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ON THE ENERGY OF BICYCLIC SIGNED DIGRAPHS

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Abstract. Among unicyclic digraphs and signed digraphs with fixed number of vertices, the digraphs and signed digraphs with minimal and maximal energy are already determined. In this paper we address the problem of finding bicyclic signed digraphs with extremal energy. We obtain minimal and maximal energy among all those n-vertex bicyclic signed digraphs which contain vertex-disjoint signed directed cycles, $n \geqslant 4$.

1. Introduction

A signed digraph (henceforth, sidigraph) is a pair $S = (D, \omega)$, where $D = (\mathscr{V}, \mathscr{A})$ is the underlying digraph and $\omega : \mathscr{A} \to \{-1, 1\}$ is the signing function. That is, each arc of D is assigned a $+1$ or a -1 sign. An arc with a $+1$ (respectively, a -1) sign is called a positive (respectively, a negative) arc. Generally, an arc with a $+1$ or a -1 sign is called a signed arc. If the direction of arcs of the underlying digraph D are removed then $S = (D, \omega)$ is called a signed graph (or sigraph).

An arc of S from u to v is denoted by uv. A directed path P_n of length $n-1$, $n \geqslant 2$, is a sidigraph on n vertices $v_1, v_2, \ldots, v_n$ with $n-1$ signed arcs $v_i v_{i+1}$, $i = 1, 2, \ldots, n-1$. A directed cycle of length n, $n \geqslant 2$, is a sidigraph having vertices $v_1, v_2, \ldots, v_n$ and signed arcs $v_i v_{i+1}$, $i = 1, 2, \ldots, n-1$ and $v_n v_1$. The sign of a sidigraph is defined as the product of signs of its arcs. A sidigraph is positive (respectively, negative) if its sign is positive (respectively, negative). A sidigraph is all-positive (respectively, all-negative) if all its arcs are positive (respectively, negative). A sidigraph is said to be cycle-balanced if each of its directed cycle is positive, otherwise non cycle-balanced. Throughout the paper, we will call cycle-balanced directed cycle a positive cycle and non cycle-balanced directed cycle a negative cycle. A positive cycle and a negative cycle, each of length n, is denoted by C_n^+ and C_n^-, respectively. Henceforth, the notation C_n is used for a signed directed cycle which may be positive or negative. A digraph which contains a unique directed cycle and its underlying graph is connected is called a unicyclic digraph. A digraph which contains exactly two directed cycles and its underlying graph is connected is called a bicyclic digraph. A digraph is strongly connected if for each pair u, v of vertices, there is a directed path from u to v and one from v to u. A sidigraph $S = (D, \omega)$ is unicyclic (bicyclic) if D is unicyclic (bicyclic). A sidigraph $S = (D, \omega)$ is strongly connected if D is strongly connected.

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The adjacency matrix of a sidigraph S whose vertices are $v_1, v_2, \dots, v_n$ is the $n \times n$ matrix $A(S) = (a_{ij})$, where

$$a_{ij} = \begin{cases} \omega(v_i v_j) & \text{if there is an arc from } v_i \text{ to } v_j \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The characteristic polynomial $\det(xI - A(S))$ of $A(S)$ is called the characteristic polynomial of the sidigraph S. The eigenvalues of $A(S)$ are called the eigenvalues of S. We observe that $A(S)$ is not necessarily a symmetric matrix. Thus, the eigenvalues of S are not necessarily real. The spectrum $\mathrm{spec}(S)$ of S is the multiset of the eigenvalues of S.

The energy of a graph is the sum of the absolute values of its eigenvalues. Energy of a simple graph was introduced by Gutman [3]. Since then, the graph energy has stimulated extensive research due to its close links to Chemistry. Several extensions of the energy of graph have been studied in the literature.

Peña and Rada [6] extended the notion of energy to digraphs. Since the adjacency matrix of a digraph is not necessarily symmetric, its eigenvalues may be complex. The energy of a digraph is the sum of the absolute values of the real parts of its eigenvalues. The authors find the unicyclic digraphs which have minimal and maximal energy among all n-vertex unicyclic digraphs, $n \geqslant 2$.

Germina et al. [2] introduced the notion of energy in signed graphs. They defined the energy of a signed graphs to be the sum of absolute values of its eigenvalues.

Very recently, Pirzada and Bhat [7] extended the concept of energy to sidigraphs. The definition of the energy of a sidigraph is similar to the definition of the energy of a digraph. The authors compute formulae for energy of signed directed cycles and prove that the energy of negative directed cyles increases monotonically with respect to their length. Khan et al. [4] study the problem of finding digraphs with minimal and maximal energy among n-vertex bicyclic digraphs, $n \geqslant 4$. Monsalve and Rada [5] study a general class of bicyclic digraphs and find the maximal value of the energy in this class.

Motivated by Pirzada and Bhat [7] and Khan et al. [4], we consider the problem of finding bycyclic sidigraphs with minimal and maximal energy among n-vertex bicyclic sidigraphs, $n \geqslant 4$. We find bicyclic sidigraphs which has minimal and maximal energy among n-vertex bicyclic sidigraphs which contain vertex-disjoint signed directed cycles, $n \geqslant 4$.

This paper is structured as follows. In Section 2, we give some known results. Our main work appears in Section 3. We introduce a class of n-vertex bicyclic sidigraphs, $n \geqslant 4$, and find sidigraphs with minimal and maximal energy in this class.

2. Known results

This section deals with some known results.

Let $z_1, z_2, \dots, z_n$ be the eigenvalues of an n-vertex sidigraph S. Then the energy of S is defined by:

$$E(S) = \sum_{k=1}^{n} \left|\mathrm{Re}(z_k)\right|,$$

where $\mathrm{Re}(z_k)$ denotes the real part of z_k.

Following result gives the spectral criterion for cycle-balanced sidigraphs.

THEOREM 1. (Acharya [1]) *A sidigraph $S = (D, \omega)$ is cycle-balanced if and only if S and D are cospectral.*

Obviously, a digraph can be regarded as all-positive sidigraph. By Theorem 1, a directed cycle and a positive directed cycle are cospectral. Thus, a directed cycle can be regarded as a positive directed cycle.

Let C_n^+ be a directed cycle of length n. Then Peña and Rada [6] show that

$$\mathrm{spec}(C_n^+) = \left\{ \exp\frac{2k\pi i}{n} \mid k = 0, 1, \ldots, n-1 \right\},$$

where $i = \sqrt{-1}$. Therefore the energy of C_n^+ is given by

$$E(C_n^+) = \sum_{k=0}^{n-1} \left| \cos\frac{2k\pi}{n} \right|.$$

One can easily observe that

$$E(C_k^+) = 2 \text{ for } k = 2, 3, 4. \tag{2}$$

Similarly, let C_n^- be a negative directed cycle of length n. Then Pirzada and Bhat [7] show that

$$\mathrm{spec}(C_n^-) = \left\{ \exp\frac{(2k+1)\pi i}{n} \mid k = 0, 1, \ldots, n-1 \right\}.$$

Therefore the energy of C_n^- is given by

$$E(C_n^-) = \sum_{k=0}^{n-1} \left| \cos\frac{(2k+1)\pi}{n} \right|. \tag{3}$$

It can easily be seen that

$$E(C_2^-) = 0, \ \ E(C_3^-) = 2 \ \text{ and } \ E(C_4^-) = 2\sqrt{2}. \tag{4}$$

Next theorem gives minimal and maximal energy among n-vertex unicyclic digraphs, $n \geqslant 2$.

