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# A NOTE ON GENERALIZED CAUCHY–SCHWARZ INEQUALITY

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*Abstract.* We generalize the well-known Cauchy-Schwarz inequality involving any number of real or complex matrices, and also give a necessary and sufficient condition for the equality. This is an improvement of the two recent literatures due to N. Harvey and D. Choi.

## 1. Introduction

In [1], N. Harvey generalized the Cauchy-Schwarz inequality to an inequality involving four vectors. Namely, for any $a,b,c,d \in \mathbb{R}^n$, it holds that

$$\|a\|^2\|b\|^2 + \|c\|^2\|d\|^2 \geqslant 2a^T cb^T d + (a^T b)^2 + (c^T d)^2 - (a^T d)^2 - (b^T c)^2.$$

Afterwards, the result was refined by D. Choi ([2]) to a stronger one:

$$\sum_{k=1}^{m} \|a_{(k)}\|^2\|b_{(k)}\|^2 \geqslant \sum_{k=1}^{m} (a_{(k)}^T b_{(k)})^2 + \frac{2}{m-1} \sum_{1\leqslant k<l\leqslant m} |a_{(k)}^T a_{(l)} b_{(k)}^T b_{(l)} - a_{(k)}^T b_{(l)} b_{(k)}^T a_{(l)}| \tag{1.1}$$

for any $a_{(k)}, b_{(k)} \in \mathbb{R}^n, k = 1,\cdots,m$, and $m \geqslant 2$. Furthermore, he also gained the complex version of the inequality.

In this paper, we will extend them into the following:

$$\sum_{k=1}^{m} \|A_{(k)}\|^2\|B_{(k)}\|^2 \geqslant \sum_{k=1}^{m} (\mathrm{tr}(A_{(k)}^T B_{(k)}))^2 + \frac{2}{m-1} \sum_{1\leqslant k<l\leqslant m} \left|\mathrm{tr}(A_{(k)}^T A_{(l)})\mathrm{tr}(B_{(k)}^T B_{(l)}) - \mathrm{tr}(A_{(k)}^T B_{(l)})\mathrm{tr}(B_{(k)}^T A_{(l)})\right| \tag{1.2}$$

for any $n\times n$ real matrix $A_{(k)}, B_{(k)}, k = 1,\cdots,m$, and $m \geqslant 2$ (Theorem 2.2). We remark that when $\{A_{(k)}, B_{(k)}, k = 1,\cdots,m\}$ are $2m$ diagonal matrices, (1.2) becomes (1.1). The complex version of Inequality (1.2) is also obtained (Theorem 2.4). The whole proof is direct as in [1] and [2], but the calculation is a bit more tedious.

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Recall that N. Harvey and D. Choi used the Euclidean norm on $\mathbb{R}^n$ or $\mathbb{C}^n$, while in this paper we generalized the inequality using the Frobenius norm on the space of $n\times n$ real or complex matrices. Noticed that an $n\times n$ matrix can be viewed as an $n^2$ dimensional vector, and the Frobenius norm of an $n\times n$ matrix is the Euclidean norm of that $n^2$ dimensional vector. Moreover, for any $n\times n$ matrix $A$ and $B$, define inner product as $\langle A,B\rangle =: \mathrm{tr}(A^TB)$. It easy to see this inner product equals to the Euclidean inner product for $A$ and $B$ as $n^2$ dimensional vectors. Thus, we remark that the inequalities obtained are also true for any finite inner product space.

In [2], D. Choi gave a necessary and sufficient condition if the equality holds in (1.1): $a_{(k),i}b_{(k),j}-a_{(k),j}b_{(k),i}=a_{(l),i}b_{(l),j}-a_{(l),j}b_{(l),i}$ for all $k,l=1,\cdots,m$, and $i,j=1,\cdots,n$. We point out that the author missed the absolute notation there. In this paper we will correct this mistake in Theorems 2.1 ∼ 2.4 below.

