Journal of
Mathematical
Inequalities
Volume 11, Number 3 (2017), 831–843 doi:10.7153/jmi-2017-11-64

ESTIMATIONS OF POWER DIFFERENCE MEAN BY HERON MEAN

MASATOSHI ITO

Dedicated to the memory of Professor Takayuki Furuta in deep sorrow

(*Communicated by J. Ichi Fujii*)

Abstract. As generalizations of arithmetic and geometric means, for positive real numbers a and b, power difference means $J_q(a,b) = \frac{q}{q+1}\frac{a^{q+1}-b^{q+1}}{a^q-b^q}$ and Heron means $K_q(a,b) = (1-q)\sqrt{ab} + q\frac{a+b}{2}$ are well known. In this paper, we obtain the greatest value $\alpha = \alpha(q)$ and the least value $\beta = \beta(q)$ such that the double inequality

$$K_\alpha(a,b) < J_q(a,b) < K_\beta(a,b)$$

holds for any $q \in \mathbb{R}$, which includes Xia, Hou, Wang and Chu's result. Moreover, from this result, we derive operator inequalities for bounded linear operators on a Hilbert space.

1. Introduction

We have been discussed and used the arithmetic and geometric means in various branches, and also many generalizations and related means are known. For example, power difference means, Heron means and so on. In what follows, we use the following notations for several means. For $q \in \mathbb{R}$ and two positive real numbers a and b,

$$A(a,b) = \frac{a+b}{2} \text{ (arithmetic mean)}, \quad G(a,b) = \sqrt{ab} \text{ (geometric mean)},$$

$$H(a,b) = \frac{2ab}{a+b} \text{ (harmonic mean)}, \quad L(a,b) = \frac{a-b}{\log a - \log b} \text{ (logarithmic mean)},$$

$$J_q(a,b) = \begin{cases} \dfrac{q}{q+1}\dfrac{a^{q+1}-b^{q+1}}{a^q-b^q} & \text{if } q \neq 0,-1, \\ \dfrac{a-b}{\log a - \log b} & \text{if } q = 0, \\ \dfrac{ab(\log a - \log b)}{a-b} & \text{if } q = -1, \end{cases} \quad \text{(power difference mean)},$$

Mathematics subject classification (2010): Primary 26E60, 47A63.
Keywords and phrases: Power difference mean, Heron mean and operator mean.
This work was supported by JSPS KAKENHI Grant Number JP16K05181.

Paper JMI-11-64

$$K_q(a,b) = (1-q)\sqrt{ab} + q\frac{a+b}{2} \text{ (Heron mean).}$$

These means are symmetric, that is, $A(a,b) = A(b,a)$, $G(a,b) = G(b,a)$ and so on. We note that $J_q(a,a) \equiv \lim_{b\to a} J_q(a,b) = a$. It is well known that

$$\begin{aligned} &H(a,b) \leqslant G(a,b) \leqslant L(a,b) \leqslant A(a,b), \\ &J_1(a,b) = K_1(a,b) = A(a,b), \quad J_0(a,b) = L(a,b), \\ &J_{\frac{-1}{2}}(a,b) = K_0(a,b) = G(a,b), \quad J_{-2}(a,b) = H(a,b), \end{aligned}$$

and also $J_q(a,b)$ and $K_q(a,b)$ are monotone increasing on $q \in \mathbb{R}$.

Many researchers investigate estimations of these means. The following relation is well known.

$$L(a,b) \leqslant K_\alpha(a,b) \text{ for all } a,b>0 \text{ if and only if } \alpha \geqslant \frac{1}{3}. \tag{1.1}$$

In [1], Bhatia proved (1.1) by using Taylor expansion. As related results to (1.1), for example, they obtain matrix norm inequalities in [1, 5, 8], and also operator inequalities for bounded linear operators on a Hilbert space in [3, 4].

We also have many related numerical inequalities to (1.1), see [7, 12, 13] and so on. We remark that $J_q(a,b)$ is called the one parameter mean in [7, 12, 13]. In [12], Xia, Hou, Wang and Chu obtained optimal inequalities between $J_q(a,b)$ and $K_q(a,b)$.

THEOREM 1.A. ([12]) *For all $a,b>0$ with $a \neq b$, we have the following inequalities.*

(i) *If $\alpha \in (0, \frac{2}{3})$, then $J_{\frac{3\alpha-1}{2}}(a,b) < K_\alpha(a,b) < J_{\frac{\alpha}{2-\alpha}}(a,b)$.*

(ii) *If $\alpha \in (\frac{2}{3}, 1)$, then $J_{\frac{\alpha}{2-\alpha}}(a,b) < K_\alpha(a,b) < J_{\frac{3\alpha-1}{2}}(a,b)$.*

The given parameters $\frac{3\alpha-1}{2}$ and $\frac{\alpha}{2-\alpha}$ in either case are best possible.