THEOREM 2. (Peña and Rada [6]) *Among all n-vertex unicyclic digraphs, the minimal energy is attained in digraphs which contain a cycle of length 2, 3 or 4. The maximal energy is attained in the cycle C_n^+ of length n.*

From the proof of Theorem 2, we derive the following inequalities:

$$E(C_r^+) > 2 \quad \text{for } r \geqslant 5 \tag{5}$$

$$E(C_{r_1}^+) \geqslant E(C_{r_2}^+) \quad \text{for } r_1 \geqslant r_2 \geqslant 5. \tag{6}$$

The inequality in (6) is strict if $r_1 > r_2$.

Let C_n^+ and C_n^- be positive and negative directed cycles, respectively, where $n \geqslant 2$. Then we have the following formulae [7].

$$E(C_n^+) = \begin{cases} 2\cot\frac{\pi}{n} & \text{if } n \equiv 0(\bmod 4) \\ 2\csc\frac{\pi}{n} & \text{if } n \equiv 2(\bmod 4) \\ \csc\frac{\pi}{2n} & \text{if } n \equiv 1(\bmod 2), \end{cases} \tag{7}$$

$$E(C_n^-) = \begin{cases} 2\csc\frac{\pi}{n} & \text{if } n \equiv 0(\bmod 4) \\ 2\cot\frac{\pi}{n} & \text{if } n \equiv 2(\bmod 4) \\ \csc\frac{\pi}{2n} & \text{if } n \equiv 1(\bmod 2). \end{cases} \tag{8}$$

Following theorem gives minimal and maximal energy among n-vertex non cycle-balanced unicyclic sidigraphs.

THEOREM 3. (Pirzada and Bhat [7]) *Energy of negative cycles increases monotonically with respect to the length. Furthermore, minimal energy is attained in C_2^-. Moreover, among all n-vertex non cycle-balanced unicyclic sidigraphs, C_n^- has the maximal energy.*

Next theorem gives few characteristics of positive and negative directed cycles.

THEOREM 4. (Pirzada and Bhat [7]) *Energy of the positive and negative cycles satisfies the following:*

(i) $E(C_n^+) = E(C_n^-)$ *if* $n \equiv 1(\bmod 2)$.

(ii) *For positive even integer n, $E(C_n^+) < E(C_n^-)$ if and only if $n \equiv 0(\bmod 4)$.*

(iii) *For positive even integer n, $E(C_n^+) > E(C_n^-)$ if and only if $n \equiv 2(\bmod 4)$.*

Following theorem gives the energy of a sidigraph with k strong components.

THEOREM 5. (Pirzada and Bhat [7]) *Let S be an n-vertex sidigraph and $S_1, S_2, \ldots, S_k$ be its strong components. Then $E(S) = \sum\limits_{i=1}^{k} E(S_i)$.*

Next lemma is important in proving many results.

LEMMA 1. (Khan et al. [4]) *Let x, a, b be real numbers such that $x \geqslant a > 0$ and $b > 0$. Then*

$$\frac{2x\pi}{bx^2 - \pi^2} \leqslant \frac{2a\pi}{ba^2 - \pi^2}. \tag{9}$$

Lemmas $2-4$ give upper bounds on the sum of energies of two positive vertex-disjoint directed cycles.

LEMMA 2. (Khan et al. [4]) *If* $n \equiv 0(\bmod\ 4)$, $m, n-m \geqslant 2$ *then the following holds for vertex-disjoint cycles:*

$$E(C_{n-m}^{+}) + E(C_{m}^{+}) \leqslant \begin{cases} \frac{2n}{\pi} & \text{if } m \equiv 0(\bmod 4) \\ \frac{2n}{\pi} + \frac{2(n-m)\pi}{6(n-m)^2-\pi^2} + \frac{2m\pi}{6m^2-\pi^2} & \text{if } m \equiv 2(\bmod 4) \\ \frac{2n}{\pi} + \frac{2(n-m)\pi}{24(n-m)^2-\pi^2} + \frac{2m\pi}{24m^2-\pi^2} & \text{if } m \equiv 1(\bmod 2). \end{cases}$$

LEMMA 3. (Khan et al. [4]) *If* $n \equiv 2(\bmod\ 4)$, $m, n-m \geqslant 2$ *then the following holds for vertex-disjoint cycles:*

$$E(C_{n-m}^{+}) + E(C_{m}^{+}) \leqslant \begin{cases} \frac{2n}{\pi} + \frac{2(n-m)\pi}{6(n-m)^2-\pi^2} & \text{if } m \equiv 0(\bmod 4) \\ \frac{2n}{\pi} + \frac{2m\pi}{6m^2-\pi^2} & \text{if } m \equiv 2(\bmod 4) \\ \frac{2n}{\pi} + \frac{2(n-m)\pi}{24(n-m)^2-\pi^2} + \frac{2m\pi}{24m^2-\pi^2} & \text{if } m \equiv 1(\bmod 2). \end{cases}$$

LEMMA 4. (Khan et al. [4]) *If* $n \equiv 1(\bmod\ 2)$, $m, n-m \geqslant 2$ *then the following holds for vertex-disjoint cycles:*

$$E(C_{n-m}^{+}) + E(C_{m}^{+}) \leqslant \begin{cases} \frac{2n}{\pi} + \frac{2(n-m)\pi}{24(n-m)^2-\pi^2} & \text{if } m \equiv 0(\bmod 4) \\ \frac{2n}{\pi} + \frac{2(n-m)\pi}{24(n-m)^2-\pi^2} + \frac{2m\pi}{6m^2-\pi^2} & \text{if } m \equiv 2(\bmod 4) \\ \frac{2n}{\pi} + \frac{2m\pi}{24m^2-\pi^2} & \text{if } m \equiv 1(\bmod 2) \text{ and } n-m \equiv 0(\bmod 4) \\ \frac{2n}{\pi} + \frac{2(n-m)\pi}{6(n-m)^2-\pi^2} + \frac{2m\pi}{24m^2-\pi^2} & \text{if } m \equiv 1(\bmod 2) \text{ and } n-m \equiv 2(\bmod 4). \end{cases}$$

Next lemma gives lower bound for the sum of energies of two vertex-disjoint positive directed cycles.