## 2. Main results and their proofs

THEOREM 2.1. *Let A, B, C and D be four $n\times n$ real matrices. Then*

$$\|A\|^2\|B\|^2+\|C\|^2\|D\|^2\geqslant(\mathrm{tr}(A^TB))^2+(\mathrm{tr}(C^TD))^2+2\left|\mathrm{tr}(A^TC)\mathrm{tr}(B^TD)-\mathrm{tr}(A^TD)\mathrm{tr}(B^TC)\right|,$$

*where the equality holds if and only if $a_{ij}b_{kl}-a_{kl}b_{ij}=\pm(c_{ij}d_{kl}-c_{kl}d_{ij})$ for all $i,j,k,l=1,\cdots,n$.*

*Proof.* Notice that $\|A\|^2=\sum\limits_{i,j}a_{ij}^2$, $\|B\|^2=\sum\limits_{i,j}b_{ij}^2$ and $\mathrm{tr}(A^TB)=\sum\limits_{i,j}a_{ij}b_{ij}$. Then we have

$$\begin{aligned}\|A\|^2\|B\|^2-(\mathrm{tr}(A^TB))^2&=\sum_{i,j}a_{ij}^2\sum_{k,l}b_{kl}^2-\sum_{i,j}a_{ij}b_{ij}\sum_{k,l}a_{kl}b_{kl}\\&=\sum_{i,j,k,l}\left(a_{ij}^2b_{kl}^2-a_{ij}b_{ij}a_{kl}b_{kl}\right)\\&=\frac{1}{2}\sum_{i,j,k,l}(a_{ij}b_{kl}-a_{kl}b_{ij})^2.\end{aligned}$$

Using the formula above, we obtain

$$\begin{aligned}&\|A\|^2\|B\|^2+\|C\|^2\|D\|^2-(\mathrm{tr}(A^TB))^2-(\mathrm{tr}(C^TD))^2\\=&\frac{1}{2}\sum_{i,j,k,l}\left[(a_{ij}b_{kl}-a_{kl}b_{ij})^2+(c_{ij}d_{kl}-c_{kl}d_{ij})^2\right]\\=&\frac{1}{2}\sum_{i,j,k,l}\left\{\left[(a_{ij}b_{kl}-a_{kl}b_{ij})\pm(c_{ij}d_{kl}-c_{kl}d_{ij})\right]^2\mp2(a_{ij}b_{kl}-a_{kl}b_{ij})(c_{ij}d_{kl}-c_{kl}d_{ij})\right\}\\=&\frac{1}{2}\sum_{i,j,k,l}\left[(a_{ij}b_{kl}-a_{kl}b_{ij})\pm(c_{ij}d_{kl}-c_{kl}d_{ij})\right]^2\mp\left(2\sum_{i,j}a_{ij}c_{ij}\sum_{k,l}b_{kl}d_{kl}-2\sum_{i,j}a_{ij}d_{ij}\sum_{k,l}b_{kl}c_{kl}\right)\\\geqslant&\mp2\left(\mathrm{tr}(A^TC)\mathrm{tr}(B^TD)-\mathrm{tr}(A^TD)\mathrm{tr}(B^TC)\right),\end{aligned}$$

which gives the desired inequality. The condition for equality is obvious. This finishes the proof. □

THEOREM 2.2. *Let $A_{(1)},\cdots,A_{(m)},B_{(1)},\cdots,B_{(m)}$ be $n\times n$ real matrices. Then, for $m\geqslant 2$, we have*

$$\sum_{k=1}^{m}\|A_{(k)}\|^2\|B_{(k)}\|^2 \geqslant \sum_{k=1}^{m}(\mathrm{tr}(A_{(k)}^TB_{(k)}))^2 + \frac{2}{m-1}\sum_{1\leqslant k<l\leqslant m}\left|\mathrm{tr}(A_{(k)}^TA_{(l)})\mathrm{tr}(B_{(k)}^TB_{(l)})-\mathrm{tr}(A_{(k)}^TB_{(l)})\mathrm{tr}(B_{(k)}^TA_{(l)})\right|,$$

*where the equality holds if and only if $a_{(k),ij}b_{(k),pq}-a_{(k),pq}b_{(k),ij}=\pm(a_{(l),ij}b_{(l),pq}-a_{(l),pq}b_{(l),ij})$ for all $k,l=1,\cdots,m$ and $i,j,p,q=1,\cdots,n$.*