In Theorem 1.A, they obtain the greatest value $p = p(\alpha)$ and the least value $q = q(\alpha)$ such that the double inequality

$$J_p(a,b) < K_\alpha(a,b) < J_q(a,b)$$

holds for any $\alpha \in (0,1)$. We remark that they proved Theorem 1.A by calculating derivatives, and also $p(\alpha) \in (\frac{-1}{2}, 1)$ and $q(\alpha) \in (0,1)$ in Theorem 1.A.

In this paper, we extend Theorem 1.A by the different way to [12]. In other words, by using Taylor expansion, we obtain the greatest value $\alpha = \alpha(q)$ and the least value $\beta = \beta(q)$ such that the double inequality

$$K_\alpha(a,b) < J_q(a,b) < K_\beta(a,b)$$

holds for any $q \in \mathbb{R}$. Moreover, from our main result, we derive operator inequalities for bounded linear operators on a Hilbert space.

2. Lemmas

In this section, as lemmas to prove our main result, we show two properties of functions $g_k(q)$ for $k=2,3,\dots$ and $q\in\mathbb{R}$ defined by

$$g_k(q)\equiv\frac{(q+1)^{2(k-1)}-q^{2(k-1)}}{\sum_{i=2}^{k}\binom{2k-1}{2(i-1)}q^{2(k-i)}} \tag{2.1}$$

and $g_k(0)\equiv\dfrac{1}{2k-1}$ for convenience' sake. Here, $\dbinom{n}{r}=\dfrac{n!}{r!(n-r)!}$ is a binomial coefficient for nonnegative integers n and r such that $0\leqslant r\leqslant n$. We remark that $g_2(q)=\dfrac{2q+1}{3}$ in particular.

LEMMA 2.1. *The limit* $g_\infty(q)\equiv\lim_{k\to\infty}g_k(q)$ *exists and* $g_\infty(q)=\begin{cases}\dfrac{2q}{q+1} & (q>0),\\ 0 & (q\leqslant 0).\end{cases}$

Proof. Firstly, we state the following relation (2.2) which is important to prove results in this paper. By putting $j=k-i$,

$$\begin{aligned}&2q\sum_{i=2}^{k}\binom{2k-1}{2(i-1)}q^{2(k-i)}=2\sum_{j=0}^{k-2}\binom{2k-1}{2(k-j-1)}q^{2j+1}\\&=2\sum_{j=0}^{k-1}\binom{2k-1}{2j+1}q^{2j+1}-2q^{2k-1}=(q+1)^{2k-1}+(q-1)^{2k-1}-2q^{2k-1}.\end{aligned} \tag{2.2}$$

If $q\neq 0$, the following holds by (2.2).

$$g_k(q)=\frac{(q+1)^{2(k-1)}-q^{2(k-1)}}{\frac{1}{2q}\{(q+1)^{2k-1}+(q-1)^{2k-1}-2q^{2k-1}\}}$$

$$=\frac{2q\left\{1-\left(\frac{q}{q+1}\right)^{2(k-1)}\right\}}{q+1+(q-1)\left(\frac{q-1}{q+1}\right)^{2k-2}-2q\left(\frac{q}{q+1}\right)^{2k-2}}\quad(\text{if } q\neq-1) \tag{2.3}$$

$$=\frac{2q\left\{\left(\frac{q+1}{q}\right)^{2(k-1)}-1\right\}}{(q+1)\left(\frac{q+1}{q}\right)^{2k-2}+(q-1)\left(\frac{q-1}{q}\right)^{2k-2}-2q}. \tag{2.4}$$

Now we divide the range of q into four cases.

Case 1 If $q>0$, then $-1<\frac{q-1}{q+1}<1$ and $0<\frac{q}{q+1}<1$. Therefore (2.3) implies $g_\infty(q)=\frac{2q}{q+1}$.

Case 2 If $\frac{-1}{2}<q<0$, then $\frac{q-1}{q+1}<-1$ and $-1<\frac{q}{q+1}<0$, so that we have $g_\infty(q)=0$.

Case 3 If $q < \frac{-1}{2}$, then $-1 < \frac{q+1}{q} < 1$ and $\frac{q-1}{q} > 1$. Therefore (2.4) implies $g_\infty(q) = 0$.

Case 4 If $q = 0$, then $g_k(0) = \frac{1}{2k-1} \to 0$ as $k \to \infty$.