LEMMA 5. (Khan et al. [4]) *Let* C_{n-2}^{+} *and* C_{2}^{+} *be two vertex-disjoint cycles,* $n \geqslant 4$. *Then we have the following inequalities:*

$$E(C_{n-2}^{+}) + E(C_{2}^{+}) \geqslant \begin{cases} \frac{2n}{\pi} - \frac{4}{\pi} - \frac{\pi^2}{3(n-2)^2} + 2 & \text{if } n \equiv 0(\bmod 4) \\ \frac{2n}{\pi} - \frac{4}{\pi} - \frac{\pi}{n-2} + 2 & \text{if } n \equiv 2(\bmod 4) \\ \frac{2n}{\pi} - \frac{4}{\pi} - \frac{\pi^2}{12(n-2)^2} + 2 & \text{if } n \equiv 1(\bmod 2). \end{cases}$$

For any real number x, with $0 < x \leqslant \frac{\pi}{2}$, the following inequalities hold:

$$\sin x \leqslant x, \quad \sin x \geqslant x - \frac{x^3}{3!}, \quad \cos x \geqslant 1 - \frac{x^2}{2} \tag{10}$$

$$\cot x \leqslant \frac{1}{x}, \quad \cot x \geqslant \frac{1}{x} - \frac{x}{2} \quad \text{if } x \neq 0. \tag{11}$$

3. Bicyclic sidigraphs

We consider a set $\mathscr{S}_n$ consisting of n-vertex bicyclic sidigraphs such that the cycles are vertex-disjoint, where $n \geqslant 4$. Let $S \in \mathscr{S}_n$ be a sidigraph with two signed directed cycles C_{r_1} and C_{r_2} of length r_1 and r_2, respectively, where $2 \leqslant r_1, r_2 \leqslant n-2$. Then Theorem 5 gives

$$E(S) = E(C_{r_1}) + E(C_{r_2}).$$

Thus, to find sidigraphs in $\mathscr{S}_n$ with minimal and maximal energy, it is enough to find lower and upper bounds on the sum of energies of vertex-disjoint signed directed cycles.

Next lemma gives lower bound for the sum of energies of two vertex-disjoint positive directed and negative directed cycles.

LEMMA 6. *Let C_{n-2}^- and C_2^+ be two vertex-disjoint cycles, $n \geqslant 4$. Then we have the following inequalities:*

$$E(C_{n-2}^-) + E(C_2^+) \geqslant \begin{cases} \frac{2n}{\pi} - \frac{4}{\pi} - \frac{\pi}{(n-2)} + 2 & \text{if } n \equiv 0 (\bmod 4) \\ \frac{2n}{\pi} - \frac{4}{\pi} - \frac{\pi^2}{3(n-2)^2} + 2 & \text{if } n \equiv 2 (\bmod 4) \\ \frac{2n}{\pi} - \frac{4}{\pi} - \frac{\pi^2}{12(n-2)^2} + 2 & \text{if } n \equiv 1 (\bmod 2). \end{cases}$$

Proof. We know that $E(C_2^+) = 2$. If $n \equiv 0 (\bmod 4)$ then (8) and (11) yield

$$\begin{aligned} E(C_{n-2}^-) + E(C_2^+) &= 2\left(\cot\frac{\pi}{n-2} + 1\right) \\ &\geqslant 2\left(\frac{n-2}{\pi} - \frac{\pi}{2(n-2)} + 1\right) \\ &= \frac{2n}{\pi} - \frac{4}{\pi} - \frac{\pi}{(n-2)} + 2. \end{aligned}$$

Next, we consider the case when $n \equiv 2 (\bmod 4)$. From (8) and (10), we get

$$\begin{aligned} E(C_{n-2}^-) + E(C_2^+) &= 2\left(\csc\frac{\pi}{n-2} + 1\right) \\ &= 2\left(\frac{1 + \sin\frac{\pi}{n-2}}{\sin\frac{\pi}{n-2}}\right) \\ &\geqslant 2\left(\frac{1 + \frac{\pi}{n-2} - \frac{\pi^3}{6(n-2)^3}}{\frac{\pi}{n-2}}\right) \\ &= \frac{2n}{\pi} - \frac{4}{\pi} - \frac{\pi^2}{3(n-2)^2} + 2. \end{aligned}$$

Finally, we consider the case when $n \equiv 1 (\bmod 2)$. In this case, (8) and (10) give

$$\begin{aligned} E(C_{n-2}^{-}) + E(C_2^{+}) &= \csc \frac{\pi}{2(n-2)} + 2 \\ &= \left(\frac{1 + 2\sin\frac{\pi}{2(n-2)}}{\sin\frac{\pi}{2(n-2)}} \right) \\ &\geqslant \frac{1 + 2\left(\frac{\pi}{2(n-2)} - \frac{\pi^3}{48(n-2)^3} \right)}{\frac{\pi}{2(n-2)}} \\ &= \frac{2(n-2)}{\pi} - \frac{\pi^2}{12(n-2)^2} + 2. \end{aligned}$$

This completes the proof. $\square$

LEMMA 7. *Let n and m be positive integers such that $n \geqslant 4$. If $m \in \{2,3\}$ or $n - m \in \{2,3\}$ then the following inequalities hold true for vertex-disjoint cycles.*

(1) *If $n \equiv 0 (\bmod 4)$ then*

$$E(C_{n-2}^{+}) + E(C_2^{+}) \geqslant E(C_{n-m}) + E(C_m).$$

(2) *If $n \equiv 2 (\bmod 4)$ then*

$$E(C_{n-2}^{-}) + E(C_2^{+}) \geqslant E(C_{n-m}) + E(C_m).$$

(3) *If $n \equiv 1 (\bmod 2)$ then*

$$E(C_{n-2}) + E(C_2^{+}) \geqslant E(C_{n-m}) + E(C_m).$$

Proof. (1). If $m \in \{2,3\}$ then by (2) and (4), we get

$$E(C_2^{+}) \geqslant E(C_m). \tag{12}$$

By Theorem 4 and (6), we have

$$E(C_{n-2}^{+}) \geqslant E(C_{n-m}). \tag{13}$$

By combining inequalities (12) and (13), we get

$$E(C_{n-2}^{+}) + E(C_2^{+}) \geqslant E(C_{n-m}) + E(C_m).$$

If $n - m \in \{2,3\}$ then above inequality can be obtained analogously.
(2) and (3) can be proved in a similar fashion. $\square$

Lemmas 8–10 give different upper bounds for the sum of energies of vertex-disjoint directed cycles of lengths $n-m, m \geqslant 2$.