*Proof.* It follows from Theorem 2.1 that

$$\begin{aligned}&\|A_{(k)}\|^2\|B_{(k)}\|^2+\|A_{(l)}\|^2\|B_{(l)}\|^2-(\mathrm{tr}(A_{(k)}^TB_{(k)}))^2-(\mathrm{tr}(A_{(l)}^TB_{(l)}))^2\\ \geqslant&2\left|\mathrm{tr}(A_{(k)}^TA_{(l)})\mathrm{tr}(B_{(k)}^TB_{(l)})-\mathrm{tr}(A_{(k)}^TB_{(l)})\mathrm{tr}(B_{(k)}^TA_{(l)})\right|\end{aligned}$$

for $1\leqslant k<l\leqslant m$. Thus, we have

$$\begin{aligned}&\sum_{k=1}^{m}\left[\|A_{(k)}\|^2\|B_{(k)}\|^2-(\mathrm{tr}(A_{(k)}^TB_{(k)}))^2\right]\\ =&\frac{1}{m-1}\sum_{1\leqslant k<l\leqslant m}\left[\|A_{(k)}\|^2\|B_{(k)}\|^2+\|A_{(l)}\|^2\|B_{(l)}\|^2-(\mathrm{tr}(A_{(k)}^TB_{(k)}))^2-(\mathrm{tr}(A_{(l)}^TB_{(l)}))^2\right]\\ \geqslant&\frac{2}{m-1}\sum_{1\leqslant k<l\leqslant m}\left|\mathrm{tr}(A_{(k)}^TA_{(l)})\mathrm{tr}(B_{(k)}^TB_{(l)})-\mathrm{tr}(A_{(k)}^TB_{(l)})\mathrm{tr}(B_{(k)}^TA_{(l)})\right| \quad \square\end{aligned}$$

In what follows, we consider the complex case. The complex Frobenius inner product is defined by $\langle A,B\rangle=\mathrm{tr}(A^*B)$. First, we give

THEOREM 2.3. *Let $A$, $B$, $C$ and $D$ be four $n\times n$ complex matrices. Then*

$$\|A\|^2\|B\|^2+\|C\|^2\|D\|^2 \geqslant |\mathrm{tr}(A^*B)|^2+|\mathrm{tr}(C^*D)|^2 +2\left|\mathrm{Re}\,(\mathrm{tr}(A^*C)\mathrm{tr}(B^*D)-\mathrm{tr}(A^*D)\mathrm{tr}(B^*C))\right|,$$

*where the equality holds if and only if $a_{ij}b_{kl}-a_{kl}b_{ij}=\pm(c_{ij}d_{kl}-c_{kl}d_{ij})$ for all $i,j,k,l=1,\cdots,n$.*

*Proof.* Since $A,B$ are complex matrices, $\|A\|^2=\sum_{i,j} a_{ij}\overline{a_{ij}}$, $\|B\|^2=\sum_{i,j} b_{ij}\overline{b_{ij}}$. Hence,

$$\begin{aligned}\|A\|^2\|B\|^2-|\mathrm{tr}(A^*B)|^2 &= \sum_{i,j} a_{ij}\overline{a_{ij}}\sum_{k,l} b_{kl}\overline{b_{kl}}-\sum_{i,j}\overline{a_{ij}}b_{ij}\sum_{k,l}a_{kl}\overline{b_{kl}}\\ &= \sum_{i,j,k,l}[a_{ij}\overline{a_{ij}}b_{kl}\overline{b_{kl}}-\overline{a_{ij}}b_{ij}a_{kl}\overline{b_{kl}}]\\ &= \frac{1}{2}\sum_{i,j,k,l}|a_{ij}b_{kl}-a_{kl}b_{ij}|^2.\end{aligned}$$