Hence the proof is complete. □

LEMMA 2.2. *Let $g_k(q)$ for $q \in \mathbb{R}$ as in (2.1). Then the following assertions hold:*

(i) *If $k \geqslant 3$, then*

$$g_2(q) - g_k(q) = \frac{2(q-1)(2q+1)(2q-1)}{3\sum_{i=2}^{k}\binom{2k-1}{2(i-1)}q^{2(k-i)}} \sum_{\substack{u,v,w \geqslant 0 \\ u+v+w=k-3}} (q+1)^{2u}(q-1)^{2v}q^{2w}.$$

(ii) *If $k \geqslant 2$ and $q > 0$, then*

$$g_k(q) - g_\infty(q) = \frac{(q-1)(2q-1)}{(q+1)\sum_{i=2}^{k}\binom{2k-1}{2(i-1)}q^{2(k-i)}} \sum_{\substack{v,w \geqslant 0 \\ v+w=k-2}} (q-1)^{2v}q^{2w}.$$

Proof. (i) We consider the case $q \neq 0$ since the case $q = 0$ holds by

$$g_2(0) - g_k(0) = \frac{1}{3} - \frac{1}{2k-1} = \frac{2(k-2)}{3(2k-1)}.$$

Since we get

$$\begin{aligned} g_2(q) - g_k(q) &= \frac{2q+1}{3} - \frac{(q+1)^{2(k-1)} - q^{2(k-1)}}{\sum_{i=2}^{k}\binom{2k-1}{2(i-1)}q^{2(k-i)}} \\ &= \frac{(2q+1)\sum_{i=2}^{k}\binom{2k-1}{2(i-1)}q^{2(k-i)} - 3\left\{(q+1)^{2(k-1)} - q^{2(k-1)}\right\}}{3\sum_{i=2}^{k}\binom{2k-1}{2(i-1)}q^{2(k-i)}}, \end{aligned}$$

we have only to show

$$\begin{aligned} h_1(q) &\equiv (2q+1)\sum_{i=2}^{k}\binom{2k-1}{2(i-1)}q^{2(k-i)} - 3\{(q+1)^{2(k-1)} - q^{2(k-1)}\} \\ &= 2(q-1)(2q+1)(2q-1) \sum_{\substack{u,v,w \geqslant 0 \\ u+v+w=k-3}} (q+1)^{2u}(q-1)^{2v}q^{2w}. \end{aligned} \tag{2.5}$$

By (2.2), the equation (2.5) holds since

$$\begin{aligned}
h_1(q) &= (2q+1)\sum_{i=2}^{k}\binom{2k-1}{2(i-1)}q^{2(k-i)} - 3\{(q+1)^{2(k-1)} - q^{2(k-1)}\} \\
&= \frac{2q+1}{2q}\left\{(q+1)^{2k-1} + (q-1)^{2k-1} - 2q^{2k-1}\right\} - 3\{(q+1)^{2(k-1)} - q^{2(k-1)}\} \\
&= \frac{1}{2q}\Big\{(2q+1)(q+1)(q+1)^{2k-2} + (2q+1)(q-1)(q-1)^{2k-2} \\
&\quad -2q(2q+1)q^{2k-2} - 6q(q+1)^{2k-2} + 6q\cdot q^{2k-2}\Big\} \\
&= \frac{1}{2q}\left\{(2q-1)(q-1)(q+1)^{2k-2} + (2q+1)(q-1)(q-1)^{2k-2} - 4q(q-1)q^{2k-2}\right\} \\
&= \frac{q-1}{2q}\left[(2q-1)\left\{(q+1)^{2(k-1)} - q^{2(k-1)}\right\} - (2q+1)\left\{q^{2(k-1)} - (q-1)^{2(k-1)}\right\}\right] \\
&\overset{(*)}{=} 2(q-1)(2q+1)(2q-1)\sum_{\substack{u,v,w\geqslant 0\\ u+v+w=k-3}}(q+1)^{2u}(q-1)^{2v}q^{2w},
\end{aligned}$$

