REMARK ON THE PAPER OF K. MURALI AND K. M. NAGARAJA

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Abstract. We show that the result of K. Murali and K. M. Nagaraja is not correct.

In the paper [1] the authors consider the function defined for \( a, b > 0, \ p, q \in \mathbb{R}, \ r + s = 1, \ r, s > 0 \) by the formula

\[
N_{p,q}(a,b;r,s) = \left( \frac{p^2}{q^2} \left( \frac{ra^p + sb^p}{ra^q + sb^q} \right) \left( \frac{a^q - b^q}{a^p - b^p} \right) \right)^{\frac{1}{q-p}}.
\]  

(1)

(this formula can be extended by continuity to an analytic function in variables \( p, q, a, b \)). The authors call \( N_{p,q} \) a mean and claim the following

CLAIM. ([1] Theorem 3.1) For fixed \((p,q) \in \mathbb{R} \times \mathbb{R}\) and \( r = s \)

1. Stolarsky’s extended type means \( N_{p,q}(a,b;r,s) \) are Schur convex with respect to \((a,b)\) if \( p + q + 3 \leq 0 \).

2. Stolarsky’s extended type means \( N_{p,q}(a,b;r,s) \) are Schur concave with respect to \((a,b)\) if \( p + q + 3 \geq 0 \).

These statement cannot be left without comment.

Firstly, it is not true that \( N_{p,q}(a,b;r,s) \) are means. It is known that in case \( p + q > 0 \) the Stolarsky means satisfy

\[
\left( \frac{p}{q} \left( \frac{a^q - b^q}{a^p - b^p} \right) \right)^{\frac{1}{q-p}} > \sqrt{ab}
\]

Suppose \( a > b \). Taking \( s \) close to 1 we can make \( \frac{ra^p + sb^p}{ra^q + sb^q} \) as close to \( b^{p-q} \) as we wish. For such \( s \) one has \( N_{p,q}(a,b;r,s) > a \).

Secondly, the statements 1. and 2. are not true.


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Note that
\[ N_{p,q}(a,b;\frac{1}{2},\frac{1}{2}) = \frac{E_{p,q}^2(a,b)}{G_{p,q}(a,b)}, \]
where
\[ E_{p,q} = \left( \frac{p}{q} \left( \frac{a^q-b^q}{a^p-b^p} \right) \right)^{\frac{1}{q-p}}, \quad G_{p,q} = \left( \frac{a^q+b^q}{a^p+b^p} \right)^{\frac{1}{q-p}} \]
are respectively the Stolarsky and Gini means. Note also that for positive \( p, q \) holds
\[ \lim_{a \to 0} E_{p,q}(a,1) = \left( \frac{p}{q} \right)^{\frac{1}{q-p}} > 0, \quad \text{and} \quad \lim_{a \to 0} G_{p,q}(a,1) = 1 \]
and they both satisfy the reciprocity identity
\[ E_{-p,-q}(a,b) = \frac{ab}{E_{p,q}(a,b)}, \quad G_{-p,-q}(a,b) = \frac{ab}{G_{p,q}(a,b)}. \]
Therefore for negative \( p, q \) we have
\[ N_{p,q}(a,1) = \frac{E_{p,q}^2(a,1)}{G_{p,q}(a,1)} = a \frac{G_{-p,-q}(a,1)}{E_{-p,-q}(a,1)} \to 0 \quad \text{as} \quad a \to 0. \]

For small \( a \) we have \((\frac{1+a}{2}, \frac{1+a}{2}) \prec (1,a)\) and \(N_{p,q}(\frac{1+a}{2}, \frac{1+a}{2}) > N_{p,q}(1,a)\), so \(N_{p,q}\) cannot be Schur convex for negative \( p, q \), so the claim 1. cannot be true.

Consider now \( q > p > 0 \) and suppose that \( N_{p,q} \) is Schur concave. Then, since for small \( a \) \((\frac{1+a}{2}, \frac{1+a}{2}) \prec (1,a)\) we should have \(\frac{1+a}{2} = N_{p,q}(\frac{1+a}{2}, \frac{1+a}{2}) \geq N_{p,q}(1,a)\). Denote \( \delta = q - p \) and take the limit as \( a \to 0 \).
\[
\frac{1}{2} \geq \left( \frac{p}{p+\delta} \right)^{2/\delta} \iff 2^{p/\delta} \leq \left( 1 + \frac{\delta}{p} \right)^{\frac{2}{\delta}}
\]
and this is impossible for large \( p \), since the right-hand side is bounded by \( e \). Therefore the claim 2. cannot be true either.

It is easy to find out why the author’s reasoning failed: they conclude that the function \( g_{p,q}(t) \) defined in [1, Lemma 3.1] is positive (negative) for all \( t > 0 \) from the fact that it is such for \( t = 0 \) (cf. [1, Lemma 3.3]).

And one more remark concerning the final conclusion: the authors write, that for \( r \neq s \) Schur convexity of \( N_{p,q} \) is an open problem. Since \((a,b) \prec (b,a) \prec (a,b)\), Schur convexity/concavity implies symmetry, and since \( N_{p,q} \) lack this property, they cannot be Schur convex/concave. This simple argument can be found e.g. in the classical book on majorization [2, p. 54].
REFERENCES


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