LEMMA 8. *If $n \equiv 0(\bmod 4)$ and $m, n-m \geqslant 2$ then the following hold for vertex-disjoint cycles:*

(1)

$$E(C_{n-m}^{+})+E(C_{m}^{+}) \leqslant \begin{cases} \frac{2n}{\pi} & \text{if } m \equiv 0(\bmod 4) \\ \frac{2n}{\pi}+\frac{2(n-m)\pi}{6(n-m)^2-\pi^2}+\frac{2m\pi}{6m^2-\pi^2} & \text{if } m \equiv 2(\bmod 4) \\ \frac{2n}{\pi}+\frac{2(n-m)\pi}{24(n-m)^2-\pi^2}+\frac{2m\pi}{24m^2-\pi^2} & \text{if } m \equiv 1(\bmod 2). \end{cases}$$

(2)

$$E(C_{n-m}^{-})+E(C_{m}^{+}) \leqslant \begin{cases} \frac{2n}{\pi}+\frac{2(n-m)\pi}{6(n-m)^2-\pi^2} & \text{if } m \equiv 0(\bmod 4) \\ \frac{2n}{\pi}+\frac{2m\pi}{6m^2-\pi^2} & \text{if } m \equiv 2(\bmod 4) \\ \frac{2n}{\pi}+\frac{2(n-m)\pi}{24(n-m)^2-\pi^2}+\frac{2m\pi}{24m^2-\pi^2} & \text{if } m \equiv 1(\bmod 2). \end{cases}$$

(3)

$$E(C_{n-m}^{+})+E(C_{m}^{-}) \leqslant \begin{cases} \frac{2n}{\pi}+\frac{2m\pi}{6m^2-\pi^2} & \text{if } m \equiv 0(\bmod 4) \\ \frac{2n}{\pi}+\frac{2(n-m)\pi}{6(n-m)^2-\pi^2} & \text{if } m \equiv 2(\bmod 4) \\ \frac{2n}{\pi}+\frac{2(n-m)\pi}{24(n-m)^2-\pi^2}+\frac{2m\pi}{24m^2-\pi^2} & \text{if } m \equiv 1(\bmod 2). \end{cases}$$

(4)

$$E(C_{n-m}^{-})+E(C_{m}^{-}) \leqslant \begin{cases} \frac{2n}{\pi}+\frac{2(n-m)\pi}{6(n-m)^2-\pi^2}+\frac{2m\pi}{6m^2-\pi^2} & \text{if } m \equiv 0(\bmod 4) \\ \frac{2n}{\pi} & \text{if } m \equiv 2(\bmod 4) \\ \frac{2n}{\pi}+\frac{2(n-m)\pi}{24(n-m)^2-\pi^2}+\frac{2m\pi}{24m^2-\pi^2} & \text{if } m \equiv 1(\bmod 2). \end{cases}$$

Proof. (1). The proof follows from Lemma 2.

(2). We first consider the case when $m \equiv 0(\bmod 4)$. In this case, $n-m \equiv 0(\bmod 4)$. By (7), (8) and (10), we get

$$\begin{aligned} E(C_{n-m}^{-})+E(C_{m}^{+}) &= 2\left(\csc\frac{\pi}{n-m}+\cot\frac{\pi}{m}\right) \\ &\leqslant 2\left(\frac{1}{\left(\frac{\pi}{n-m}-\frac{\pi^3}{6(n-m)^3}\right)}+\frac{m}{\pi}\right) \\ &= \frac{2n}{\pi}+\frac{2(n-m)\pi}{6(n-m)^2-\pi^2}. \end{aligned}$$

Secondly, we consider the case when $m \equiv 2(\bmod 4)$. In this case $n-m \equiv 2(\bmod 4)$. By (7), (8) and (10), we get

$$\begin{aligned} E(C_{n-m}^{-})+E(C_{m}^{+}) &= 2\left(\cot\frac{\pi}{n-m}+\csc\frac{\pi}{m}\right) \\ &\leqslant 2\left(\frac{n-m}{\pi}+\frac{1}{\frac{\pi}{m}-\frac{\pi^3}{6m^3}}\right) \\ &= \frac{2n}{\pi}+\frac{2m\pi}{6m^2-\pi^2}. \end{aligned}$$

Finally, we take $m \equiv 1(\bmod 2)$. Then $n-m \equiv 1(\bmod 2)$. By (7), (8) and (10), we get

$$\begin{aligned} E(C_{n-m}^{-})+E(C_m^{+}) &= \csc\frac{\pi}{2(n-m)}+\csc\frac{\pi}{2m} \\ &\leqslant \frac{1}{(\frac{\pi}{2(n-m)})(1-\frac{\pi^2}{24(n-m)^2})}+\frac{1}{(\frac{\pi}{2m})(1-\frac{\pi^2}{24m^2})} \\ &= \frac{2n}{\pi}+2\left(\frac{(n-m)\pi}{24(n-m)^2-\pi^2}+\frac{m\pi}{24m^2-\pi^2}\right). \end{aligned}$$

Analogously, one can prove (3) and (4). □

LEMMA 9. *If $n \equiv 2(\bmod 4)$ and $m, n-m \geqslant 2$ then the following hold for vertex-disjoint cycles:*

(1)

$$E(C_{n-m}^{+})+E(C_m^{+}) \leqslant \begin{cases} \frac{2n}{\pi}+\frac{2(n-m)\pi}{6(n-m)^2-\pi^2} & \text{if } m \equiv 0(\bmod 4) \\ \frac{2n}{\pi}+\frac{2m\pi}{6m^2-\pi^2} & \text{if } m \equiv 2(\bmod 4) \\ \frac{2n}{\pi}+\frac{2(n-m)\pi}{24(n-m)^2-\pi^2}+\frac{2m\pi}{24m^2-\pi^2} & \text{if } m \equiv 1(\bmod 2). \end{cases}$$

(2)

$$E(C_{n-m}^{-})+E(C_m^{+}) \leqslant \begin{cases} \frac{2n}{\pi} & \text{if } m \equiv 0(\bmod 4) \\ \frac{2n}{\pi}+\frac{2(n-m)\pi}{6(n-m)^2-\pi^2}+\frac{2m\pi}{6m^2-\pi^2} & \text{if } m \equiv 2(\bmod 4) \\ \frac{2n}{\pi}+\frac{2(n-m)\pi}{24(n-m)^2-\pi^2}+\frac{2m\pi}{24m^2-\pi^2} & \text{if } m \equiv 1(\bmod 2). \end{cases}$$

(3)

$$(3)\ \ E(C_{n-m}^{+})+E(C_m^{-}) \leqslant \begin{cases} \frac{2n}{\pi}+\frac{2(n-m)\pi}{6(n-m)^2-\pi^2}+\frac{2m\pi}{6m^2-\pi^2} & \text{if } m \equiv 0(\bmod 4) \\ \frac{2n}{\pi} & \text{if } m \equiv 2(\bmod 4) \\ \frac{2n}{\pi}+\frac{2(n-m)\pi}{24(n-m)^2-\pi^2}+\frac{2m\pi}{24m^2-\pi^2} & \text{if } m \equiv 1(\bmod 2). \end{cases}$$

(4)

$$E(C_{n-m}^{-})+E(C_m^{-}) \leqslant \begin{cases} \frac{2n}{\pi}+\frac{2m\pi}{6m^2-\pi^2} & \text{if } m \equiv 0(\bmod 4) \\ \frac{2n}{\pi}+\frac{2(n-m)\pi}{6(n-m)^2-\pi^2} & \text{if } m \equiv 2(\bmod 4) \\ \frac{2n}{\pi}+\frac{2(n-m)\pi}{24(n-m)^2-\pi^2}+\frac{2m\pi}{24m^2-\pi^2} & \text{if } m \equiv 1(\bmod 2). \end{cases}$$