and the last equality $(*)$ holds since

$$\begin{aligned}
&(2q-1)\left\{(q+1)^{2(k-1)} - q^{2(k-1)}\right\} - (2q+1)\left\{q^{2(k-1)} - (q-1)^{2(k-1)}\right\} \\
&= (2q-1)\{(q+1)^2 - q^2\}\left\{(q+1)^{2(k-2)} + (q+1)^{2(k-3)}q^2 + \cdots + (q+1)^2q^{2(k-3)} + q^{2(k-2)}\right\} \\
&\quad -(2q+1)\{q^2 - (q-1)^2\}\left\{q^{2(k-2)} + q^{2(k-3)}(q-1)^2 + \cdots + q^2(q-1)^{2(k-3)} + (q-1)^{2(k-2)}\right\} \\
&= (2q+1)(2q-1)\sum_{i=1}^{k-2}\{(q+1)^{2i} - (q-1)^{2i}\}\,q^{2(k-2-i)} \\
&= (2q+1)(2q-1)\sum_{i=1}^{k-2}\{(q+1)^2 - (q-1)^2\} \\
&\quad \times\left\{(q+1)^{2(i-1)} + (q+1)^{2(i-2)}(q-1)^2 + \cdots + (q-1)^{2(i-1)}\right\}q^{2(k-2-i)} \\
&= 4q(2q+1)(2q-1)\sum_{i=1}^{k-2}\left\{\sum_{j=0}^{i-1}(q+1)^{2j}(q-1)^{2(i-1-j)}\right\}q^{2(k-2-i)} \\
&= 4q(2q+1)(2q-1)\sum_{\substack{u,v,w\geqslant 0\\ u+v+w=k-3}}(q+1)^{2u}(q-1)^{2v}q^{2w}.
\end{aligned}$$

Therefore the desired result holds.

(ii) We get

$$g_k(q)-g_\infty(q)=\frac{(q+1)^{2(k-1)}-q^{2(k-1)}}{\sum_{i=2}^{k}\binom{2k-1}{2(i-1)}q^{2(k-i)}}-\frac{2q}{q+1}$$
$$=\frac{(q+1)^{2k-1}-(q+1)q^{2k-2}-2q\sum_{i=2}^{k}\binom{2k-1}{2(i-1)}q^{2(k-i)}}{(q+1)\sum_{i=2}^{k}\binom{2k-1}{2(i-1)}q^{2(k-i)}},$$

so we have only to show

$$\begin{aligned}h_2(q)&\equiv(q+1)^{2k-1}-(q+1)q^{2k-2}-2q\sum_{i=2}^{k}\binom{2k-1}{2(i-1)}q^{2(k-i)}\\&=(q-1)(2q-1)\sum_{\substack{v,w\geqslant 0\\ v+w=k-2}}(q-1)^{2v}q^{2w}.\end{aligned}\tag{2.6}$$

By (2.2), the equation (2.6) holds since

$$\begin{aligned}h_2(q)&=(q+1)^{2k-1}-(q+1)q^{2k-2}-2q\sum_{i=2}^{k}\binom{2k-1}{2(i-1)}q^{2(k-i)}\\&=(q+1)^{2k-1}-q^{2k-1}-q^{2k-2}-\left\{(q+1)^{2k-1}+(q-1)^{2k-1}-2q^{2k-1}\right\}\\&=(q-1)\left\{q^{2(k-1)}-(q-1)^{2(k-1)}\right\}\\&=(q-1)(2q-1)\left\{q^{2(k-2)}+q^{2(k-3)}(q-1)^2+\cdots+q^2(q-1)^{2(k-3)}+(q-1)^{2(k-2)}\right\}\\&=(q-1)(2q-1)\sum_{\substack{v,w\geqslant 0\\ v+w=k-2}}(q-1)^{2v}q^{2w}.\end{aligned}$$

Hence the proof is complete. □

3. Main result

Theorem 1.A can be written by the following form as the result estimating power difference mean by Heron mean, but the estimations are partial.

THEOREM 1.A'. ([12]) *For all $a,b>0$ with $a\neq b$, we have the following inequalities.*

(i) *If $q\in(0,\frac{1}{2})$, then $K_{\frac{2q}{q+1}}(a,b)<J_q(a,b)<K_{\frac{2q+1}{3}}(a,b)$.*

(ii) *If $q\in(\frac{1}{2},1)$, then $K_{\frac{2q+1}{3}}(a,b)<J_q(a,b)<K_{\frac{2q}{q+1}}(a,b)$.*

(iii) *If $q\in(\frac{-1}{2},0]$, then $J_q(a,b)<K_{\frac{2q+1}{3}}(a,b)$.*

The given parameters $\frac{2q+1}{3}$ and $\frac{2q}{q+1}$ in either case are best possible.

Here, we obtain estimations of power difference mean by Heron mean for all $q \in \mathbb{R}$.

