Journal of Mathematical Inequalities
Volume 11, Number 3 (2017), 897–899 doi:10.7153/jmi-2017-11-70

REMARK ON THE PAPER OF K. MURALI AND K. M. NAGARAJA

ALFRED WITKOWSKI

(*Communicated by J. Pečarić*)

Abstract. We show that the result of K. Murali and K. M. Nagaraja is not correct.

In the paper [1] the authors consider the function defined for $a,b>0$, $p,q\in\mathbb{R}$, $r+s=1$, $r,s>0$ by the formula

$$N_{p,q}(a,b;r,s)=\left(\frac{p^2}{q^2}\left(\frac{ra^p+sb^p}{ra^q+sb^q}\right)\left(\frac{a^q-b^q}{a^p-b^p}\right)^2\right)^{\frac{1}{q-p}}. \tag{1}$$

(this formula can be extended by continuity to an analytic function in variables p,q,a,b).

The authors call $N_{p,q}$ a mean and claim the following

CLAIM. ([1] Theorem 3.1) *For fixed* $(p,q)\in\mathbb{R}\times\mathbb{R}$ *and* $r=s$

1. *Stolarsky's extended type means* $N_{p,q}(a,b;r,s)$ *are Schur convex with respect to* (a,b) *if* $p+q+3\leqslant 0$.
2. *Stolarsky's extended type means* $N_{p,q}(a,b;r,s)$ *are Schur concave with respect to* (a,b) *if* $p+q+3\geqslant 0$.

These statement cannot be left without comment.

Firstly, it is not true that $N_{p,q}(a,b;r,s)$ are means. It is known that in case $p+q>0$ the Stolarsky means satisfy

$$\left(\frac{p}{q}\left(\frac{a^q-b^q}{a^p-b^p}\right)\right)^{\frac{1}{q-p}}>\sqrt{ab}$$

Suppose $a>b$. Taking s close to 1 we can make $\frac{ra^p+sb^p}{ra^q+sb^q}$ as close to b^{p-q} as we wish. For such s one has $N_{p,q}(a,b;r,s)>a$.

Secondly, the statements 1. and 2. are not true.

Mathematics subject classification (2010): 26D15.
Keywords and phrases: mean, Schur convexity.

Note that

$$N_{p,q}(a,b;\tfrac{1}{2},\tfrac{1}{2}) = \frac{E^2_{p,q}(a,b)}{G_{p,q}(a,b)},$$

where

$$E_{p,q} = \left(\frac{p}{q}\left(\frac{a^q-b^q}{a^p-b^p}\right)\right)^{\frac{1}{q-p}}, \quad G_{p,q} = \left(\frac{a^q+b^q}{a^p+b^p}\right)^{\frac{1}{q-p}}$$

are respectively the Stolarsky and Gini means. Note also that for positive p,q holds

$$\lim_{a\to 0} E_{p,q}(a,1) = \left(\frac{p}{q}\right)^{\frac{1}{q-p}} > 0, \text{ and } \lim_{a\to 0} G_{p,q}(a,1) = 1$$

and they both satisfy the reciprocity identity

$$E_{-p,-q}(a,b) = \frac{ab}{E_{p,q}(a,b)}, \quad G_{-p,-q}(a,b) = \frac{ab}{G_{p,q}(a,b)}.$$

Therefore for negative p,q we have

$$N_{p,q}(a,1) = \frac{E^2_{p,q}(a,1)}{G_{p,q}(a,1)} = a\frac{G_{-p,-q}(a,1)}{E^2_{-p,-q}(a,1)} \to 0 \quad \text{as } a \to 0.$$

For small a we have $(\frac{1+a}{2},\frac{1+a}{2}) \prec (1,a)$ and $N_{p,q}(\frac{1+a}{2},\frac{1+a}{2}) > N_{p,q}(1,a)$, so $N_{p,q}$ cannot be Schur convex for negative p,q, so the claim 1. cannot be true.

Consider now $q > p > 0$ and suppose that $N_{p,q}$ is Schur concave. Then, since for small a $(\frac{1+a}{2},\frac{1+a}{2}) \prec (1,a)$ we should have $\frac{1+a}{2} = N_{p,q}(\frac{1+a}{2},\frac{1+a}{2}) \geqslant N_{p,q}(1,a)$. Denote $\delta = q-p$ and take the limit as $a \to 0$.

$$\frac{1}{2} \geqslant \left(\frac{p}{p+\delta}\right)^{2/\delta} \Leftrightarrow 2^{p/2} \leqslant \left(1+\frac{\delta}{p}\right)^{\frac{p}{\delta}}$$

and this is impossible for large p, since the right-hand side is bounded by e. Therefore the claim 2. cannot be true either.

It is easy to find out why the author's reasoning failed: they conclude that the function $g_{p,q}(t)$ defined in [1, Lemma 3.1] is positive (negative) for all $t > 0$ from the fact that it is such for $t = 0$ (cf. [1, Lemma 3.3]).

And one more remark concerning the final conclusion: the authors write, that for $r \neq s$ Schur convexity of $N_{p,q}$ is an open problem. Since $(a,b) \prec (b,a) \prec (a,b)$, Schur convexity/concavity implies symmetry, and since $N_{p,q}$ lack this property, they cannot be Schur convex/concave. This simple argument can be found e.g. in the classical book on majorization [2, p. 54]

REFERENCES

[1] K. MURALI AND K. M. NAGARAJA, *Schur convexity of Stolarsky's extended mean values*, J. Math. Inequal. **10**, 3 (2016), 725–735.
[2] A. W. MARSHALL AND I. OLKIN, *Inequalities: Theory of Majorization and Its Applications*, Academic Press, New York, 1979.

(Received May 27, 2016)

Alfred Witkowski
Institute of Mathematics and Physics
UTP University of Science and Technology
al. prof. Kaliskiego 7, 85-796 Bydgoszcz, Poland
e-mail: alfred.witkowski@utp.edu.pl