Proof. (1). The proof follows from Lemma 3.
(2). Let $m \equiv 0(\bmod 4)$. Then $n-m \equiv 2(\bmod 4)$. By (7), (8) and (10), we get

$$\begin{aligned} E(C_{n-m}^{-})+E(C_m^{+}) &= 2\left(\cot\frac{\pi}{n-m}+\cot\frac{\pi}{m}\right) \\ &\leqslant 2\left(\frac{n-m}{\pi}+\frac{m}{\pi}\right) \\ &= \frac{2n}{\pi}. \end{aligned}$$

Next, we take $m \equiv 2(\bmod 4)$. Then $n-m \equiv 0(\bmod 4)$. By (7), (8) and (10), we get

$$\begin{aligned} E(C_{n-m}^{-})+E(C_{m}^{+}) &= 2\left(\csc\frac{\pi}{n-m}+\csc\frac{\pi}{m}\right) \\ &\leqslant 2\left(\frac{1}{(\frac{\pi}{n-m})(1-\frac{\pi^2}{6(n-m)^2})}+\frac{1}{(\frac{\pi}{m})(1-\frac{\pi^2}{6m^2})}\right) \\ &= \frac{2n}{\pi}+2\left(\frac{(n-m)\pi}{6(n-m)^2-\pi^2}+\frac{m\pi}{6m^2-\pi^2}\right). \end{aligned}$$

Finally, let $m \equiv 1(\bmod 2)$. Then $n-m \equiv 1(\bmod 2)$. By (7), (8) and (10), we get

$$\begin{aligned} E(C_{n-m}^{-})+E(C_{m}^{+}) &= \csc\frac{\pi}{2(n-m)}+\csc\frac{\pi}{2m} \\ &\leqslant \frac{1}{(\frac{\pi}{2(n-m)})(1-\frac{\pi^2}{24(n-m)^2})}+\frac{1}{(\frac{\pi}{2m})(1-\frac{\pi^2}{24m^2})} \\ &= \frac{2n}{\pi}+2\left(\frac{(n-m)\pi}{24(n-m)^2-\pi^2}+\frac{m\pi}{24m^2-\pi^2}\right). \end{aligned}$$

(3) and (4) can be proved in a similar fashion. □

LEMMA 10. *If $n \equiv 1(\bmod 2)$ and $m, n-m \geqslant 2$ then the following hold for vertex-disjoint cycles:*

(1)

$$E(C_{n-m}^{+})+E(C_{m}^{+}) \leqslant \begin{cases} \frac{2n}{\pi}+\frac{2(n-m)\pi}{24(n-m)^2-\pi^2} & \text{if } m \equiv 0(\bmod 4) \\ \frac{2n}{\pi}+\frac{2(n-m)\pi}{24(n-m)^2-\pi^2}+\frac{2m\pi}{6m^2-\pi^2} & \text{if } m \equiv 2(\bmod 4) \\ \frac{2n}{\pi}+\frac{2m\pi}{24m^2-\pi^2} & \text{if } m \equiv 1(\bmod 2) \text{ and } n-m \equiv 0(\bmod 4) \\ \frac{2n}{\pi}+\frac{2(n-m)\pi}{6(n-m)^2-\pi^2}+\frac{2m\pi}{24m^2-\pi^2} & \text{if } m \equiv 1(\bmod 2) \text{ and } n-m \equiv 2(\bmod 4). \end{cases}$$

(2)

$$E(C_{n-m}^{-})+E(C_{m}^{+}) \leqslant \begin{cases} \frac{2n}{\pi}+\frac{2(n-m)\pi}{24(n-m)^2-\pi^2} & \text{if } m \equiv 0(\bmod 4) \\ \frac{2n}{\pi}+\frac{2(n-m)\pi}{24(n-m)^2-\pi^2}+\frac{2m\pi}{6m^2-\pi^2} & \text{if } m \equiv 2(\bmod 4) \\ \frac{2n}{\pi}+\frac{2(n-m)\pi}{6(n-m)^2-\pi^2}+\frac{2m\pi}{24m^2-\pi^2} & \text{if } m \equiv 1(\bmod 2) \text{ and } n-m \equiv 0(\bmod 4) \\ \frac{2n}{\pi}+\frac{2m\pi}{24m^2-\pi^2} & \text{if } m \equiv 1(\bmod 2) \text{ and } n-m \equiv 2(\bmod 4). \end{cases}$$

(3)

$$E(C_{n-m}^{+})+E(C_{m}^{-}) \leqslant \begin{cases} \frac{2n}{\pi}+\frac{2(n-m)\pi}{24(n-m)^2-\pi^2}+\frac{2m\pi}{6m^2-\pi^2} & \text{if } m \equiv 0(\bmod 4) \\ \frac{2n}{\pi}+\frac{2(n-m)\pi}{24(n-m)^2-\pi^2} & \text{if } m \equiv 2(\bmod 4) \\ \frac{2n}{\pi}+\frac{2m\pi}{24m^2-\pi^2} & \text{if } m \equiv 1(\bmod 2) \text{ and } n-m \equiv 0(\bmod 4) \\ \frac{2n}{\pi}+\frac{2(n-m)\pi}{6(n-m)^2-\pi^2}+\frac{2m\pi}{24m^2-\pi^2} & \text{if } m \equiv 1(\bmod 2) \text{ and } n-m \equiv 2(\bmod 4). \end{cases}$$

(4)

$$E(C_{n-m}^{-})+E(C_m^{-}) \leqslant \begin{cases} \frac{2n}{\pi}+\frac{2(n-m)\pi}{24(n-m)^2-\pi^2}+\frac{2m\pi}{6m^2-\pi^2} & \text{if } m \equiv 0(\bmod 4) \\ \frac{2n}{\pi}+\frac{2(n-m)\pi}{24(n-m)^2-\pi^2} & \text{if } m \equiv 2(\bmod 4) \\ \frac{2n}{\pi}+\frac{2(n-m)\pi}{6(n-m)^2-\pi^2}+\frac{2m\pi}{24m^2-\pi^2} & \text{if } m \equiv 1(\bmod 2) \text{ and } n-m \equiv 0(\bmod 4) \\ \frac{2n}{\pi}+\frac{2m\pi}{24m^2-\pi^2} & \text{if } m \equiv 1(\bmod 2) \text{ and } n-m \equiv 2(\bmod 4). \end{cases}$$

Proof. (1). It follows from Lemma 4.
(2). Let $m \equiv 0(\bmod 4)$. Then $n-m \equiv 1(\bmod 2)$. By (7), (8) and (10), we get

$$\begin{aligned} E(C_{n-m}^{-})+E(C_m^{+}) &= \csc\frac{\pi}{2(n-m)}+2\cot\frac{\pi}{m} \\ &\leqslant \left(\frac{1}{\frac{\pi}{2(n-m)}(1-\frac{\pi^2}{24(n-m)^2})}\right)+2\left(\frac{m}{\pi}\right) \\ &= \frac{2n}{\pi}+\frac{2(n-m)\pi}{24(n-m)^2-\pi^2}. \end{aligned}$$