THEOREM 3.1. *For all $a,b>0$ with $a \neq b$, we have the following.*

(i) *Let $q \in (0,\frac{1}{2}) \cup (1,\infty)$. Then*

$$K_{\frac{2q}{q+1}}(a,b) < J_q(a,b) < K_{\frac{2q+1}{3}}(a,b).$$

(ii) *Let $q \in (\frac{1}{2},1)$. Then*

$$K_{\frac{2q+1}{3}}(a,b) < J_q(a,b) < K_{\frac{2q}{q+1}}(a,b).$$

(iii) *Let $q \in (\frac{-1}{2},0]$. Then*

$$G(a,b) = K_0(a,b) < J_q(a,b) < K_{\frac{2q+1}{3}}(a,b).$$

(iv) *Let $q \in (-\infty,\frac{-1}{2})$. Then*

$$K_{\frac{2q+1}{3}}(a,b) < J_q(a,b) < K_0(a,b) = G(a,b).$$

The given parameters of $K_\alpha(a,b)$ in each case are best possible.

We remark that equalities hold between $J_q(a,b)$ and $K_\alpha(a,b)$ in the following cases.

$$J_q(a,b) = K_{\frac{2q+1}{3}}(a,b) = K_{\frac{2q}{q+1}}(a,b) \quad \text{for } q = \tfrac{1}{2}, 1.$$
$$J_q(a,b) = K_{\frac{2q+1}{3}}(a,b) = K_0(a,b) \quad \text{for } q = \tfrac{-1}{2}.$$

To prove Theorem 3.1, we shall show the following propositions.

PROPOSITION 3.2. *The following statements hold:*

(i) *Let $q \in (\frac{-1}{2},\frac{1}{2}) \cup (1,\infty)$. Then*

$$J_q(x,1) < K_\alpha(x,1) \text{ for all } x>0 \text{ with } x \neq 1 \text{ if and only if } \alpha \geqslant \tfrac{2q+1}{3}.$$

(ii) *Let $q \in (-\infty,\frac{-1}{2}) \cup (\frac{1}{2},1)$. Then*

$$J_q(x,1) > K_\alpha(x,1) \text{ for all } x>0 \text{ with } x \neq 1 \text{ if and only if } \alpha \leqslant \tfrac{2q+1}{3}.$$

PROPOSITION 3.3. *The following statements hold:*

(i-1) *Let* $q \in (0, \frac{1}{2}) \cup (1, \infty)$. *Then*

$$J_q(x,1) > K_\alpha(x,1) \text{ for all } x > 0 \text{ with } x \neq 1 \text{ if and only if } \alpha \leqslant \tfrac{2q}{q+1}.$$

(i-2) *Let* $q \in (\frac{-1}{2}, 0]$. *Then*

$$J_q(x,1) > K_\alpha(x,1) \text{ for all } x > 0 \text{ with } x \neq 1 \text{ if and only if } \alpha \leqslant 0.$$

(ii-1) *Let* $q \in (\frac{1}{2}, 1)$. *Then*

$$J_q(x,1) < K_\alpha(x,1) \text{ for all } x > 0 \text{ with } x \neq 1 \text{ if and only if } \alpha \geqslant \tfrac{2q}{q+1}.$$

(ii-2) *Let* $q \in (-\infty, \frac{-1}{2})$. *Then*

$$J_q(x,1) < K_\alpha(x,1) \text{ for all } x > 0 \text{ with } x \neq 1 \text{ if and only if } \alpha \geqslant 0.$$

Proof of Proposition 3.2. (i) Let $q \in (\frac{-1}{2}, \frac{1}{2}) \cup (1, \infty)$. Firstly we show that $\alpha \geqslant \frac{2q+1}{3}$ ensures

$$J_q(x,1) = \frac{q}{q+1}\frac{x^{q+1}-1}{x^q-1} < (1-\alpha)\sqrt{x} + \alpha\frac{x+1}{2} = K_\alpha(x,1) \quad \text{for all } x > 0 \text{ with } x \neq 1. \tag{3.1}$$

If $q \neq 0$, by putting $x = e^{2t}$, (3.1) holds if and only if

$$\frac{q}{q+1}\frac{e^{(q+1)t} - e^{-(q+1)t}}{e^{qt} - e^{-qt}} < (1-\alpha) + \alpha\frac{e^t + e^{-t}}{2} \quad \text{for all } t \in \mathbb{R} \setminus \{0\}. \tag{3.2}$$

Since both sides of (3.2) are even functions, we have only to consider the case $t > 0$. Then, since $\frac{e^{qt} - e^{-qt}}{q} > 0$, (3.2) for $t > 0$ is equivalent to

$$\begin{aligned} f(t) &\equiv \frac{e^{qt} - e^{-qt}}{q}\left\{(1-\alpha) + \alpha\frac{e^t + e^{-t}}{2}\right\} - \frac{e^{(q+1)t} - e^{-(q+1)t}}{q+1} \\ &= \frac{2}{q}\sinh(qt)\{(1-\alpha) + \alpha\cosh t\} - \frac{2}{q+1}\sinh((q+1)t) > 0 \quad \text{for all } t > 0. \end{aligned} \tag{3.3}$$