Therefore,

$$\begin{aligned}&\|A\|^2\|B\|^2+\|C\|^2\|D\|^2-|\mathrm{tr}(A^*B)|^2-|\mathrm{tr}(C^*D)|^2\\ =&\frac{1}{2}\sum_{i,j,k,l}\left[|a_{ij}b_{kl}-a_{kl}b_{ij}|^2+|c_{ij}d_{kl}-c_{kl}d_{ij}|^2\right]\\ =&\frac{1}{2}\sum_{i,j,k,l}\left[|(a_{ij}b_{kl}-a_{kl}b_{ij})\pm(c_{ij}d_{kl}-c_{kl}d_{ij})|^2\mp 2\mathrm{Re}(\overline{a_{ij}b_{kl}-a_{kl}b_{ij}})(c_{ij}d_{kl}-c_{kl}d_{ij})\right]\\ =&\frac{1}{2}\sum_{i,j,k,l}|(a_{ij}b_{kl}-a_{kl}b_{ij})\pm(c_{ij}d_{kl}-c_{kl}d_{ij})|^2\\ &\mp 2\mathrm{Re}\left(\sum_{i,j}\overline{a_{ij}}c_{ij}\sum_{k,l}\overline{b_{kl}}d_{kl}-\sum_{i,j}\overline{a_{ij}}d_{ij}\sum_{k,l}\overline{b_{kl}}c_{kl}\right)\\ \geqslant&\mp 2\mathrm{Re}(\mathrm{tr}(A^*C)\mathrm{tr}(B^*D)-\mathrm{tr}(A^*D)\mathrm{tr}(B^*C)).\end{aligned}$$

Then the desired inequality follows. □

By a similar argument, we further obtain

THEOREM 2.4. *Let $A_{(1)},\cdots,A_{(m)},B_{(1)},\cdots,B_{(m)}$ be $n\times n$ complex matrices. Then, for $m\geqslant 2$, we have*

$$\begin{aligned}\sum_{k=1}^{m}\|A_{(k)}\|^2\|B_{(k)}\|^2\geqslant&\sum_{k=1}^{m}|\mathrm{tr}(A_{(k)}^*B_{(k)})|^2\\ &+\frac{2}{m-1}\sum_{1\leqslant k<l\leqslant m}\left|\mathrm{Re}(\mathrm{tr}(A_{(k)}^*A_{(l)})\mathrm{tr}(B_{(k)}^*B_{(l)})-\mathrm{tr}(A_{(k)}^*B_{(l)})\mathrm{tr}(B_{(k)}^*A_{(l)}))\right|,\end{aligned}$$

*where the equality holds if and only if $a_{(k),ij}b_{(k),pq}-a_{(k),pq}b_{(k),ij}=\pm(a_{(l),ij}b_{(l),pq}-a_{(l),pq}b_{(l),ij})$ for all $k,l=1,\cdots,m$ and $i,j,p,q=1,\cdots,n$.*

Finally, we unify the two Frobenius inner product (real and complex) into one representation by $\langle\cdot,\cdot\rangle$. Then Theorems 2.1$\sim$2.4 can be written as

THEOREM 2.5. *Let $A_{(1)},\cdots,A_{(m)},B_{(1)},\cdots,B_{(m)}$ be $n\times n$ real or complex matri-*

*ces. Then, for $m \geqslant 2$, we have*

$$\sum_{k=1}^{m} \|A_{(k)}\|^2 \|B_{(k)}\|^2 \geqslant \sum_{k=1}^{m} |\langle A_{(k)}, B_{(k)} \rangle|^2 + \frac{2}{m-1} \sum_{1 \leqslant k < l \leqslant m} \left| \mathrm{Re}(\langle A_{(k)}, A_{(l)} \rangle \langle B_{(k)}, B_{(l)} \rangle - \langle A_{(k)}, B_{(l)} \rangle \langle B_{(k)}, A_{(l)} \rangle) \right|,$$

*where the equality holds if and only if $a_{(k),ij} b_{(k),pq} - a_{(k),pq} b_{(k),ij} = \pm(a_{(l),ij} b_{(l),pq} - a_{(l),pq} b_{(l),ij})$ for all $k, l = 1, \cdots, m$ and $i, j, p, q = 1, \cdots, n$.*

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