Next, we consider the case when $m \equiv 2(\bmod 4)$. Then $n-m \equiv 1(\bmod 2)$. By (7), (8) and (10), we get

$$\begin{aligned} E(C_{n-m}^{-})+E(C_m^{+}) &= \csc\frac{\pi}{2(n-m)}+2\csc\frac{\pi}{m} \\ &\leqslant \left(\frac{1}{\frac{\pi}{2(n-m)}(1-\frac{\pi^2}{24(n-m)^2})}\right)+2\left(\frac{1}{\frac{\pi}{m}(1-\frac{\pi^2}{6m^2})}\right) \\ &= \frac{2n}{\pi}+\frac{2(n-m)\pi}{24(n-m)^2-\pi^2}+\frac{2m\pi}{6m^2-\pi^2}. \end{aligned}$$

Next, we let $m \equiv 1(\bmod 2)$ and $n-m \equiv 0(\bmod 4)$. By (7), (8) and (10), we get

$$\begin{aligned} E(C_{n-m}^{-})+E(C_m^{+}) &= 2\csc\frac{\pi}{n-m}+\csc\frac{\pi}{2m} \\ &\leqslant 2\left(\frac{1}{\frac{\pi}{n-m}(1-\frac{\pi^2}{6(n-m)^2})}\right)+\frac{1}{\frac{\pi}{2m}(1-\frac{\pi^2}{24m^2})} \\ &= \frac{2n}{\pi}+\frac{2(n-m)\pi}{6(n-m)^2-\pi^2}+\frac{2m\pi}{24m^2-\pi^2}. \end{aligned}$$

Finally, we let $m \equiv 1(\bmod 2)$ and $n-m \equiv 2(\bmod 4)$. By (7) and (10), we get

$$\begin{aligned} E(C_{n-m}^{-})+E(C_m^{+}) &= 2\cot\frac{\pi}{n-m}+\csc\frac{\pi}{2m} \\ &\leqslant 2\left(\frac{n-m}{\pi}\right)+\frac{1}{\frac{\pi}{2m}(1-\frac{\pi^2}{24m^2})} \\ &= \frac{2n}{\pi}+\frac{2m\pi}{24m^2-\pi^2}. \end{aligned}$$

One can prove (3) and (4) analogously. □

By using Lemmas 6 – 10, we prove next result for vertex-disjoint cycles.

LEMMA 11. *For $m, n-m \geqslant 4$, the following hold for vertex-disjoint cycles.*

(1) *If* $n \equiv 0 (\bmod 4)$ *then*

$$E(C^{+}_{n-2}) + E(C^{+}_{2}) \geqslant E(C_{n-m}) + E(C_m). \tag{14}$$

(2) *If* $n \equiv 2 (\bmod 4)$ *then*

$$E(C^{-}_{n-2}) + E(C^{+}_{2}) \geqslant E(C_{n-m}) + E(C_m). \tag{15}$$

(3) *If* $n \equiv 1 (\bmod 2)$ *then*

$$E(C_{n-2}) + E(C^{+}_{2}) \geqslant E(C_{n-m}) + E(C_m). \tag{16}$$

Proof. (1). Let $n \equiv 0 (\bmod 4)$. In this case, $n-2 \geqslant 6$. This together with Lemma 5 gives

$$\begin{aligned} E(C^{+}_{n-2}) + E(C^{+}_{2}) &\geqslant \frac{2n}{\pi} + 2 - \frac{4}{\pi} - \frac{\pi^2}{3(n-2)^2} \\ &\geqslant \frac{2n}{\pi} + 2 - \frac{4}{\pi} - \frac{\pi^2}{3(6)^2} \\ &\geqslant \frac{2n}{\pi} + 0.62. \end{aligned} \tag{17}$$

On the other hand, let $m \equiv 0 (\bmod 4)$. Then $m, n-m \geqslant 4$. Lemma 8 implies

$$E(C^{+}_{n-m}) + E(C^{+}_{m}) \leqslant \frac{2n}{\pi}. \tag{18}$$

Furthermore, applying Lemma 1 and Lemma 8, we get

$$\begin{aligned} E(C^{-}_{n-m}) + E(C^{+}_{m}) &\leqslant \frac{2n}{\pi} + \frac{2(n-m)\pi}{6(n-m)^2 - \pi^2} \\ &\leqslant \frac{2n}{\pi} + \frac{8\pi}{6(4)^2 - \pi^2} \\ &\leqslant \frac{2n}{\pi} + 0.30. \end{aligned} \tag{19}$$

Similarly, we obtain

$$E(C^{+}_{n-m}) + E(C^{-}_{m}) \leqslant \frac{2n}{\pi} + 0.30, \tag{20}$$

$$E(C^{-}_{n-m}) + E(C^{-}_{m}) \leqslant \frac{2n}{\pi} + 0.59. \tag{21}$$

Then the required inequality (14) follows from (17) and (18) − (21).

If $m \equiv 2(\bmod 4)$ then $m, n-m \geqslant 6$. By Lemma 1 and Lemma 2, we have

$$\begin{aligned} E(C_{n-m}^{+}) + E(C_m^{+}) &\leqslant \frac{2n}{\pi} + \frac{2m\pi}{6m^2 - \pi^2} + \frac{2(n-m)\pi}{6(n-m)^2 - \pi^2} \\ &\leqslant \frac{2n}{\pi} + \frac{12\pi}{6^3 - \pi^2} + \frac{12\pi}{6^3 - \pi^2} \\ &\leqslant \frac{2n}{\pi} + 0.37. \end{aligned} \tag{22}$$

Similarly, Lemma 1 and Lemma 8 imply

$$E(C_{n-m}^{-}) + E(C_m^{+}) \leqslant \frac{2n}{\pi} + 0.19, \tag{23}$$

$$E(C_{n-m}^{+}) + E(C_m^{-}) \leqslant \frac{2n}{\pi} + 0.19, \tag{24}$$

$$E(C_{n-m}^{-}) + E(C_m^{-}) \leqslant \frac{2n}{\pi}. \tag{25}$$

The required inequality (14) follows from (17) and (22) − (25).