Therefore we prove (3.3). By Taylor expansion, we have

$$\begin{aligned}
f(t) &= \frac{2}{q}\left(qt+\frac{q^3t^3}{3!}+\frac{q^5t^5}{5!}+\cdots\right)\left\{(1-\alpha)+\alpha\left(1+\frac{t^2}{2!}+\frac{t^4}{4!}+\cdots\right)\right\}\\
&\quad -\frac{2}{q+1}\left\{(q+1)t+\frac{(q+1)^3t^3}{3!}+\frac{(q+1)^5t^5}{5!}+\cdots\right\}\\
&= 2\left(t+\frac{q^2}{3!}t^3+\frac{q^4}{5!}t^5+\cdots\right)\left(1+\frac{\alpha}{2!}t^2+\frac{\alpha}{4!}t^4+\cdots\right)\\
&\quad -2\left\{t+\frac{(q+1)^2}{3!}t^3+\frac{(q+1)^4}{5!}t^5+\cdots\right\}\\
&= 2\sum_{k=2}^{\infty}\left\{\frac{q^{2(k-1)}}{(2k-1)!}+\sum_{i=2}^{k}\frac{q^{2(k-i)}\alpha}{(2i-2)!(2k+1-2i)!}-\frac{(q+1)^{2(k-1)}}{(2k-1)!}\right\}t^{2k-1}\\
&= 2\sum_{k=2}^{\infty}\phi_{k,q}(\alpha)t^{2k-1},
\end{aligned}$$

where

$$\phi_{k,q}(\alpha)\equiv\frac{q^{2(k-1)}}{(2k-1)!}+\sum_{i=2}^{k}\frac{q^{2(k-i)}\alpha}{(2i-2)!(2k+1-2i)!}-\frac{(q+1)^{2(k-1)}}{(2k-1)!}\quad\text{for } k=2,3,\ldots. \tag{3.4}$$

Then $\phi_{k,q}(\alpha)>0$ if and only if

$$\alpha>\frac{(q+1)^{2(k-1)}-q^{2(k-1)}}{\sum_{i=2}^{k}\binom{2k-1}{2(i-1)}q^{2(k-i)}}=g_k(q).$$

If $q=0$, by the similar argument, we can get

$$\phi_{k,0}(\alpha)\equiv\frac{\alpha}{(2k-2)!}-\frac{1}{(2k-1)!}\quad\text{for } k=2,3,\ldots, \tag{3.5}$$

so that $\phi_{k,0}(\alpha)>0$ if and only if $\alpha>\frac{1}{2k-1}=g_k(0)$.

By (i) in Lemma 2.2, $q\in(\frac{-1}{2},\frac{1}{2})\cup(1,\infty)$ ensures that $g_2(q)>g_k(q)$ for all $k\geqslant 3$. Therefore, if $\alpha\geqslant\frac{2q+1}{3}=g_2(q)$, then $\phi_{2,q}(\alpha)\geqslant 0$ and $\phi_{k,q}(\alpha)>0$ for all $k\geqslant 3$, that is, (3.3) holds.

On the other hand, if $\alpha<\frac{2q+1}{3}=g_2(q)$, then $\phi_{2,q}(\alpha)<0$ holds, that is, $f(t)<0$ for sufficiently small $t>0$. Therefore (3.3) assures $\alpha\geqslant\frac{2q+1}{3}$.

We can prove (ii) similarly, so the proof is complete. □

Proof of Proposition 3.3. (i) Let $q\in(\frac{-1}{2},\frac{1}{2})\cup(1,\infty)$. Then by the same way to the proof of Proposition 3.2, we have only to consider the case that

$$f(t)=2\sum_{k=2}^{\infty}\phi_{k,q}(\alpha)t^{2k-1}<0\quad\text{holds for all } t>0, \tag{3.6}$$

that is, $\alpha < g_k(q)$ for $k = 2,3,\dots$, where $\phi_{k,q}(\alpha)$ is defined in (3.4) and (3.5), and also $g_k(q)$ is in (2.1).

(i-1) Let $q \in (0,\frac{1}{2}) \cup (1,\infty)$. By (ii) in Lemma 2.2, $q \in (0,\frac{1}{2}) \cup (1,\infty)$ ensures that $g_k(q) > g_\infty(q)$ for all $k \geqslant 2$, so that (3.6) holds if $\alpha \leqslant \frac{2q}{q+1} = g_\infty(q)$ by Lemma 2.1.

