_\beta(f\chi_l)\chi_k\|_{L^{p(\cdot)}(w)} \right)^q \right)^{1/q}.$$

For every $k \in \mathbb{Z}$, $x \in D_k$, $l \geqslant k+2$, $(y,t) \in \Gamma(x)$ and $z \in D_l$ with $|y - z| < t$, we see that

$$t = \frac{1}{2}(t+t) > \frac{1}{2}(|x-y| + |y-z|) \geqslant \frac{1}{2}|x-z| \geqslant \frac{1}{2}(|z| - |x|) > \frac{1}{2}(2^{l-1} - 2^k) \geqslant 2^{l-3}.$$

Thus we have

$$
\begin{aligned}
|S_\beta(f\chi_l)(x)| &= \left(\iint_{\Gamma(x)} \left(\sup_{\varphi\in\mathscr{C}_\beta} |(f\chi_l)*\varphi_t(y)| \right)^2 \frac{dy\,dt}{t^{n+1}} \right)^{1/2} \\
&\leqslant C \left(\iint_{\Gamma(x)} \left(t^{-n} \int_{\{z\in D_l\,:\,|y-z|<t\}} |f(z)|\,dz \right)^2 \frac{dy\,dt}{t^{n+1}} \right)^{1/2} \\
&\leqslant C \left(\int_{2^{l-3}}^{\infty} \int_{\{y\,:\,|x-y|<t\}} t^{-3n-1} \left(\int_{D_l} |f(z)|\,dz \right)^2 dy\,dt \right)^{1/2} \\
&= C \left(\int_{D_l} |f(z)|\,dz \right) \left(\int_{2^{l-3}}^{\infty} t^{-2n-1}\,dt \right)^{1/2} \\
&= C|B_l|^{-1} \int_{D_l} |f(z)|\,dz.
\end{aligned}
$$

By virtue of the generalized Hölder inequality and Lemma 2, we have

$$
\begin{aligned}
|S_\beta(f\chi_l)(x)| &\leqslant C|B_l|^{-1} \|fw^{1/p(\cdot)}\chi_l\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|w^{-1/p(\cdot)}\chi_l\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \\
&\leqslant C\|fw^{1/p(\cdot)}\chi_l\|_{L^{p(\cdot)}(\mathbb{R}^n)} \cdot |B_l|^{-1} \|w^{-1/p(\cdot)}\chi_{B_l}\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \\
&\leqslant C\|f\chi_l\|_{L^{p(\cdot)}(w)} \cdot \|\chi_{B_l}\|_{L^{p(\cdot)}(w)}^{-1}.
\end{aligned}
$$

Hence we obtain

$$
\begin{aligned}
T_3 &\leqslant C \left(\sum_{k=-\infty}^{\infty} 2^{\alpha kq} \left(\sum_{l=k+2}^{\infty} \|f\chi_l\|_{L^{p(\cdot)}(w)} \frac{\|\chi_k\|_{L^{p(\cdot)}(w)}}{\|\chi_{B_l}\|_{L^{p(\cdot)}(w)}} \right)^q \right)^{1/q} \\
&\leqslant C \left(\sum_{k=-\infty}^{\infty} 2^{\alpha kq} \left(\sum_{l=k+2}^{\infty} \|f\chi_l\|_{L^{p(\cdot)}(w)} \frac{\|\chi_{B_k}\|_{L^{p(\cdot)}(w)}}{\|\chi_{B_l}\|_{L^{p(\cdot)}(w)}} \right)^q \right)^{1/q} \\
&\leqslant C \left(\sum_{k=-\infty}^{\infty} 2^{\alpha kq} \left(\sum_{l=k+2}^{\infty} \|f\chi_l\|_{L^{p(\cdot)}(w)} 2^{\delta n(k-l)} \right)^q \right)^{1/q}.
\end{aligned} \tag{10}
$$

Applying Lemma 7 to (10), we get

$$
\begin{aligned}
T_3 &\leqslant C \left(\sum_{k=-\infty}^{\infty} \left(\sum_{l=k+2}^{\infty} 2^{-(\alpha+\delta n)|l-k|} 2^{\alpha l} \|f\chi_l\|_{L^{p(\cdot)}(w)} \right)^q \right)^{1/q} \\
&\leqslant C \left(\sum_{l=-\infty}^{\infty} 2^{\alpha lq} \|f\chi_l\|_{L^{p(\cdot)}(w)}^q \right)^{1/q} = C\|f\|_{\dot{K}_{p(\cdot)}^{\alpha,q}(w)}.
\end{aligned}
$$

Consequently we have finished all estimates and proved the theorem. $\square$

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