If $m \equiv 1(\bmod 2)$ then $m \geqslant 5$ and $n-m \geqslant 7$. Lemma 1 and Lemma 2 imply

$$\begin{aligned} E(C_{n-m}^{+}) + E(C_m^{+}) &\leqslant \frac{2n}{\pi} + \frac{2(n-m)\pi}{24(n-m)^2 - \pi^2} + \frac{2m\pi}{24m^2 - \pi^2} \\ &\leqslant \frac{2n}{\pi} + \frac{14\pi}{24(7)^2 - \pi^2} + \frac{10\pi}{24(5)^2 - \pi^2} \\ &\leqslant \frac{2n}{\pi} + 0.10. \end{aligned} \tag{26}$$

Analogously, Lemma 1 and Lemma 8 give

$$E(C_{n-m}^{-}) + E(C_m^{+}) \leqslant \frac{2n}{\pi} + 0.10, \tag{27}$$

$$E(C_{n-m}^{+}) + E(C_m^{-}) \leqslant \frac{2n}{\pi} + 0.10, \tag{28}$$

$$E(C_{n-m}^{-}) + E(C_m^{-}) \leqslant \frac{2n}{\pi} + 0.10. \tag{29}$$

The required inequality (14) follows from (17) and (26) − (29).

(2). Let $n \equiv 2(\bmod 4)$. In this case, $n-2 \geqslant 8$. This together with Lemma 6 gives

$$\begin{aligned} E(C_{n-2}^{-}) + E(C_2^{+}) &\geqslant \frac{2n}{\pi} - \frac{4}{\pi} + 2 - \frac{\pi^2}{3(n-2)^2} \\ &\geqslant \frac{2n}{\pi} - \frac{4}{\pi} + 2 - \frac{\pi^2}{3(8)^2} \\ &\geqslant \frac{2n}{\pi} + 0.66. \end{aligned} \tag{30}$$

On the other hand, if $m \equiv 0(\bmod 4)$ then $n-m \geqslant 6$. Lemma 1 and Lemma 9 imply

$$\begin{aligned} E(C_{n-m}^{+})+E(C_{m}^{+}) &\leqslant \frac{2n}{\pi}+\frac{2(n-m)\pi}{6(n-m)^{2}-\pi^{2}} \\ &\leqslant \frac{2n}{\pi}+\frac{12\pi}{(6)^{3}-\pi^{2}} \\ &\leqslant \frac{2n}{\pi}+0.19. \end{aligned} \tag{31}$$

Similarly, by Lemma 1 and Lemma 9 we get the following inequalities:

$$E(C_{n-m}^{-})+E(C_{m}^{+}) \leqslant \frac{2n}{\pi}, \tag{32}$$

$$E(C_{n-m}^{+})+E(C_{m}^{-}) \leqslant \frac{2n}{\pi}+0.37, \tag{33}$$

$$E(C_{n-m}^{-})+E(C_{m}^{-}) \leqslant \frac{2n}{\pi}+0.30. \tag{34}$$

The required inequality (15) follows from (30) and (31) – (34).

If $m \equiv 2(\bmod 4)$ then $n-m \geqslant 4$ and $m \geqslant 6$. Lemma 1 and Lemma 9 give

$$\begin{aligned} E(C_{n-m}^{+})+E(C_{m}^{+}) &\leqslant \frac{2n}{\pi}+\frac{2m\pi}{6m^{2}-\pi^{2}} \\ &\leqslant \frac{2n}{\pi}+\frac{12\pi}{(6)^{3}-\pi^{2}} \\ &\leqslant \frac{2n}{\pi}+0.19. \end{aligned} \tag{35}$$

Analogously, one can obtain the following inequalities by using Lemma 1 and Lemma 9:

$$E(C_{n-m}^{-})+E(C_{m}^{+}) \leqslant \frac{2n}{\pi}+0.48, \tag{36}$$

$$E(C_{n-m}^{+})+E(C_{m}^{-}) \leqslant \frac{2n}{\pi}, \tag{37}$$

$$E(C_{n-m}^{-})+E(C_{m}^{-}) \leqslant \frac{2n}{\pi}+0.30. \tag{38}$$

The required inequality (15) follows from (30) and (35) – (38).

If $m \equiv 1(\bmod 2)$ then $m, n-m \geqslant 5$. By Lemma 1 and Lemma 9, we obtain

$$\begin{aligned} E(C_{n-m}^{+})+E(C_{m}^{+}) &\leqslant \frac{2n}{\pi}+\frac{2m\pi}{24m^{2}-\pi^{2}}+\frac{2(n-m)\pi}{24(n-m)^{2}-\pi^{2}} \\ &\leqslant \frac{2n}{\pi}+\frac{10\pi}{24(5)^{2}-\pi^{2}}+\frac{10\pi}{24(5)^{2}-\pi^{2}} \\ &\leqslant \frac{2n}{\pi}+0.11. \end{aligned} \tag{39}$$

Since m and $n-m$ both are odd, by Theorem 4 and (39), we obtain

$$\begin{aligned} E(C_{n-m}^{-})+E(C_{m}^{+}) &= E(C_{n-m}^{+})+E(C_{m}^{-}) \\ &= E(C_{n-m}^{-})+E(C_{m}^{-}) \\ &\leqslant \frac{2n}{\pi}+0.11. \end{aligned} \tag{40}$$

Inequalities (30), (39) and (40) give (15).

(3). If $n \equiv 1(\bmod 2)$ then $n-2 \geqslant 7$. By Theorem 4 and Lemma 6, we obtain

$$\begin{aligned} E(C_{n-2})+E(C_{2}^{+}) &= E(C_{n-2}^{-})+E(C_{2}^{+}) \\ &\geqslant \frac{2n}{\pi}-\frac{4}{\pi}+2-\frac{\pi^2}{12(n-2)^2} \\ &\geqslant \frac{2n}{\pi}+0.70. \end{aligned} \tag{41}$$

On the other hand, if $m \equiv 0(\bmod 4)$ then $n-m \geqslant 5$. Lemma 1 and Lemma 10 give

$$\begin{aligned} E(C_{n-m}^{+})+E(C_{m}^{+}) &\leqslant \frac{2n}{\pi}+\frac{2(n-m)\pi}{24(n-m)^2-\pi^2} \\ &\leqslant \frac{2n}{\pi}+\frac{2(5)\pi}{24(5)^2-\pi^2} \\ &\leqslant \frac{2n}{\pi}+0.054. \end{aligned} \tag{42}$$

Similarly, we can show the following inequalities:

$$E(C_{n-m}^{-})+E(C_{m}^{+}) \leqslant \frac{2n}{\pi}+0.054, \tag{43}$$

$$E(C_{n-m}^{+})+E(C_{m}^{-}) \leqslant \frac{2n}{\pi}+0.35, \tag{44}$$

$$E(C_{n-m}^{-})+E(C_{m}^{-}) \leqslant \frac{2n}{\pi}+0.35. \tag{45}$$

Inequality (16) follows from (41) and (42) – (45).