On the other hand, for any $\varepsilon > 0$, there exists a natural number n_0 such that $n \geqslant n_0$ implies $g_\infty(q) < g_n(q) < g_\infty(q) + \varepsilon$. If $\alpha_\varepsilon \equiv g_\infty(q) + \varepsilon > \frac{2q}{q+1} = g_\infty(q)$, then $\phi_{n,q}(\alpha_\varepsilon) > 0$ holds for $n \geqslant n_0$, that is, $f(t) > 0$ for sufficiently large t. Therefore (3.6) assures $\alpha \leqslant \frac{2q}{q+1}$.

(i-2) Let $q \in (\frac{-1}{2}, 0]$. Then $g_k(q) > g_\infty(q) = 0$ for all $k \geqslant 2$ by Lemma 2.1. Therefore (3.6) holds if $\alpha \leqslant 0$. We can show "only if" part by the same way to (i-1).

We can prove (ii-1) and (ii-2) similarly, so the proof is complete. □

Proof of Theorem 3.1. By putting $x = \frac{a}{b}$ in Propositions 3.2 and 3.3, we immediately obtain the desired result. □

REMARK. By (3.3) in the proof of Proposition 3.2, we obtain the following inequalities on hyperbolic functions. We remark that we can produce related inequalities from other results in Propositions 3.2 and 3.3.

PROPOSITION 3.4. *Let $q > 1$. Then the following inequalities hold.*

(i) *If $\alpha \geqslant \frac{2q+1}{3}$, then*

$$\left(\alpha - \frac{2q}{q+1}\right)\sinh((q+1)t) + 2(1-\alpha)\sinh(qt) + \alpha\sinh((q-1)t) > 0$$

holds for all $t > 0$.

(ii) $$\frac{2q-1}{q+1}\sinh((q+1)t) + \frac{2q+1}{q-1}\sinh((q-1)t) > 4\sinh(qt) \quad \textit{for all } t > 0.$$

(iii) $$\frac{\sinh((q+1)t)}{q+1} - \frac{\sinh(qt)}{q} > \frac{\sinh(qt)}{q} - \frac{\sinh((q-1)t)}{q-1} \quad \textit{for all } t > 0.$$

Proof. (i) is shown by applying the product-to-sum formula to (3.3). We have (ii) and (iii) by putting $\alpha = \frac{2q+1}{3}$ and $\alpha = q\left(> \frac{2q+1}{3}\right)$ in (i), respectively. □

4. Operator inequalities

Here, an operator means a bounded linear operator on a Hilbert space $\mathscr{H}$. An operator T is said to be positive (denoted by $T \geqslant 0$) if $(Tx,x) \geqslant 0$ for all $x \in \mathscr{H}$, and also an operator T is said to be strictly positive (denoted by $T > 0$) if T is positive and invertible. A real-valued function f defined on $J \subset \mathbb{R}$ is said to be operator monotone if

$$A \leqslant B \text{ implies } f(A) \leqslant f(B)$$

for selfadjoint operators A and B whose spectra $\sigma(A), \sigma(B) \subset J$, where $A \leqslant B$ means $B - A \geqslant 0$.

Kubo and Ando [10] investigated an axiomatic approach for operator means. In [10], they obtained that there exists a one-to-one correspondence between an operator mean $\mathfrak{M}$ and an operator monotone function $f \geqslant 0$ on $[0, \infty)$ with $f(1) = 1$. We remark that f is called the representing function of $\mathfrak{M}$, and also an operator mean $\mathfrak{M}$ can be defined by

$$\mathfrak{M}(A,B) = A^{\frac{1}{2}} f(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}) A^{\frac{1}{2}} \tag{4.1}$$

if $A > 0$ and $B \geqslant 0$.

For $A, B > 0$, arithmetic mean $\mathfrak{A}(A,B)$, geometric mean $\mathfrak{G}(A,B)$, harmonic mean $\mathfrak{H}(A,B)$ and logarithmic mean $\mathfrak{L}(A,B)$ are typical examples of operator means, and their representing functions are

$$A(x,1) = \frac{x+1}{2}, \quad G(x,1) = \sqrt{x}, \quad H(x,1) = \frac{2x}{x+1} \quad \text{and} \quad L(x,1) = \frac{x-1}{\log x}.$$

We remark that we often denote $\mathfrak{A}(A,B)$, $\mathfrak{G}(A,B)$ and $\mathfrak{H}(A,B)$ by $A \nabla B$, $A \sharp B$ and $A \,!\, B$, that is,

$$A \nabla B = \frac{A+B}{2}, \quad A \sharp B = A^{\frac{1}{2}} (A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^{\frac{1}{2}} A^{\frac{1}{2}} \quad \text{and} \quad A \,!\, B = \left(\frac{A^{-1} + B^{-1}}{2} \right)^{-1}.$$

Now it is permitted to consider binary operations given by (4.1) for general real-valued functions. Power difference mean $\mathfrak{J}_q(A,B)$ and Heron mean $\mathfrak{K}_q(A,B)$ are given by $J_q(x,1)$ and $K_q(x,1)$, respectively. For $-2 \leqslant q \leqslant 1$, it is known in [2, 6, 9, 11] that $\mathfrak{J}_q(A,B)$ is increasing on q and $\mathfrak{J}_q(A,B)$ is an operator mean. Obviously $\mathfrak{K}_q(A,B)$ is an operator mean for $0 \leqslant q \leqslant 1$.

Fujii, Furuichi and Nakamoto [3] showed the following result on an estimation of Heron mean for positive operators.

PROPOSITION 4.A. ([3]) *Let A and B be positive invertible operators and $r \in \mathbb{R}$. Then the following inequalities hold:*

(i) *If $r \geqslant 2$, then $rA \sharp B + (1-r) A \nabla B \leqslant A \,!\, B$.*

(ii) *If $r \leqslant 1$, then $rA \sharp B + (1-r) A \nabla B \geqslant A \,!\, B$.*

The conditions on r is optimal, that is,

$$\inf\{r \mid rA \sharp B + (1-r) A \nabla B \leqslant A \,!\, B\} = 2 \ \text{ and}$$
$$\sup\{r \mid rA \sharp B + (1-r) A \nabla B \geqslant A \,!\, B\} = 1.$$

By Propositions 3.2 and 3.3, we can obtain an extension of Proposition 4.A.

THEOREM 4.1. *Let A and B be positive invertible operators.*

(i) *Let $q \in (0, \frac{1}{2}) \cup (1, \infty)$. Then*

$$\mathfrak{K}_{\frac{2q}{q+1}}(A,B) \leqslant \mathfrak{J}_q(A,B) \leqslant \mathfrak{K}_{\frac{2q+1}{3}}(A,B).$$

(ii) *Let $q \in (\frac{1}{2}, 1)$. Then*

$$\mathfrak{K}_{\frac{2q+1}{3}}(A,B) \leqslant \mathfrak{J}_q(A,B) \leqslant \mathfrak{K}_{\frac{2q}{q+1}}(A,B).$$

(iii) *Let $q \in (\frac{-1}{2}, 0]$. Then*

$$\mathfrak{G}(A,B) = \mathfrak{K}_0(A,B) \leqslant \mathfrak{J}_q(A,B) \leqslant \mathfrak{K}_{\frac{2q+1}{3}}(A,B).$$

(iv) *Let $q \in (-\infty, \frac{-1}{2})$. Then*

$$\mathfrak{K}_{\frac{2q+1}{3}}(A,B) \leqslant \mathfrak{J}_q(A,B) \leqslant \mathfrak{K}_0(A,B) = \mathfrak{G}(A,B).$$

The given parameters of $\mathfrak{K}_\alpha(A,B)$ in each case are best possible.

Proof. We have Theorem 4.1 by putting $x = A^{\frac{-1}{2}}BA^{\frac{-1}{2}}$ and applying the standard operational calculus in Propositions 3.2 and 3.3. $\square$

Theorem 4.1 implies the following inequalities by putting $q = 0, -2$.

COROLLARY 4.2. *Let A and B be positive invertible operators. Then the following hold.*

(i) $\mathfrak{G}(A,B) = \mathfrak{K}_0(A,B) \leqslant \mathfrak{L}(A,B) \leqslant \mathfrak{K}_{\frac{1}{3}}(A,B)$.

(ii) $\mathfrak{K}_{-1}(A,B) \leqslant \mathfrak{H}(a,b) \leqslant \mathfrak{K}_0(A,B) = \mathfrak{G}(A,B)$.

The given parameters of $\mathfrak{K}_\alpha(A,B)$ in each case are best possible.

In Corollary 4.2, the second inequality in (i) is an operator version of (1.1), and also (ii) is just Proposition 4.A.

Acknowledgements. The author would like to express his cordial thanks to the editor and the anonymous referees for their valuable comments and suggestions.

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(Received June 23, 2016)

Masatoshi Ito
Maebashi Institute of Technology
460-1 Kamisadorimachi, Maebashi, Gunma 371-0816, Japan
e-mail: m-ito@maebashi-it.ac.jp