If $m \equiv 2(\bmod 4)$ then $m \geqslant 6$ and $n-m \geqslant 5$. From Lemma 1 and Lemma 10, we obtain

$$\begin{aligned} E(C_{n-m}^{+})+E(C_{m}^{+}) &\leqslant \frac{2n}{\pi}+\frac{2m\pi}{6m^2-\pi^2}+\frac{2(n-m)\pi}{24(n-m)^2-\pi^2} \\ &\leqslant \frac{2n}{\pi}+\frac{12\pi}{(6)^3-\pi^2}+\frac{10\pi}{24(5)^2-\pi^2} \\ &\leqslant \frac{2n}{\pi}+0.24. \end{aligned} \tag{46}$$

Similarly, we can show the following inequalities:

$$E(C_{n-m}^{-})+E(C_{m}^{+}) \leqslant \frac{2n}{\pi}+0.24, \tag{47}$$

$$E(C_{n-m}^{+})+E(C_{m}^{-}) \leqslant \frac{2n}{\pi}+0.054, \tag{48}$$

$$E(C_{n-m}^{-})+E(C_{m}^{-}) \leqslant \frac{2n}{\pi}+0.054. \tag{49}$$

Inequality (16) follow from (41) and (46) – (49).

If $m \equiv 1(\bmod 2)$ and $n-m \equiv 0(\bmod 4)$ then $m \geqslant 5$ and $n-m \geqslant 4$. Lemma 1 and Lemma 10 imply

$$\begin{aligned} E(C_{n-m}^{+})+E(C_{m}^{+}) &\leqslant \frac{2n}{\pi}+\frac{2m\pi}{24m^2-\pi^2} \\ &\leqslant \frac{2n}{\pi}+\frac{10\pi}{24(5)^2-\pi^2} \\ &\leqslant \frac{2n}{\pi}+0.054. \end{aligned} \tag{50}$$

Similarly, we can show the following inequalities:

$$E(C_{n-m}^{-})+E(C_{m}^{+}) \leqslant \frac{2n}{\pi}+0.35, \tag{51}$$

$$E(C_{n-m}^{+})+E(C_{m}^{-}) \leqslant \frac{2n}{\pi}+0.054, \tag{52}$$

$$E(C_{n-m}^{-})+E(C_{m}^{-}) \leqslant \frac{2n}{\pi}+0.35. \tag{53}$$

Inequality (16) follows from (41) and (50) – (53).

If $m \equiv 1(\bmod 2)$ and $n-m \equiv 2(\bmod 4)$ then $m \geqslant 5$ and $n-m \geqslant 6$. Lemma 1 and Lemma 10 give

$$\begin{aligned} E(C_{n-m}^{+})+E(C_{m}^{+}) &\leqslant \frac{2n}{\pi}+\frac{2m\pi}{24m^2-\pi^2}+\frac{2(n-m)\pi}{6(n-m)^2-\pi^2} \\ &\leqslant \frac{2n}{\pi}+\frac{10\pi}{24(5)^2-\pi^2}+\frac{12\pi}{(6)^3-\pi^2} \\ &\leqslant \frac{2n}{\pi}+0.24. \end{aligned} \tag{54}$$

Similarly, we can show the following inequalities:

$$E(C_{n-m}^{-})+E(C_{m}^{+}) \leqslant \frac{2n}{\pi}+0.054, \tag{55}$$

$$E(C_{n-m}^{+})+E(C_{m}^{-}) \leqslant \frac{2n}{\pi}+0.24, \tag{56}$$

$$E(C_{n-m}^{-})+E(C_{m}^{-}) \leqslant \frac{2n}{\pi}+0.054. \tag{57}$$

Inequality (16) follows from (41) and (54) – (57). □

Combining Lemma 7 and Lemma 11, we have the following theorem.

THEOREM 6. *For $m, n-m \geqslant 2$, the following hold for vertex-disjoint cycles:*

(1) *If* $n \equiv 0(\bmod 4)$ *then*

$$E(C_{n-2}^{+}) + E(C_{2}^{+}) \geqslant E(C_{n-m}) + E(C_{m}). \tag{58}$$

(2) *If* $n \equiv 2(\bmod 4)$ *then*

$$E(C_{n-2}^{-}) + E(C_{2}^{+}) \geqslant E(C_{n-m}) + E(C_{m}). \tag{59}$$

(3) *If* $n \equiv 1(\bmod 2)$ *then*

$$E(C_{n-2}) + E(C_{2}^{+}) \geqslant E(C_{n-m}) + E(C_{m}). \tag{60}$$

The following theorem gives the sidigraphs in $\mathscr{S}_n$ with minimal and maximal energy.

THEOREM 7. *Let $S \in \mathscr{S}_n$ with signed directed cycles C_{r_1} and C_{r_2}, where $2 \leqslant r_1, r_2 \leqslant n-2$.*

(1) *S has minimal energy when $C_{r_1} = C_{r_2} = C_2^{-}$.*

(2) *If $n \equiv 0(\bmod 4)$ then S has maximal energy when $C_{r_1} = C_{n-2}^{+}$ and $C_{r_2} = C_2^{+}$.*

(3) *If $n \equiv 2(\bmod 4)$ then S has maximal energy when $C_{r_1} = C_{n-2}^{-}$ and $C_{r_2} = C_2^{+}$.*

(4) *If $n \equiv 1(\bmod 2)$ then S has maximal energy when $C_{r_1} = C_{n-2}$ and $C_{r_2} = C_2^{+}$.*

Proof. Let $S \in \mathscr{S}_n$ with signed directed cycles C_{r_1} and C_{r_2}, where $2 \leqslant r_1, r_2 \leqslant n-2$. From Theorem 5, it follows that

$$E(S) = E(C_{r_1}) + E(C_{r_2}). \tag{61}$$

(1). As $E(C_2^{-}) = 0$, it is easily seen from (61) that S has minimal energy when $C_{r_1} = C_{r_2} = C_2^{-}$.

(2). If $n \equiv 0(\bmod 4)$ then let $C_{r_1} = C_{n-2}^{+}$ and $C_{r_2} = C_2^{+}$. We take any sidigraph $H \in \mathscr{S}_n$ with signed directed cycles C_{s_1} and C_{s_2}, where $2 \leqslant s_1, s_2 \leqslant n-2$. From (61) and Theorem 6 (1), it holds that

$$\begin{aligned} E(S) &= E(C_{n-2}^{+}) + E(C_2^{+}) \\ &\geqslant E(C_{n-s_1}) + E(C_{s_1}). \end{aligned} \tag{62}$$

Since $n - s_1 \geqslant s_2$, we get from (6) and Theorem 3 that $E(C_{n-s_1}) \geqslant E(C_{s_2})$. Thus

$$\begin{aligned} E(C_{n-s_1}) + E(C_{s_1}) &\geqslant E(C_{s_2}) + E(C_{s_1}) \\ &= E(H). \end{aligned} \tag{63}$$

Inequalities (62) and (63) show that $E(S) \geqslant E(H)$. Thus S has the maximal energy among all sidigraphs of $\mathscr{S}_n$.

(3) and (4) can be proved analogously. □

4. Conclusion

This paper focuses on a class $\mathscr{S}_n$ of those n-vertex bicyclic sidigraphs which contain vertex-disjoint directed cycles, $n \geqslant 4$. We obtained sidigraphs in $\mathscr{S}_n$ which have minimal and maximal energy. It will be worthwhile to consider a more general class of bicyclic sidigraphs and to find sidigraphs in this class with extremal energy. We leave this problem to future work.

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