

THE LOG–MINKOWSKI INEQUALITIES FOR QUERMASSEINTEGRALS

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Abstract. Recently, Stancu established the log-Minkowski inequality for non-symmetric convex bodies. In this article, we establish the log-Minkowski inequality for quermassintegrals, which is more general than Stancu’s results.

1. Introduction

Böröczky et al. [1] established the plane log-Brunn-Minkowski inequality and the plane log-Minkowski inequality for origin-symmetric convex bodies. For $n \geq 3$, they [1] conjectured that there exists the log-Brunn-Minkowski inequality and log-Minkowski inequality for origin-symmetric convex bodies in \mathbb{R}^n , and showed that these two inequalities are equivalent. Saroglou [13] established the log-Minkowski inequality for unconditional convex bodies. Ma [12] gave a new proof of the plane log-Minkowski inequality for origin-symmetric convex bodies.

Böröczky et al. [1] also pointed out that while the log-Minkowski inequality holds for two origin-symmetric cubes, there exists a translate of one of the cubes which makes the inequality false. It means that the log-Minkowski inequality cannot hold for all convex bodies.

Recently, Stancu [15] established the log-Minkowski inequality for general convex bodies without the symmetry assumption.

THEOREM A. *Let K and L be two convex bodies in \mathbb{R}^n that contain the origin in their interiors. Then*

$$\int_{S^{n-1}} \ln \frac{h_K}{h_L} d\bar{v}_L \geq \frac{\left(\frac{h_K}{h_L}\right)_{\text{average}}}{\left(\frac{h_K}{h_L}\right)_{\text{max}}} \cdot \frac{1}{n} \ln \frac{V(K)}{V(L)} + \ln \left[\left(\frac{h_K}{h_L}\right)_{\text{min}} \right] \cdot \left[1 - \frac{\left(\frac{h_K}{h_L}\right)_{\text{average}}}{\left(\frac{h_K}{h_L}\right)_{\text{max}}} \right],$$

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with equality if and only if K is homothetic to L . Here $d\bar{v}_L$ denotes the cone-volume probability measure of L . $\left(\frac{h_K}{h_L}\right)_{average} := \frac{\int_{S^{n-1}} \frac{h_K}{h_L} dv_L}{\int_{S^{n-1}} dv_L}$, $\left(\frac{h_K}{h_L}\right)_{max} := \max_{u \in \text{supp } v_L} \frac{h_K}{h_L}$, $\left(\frac{h_K}{h_L}\right)_{min} := \min_{u \in \text{supp } v_K} \frac{h_K}{h_L}$. Here $\text{supp } v_K$ and $\text{supp } v_L$ will be denoted by the support of the cone-volume measure of v_K and v_L , respectively.

By adapting the proofs in [15], we will generalize Stancu’s results. The main purpose of this paper is to establish the log-Minkowski inequality for quermassintegrals as follows.

THEOREM 1.1. *Let K and L be two convex bodies in \mathbb{R}^n that contain the origin in their interiors. For $i = 0, \dots, n - 1$, then*

$$\int_{S^{n-1}} \ln \frac{h_K}{h_L} d\bar{v}_{i,L} \geq \frac{\left(\frac{h_K}{h_L}\right)_{i,average}}{\left(\frac{h_K}{h_L}\right)_{i,max}} \cdot \frac{1}{n-i} \ln \frac{W_i(K)}{W_i(L)} + \ln \left[\left(\frac{h_K}{h_L}\right)_{i,min} \right] \cdot \left[1 - \frac{\left(\frac{h_K}{h_L}\right)_{i,average}}{\left(\frac{h_L}{h_K}\right)_{i,max}} \right],$$

with equality if and only if K is homothetic to L . Here $d\bar{v}_{i,L}$ denotes the mixed cone-

volume probability measure of L . $\left(\frac{h_K}{h_L}\right)_{i,average} := \frac{\int_{S^{n-1}} \frac{h_K}{h_L} dv_{i,L}}{\int_{S^{n-1}} dv_{i,L}}$, $\left(\frac{h_K}{h_L}\right)_{i,max} := \max_{u \in \text{supp } v_{i,L}} \frac{h_K}{h_L}$, $\left(\frac{h_K}{h_L}\right)_{i,min} := \min_{u \in \text{supp } v_{i,K}} \frac{h_K}{h_L}$.

2. Notation and background material

For general reference for the theory of convex (star) bodies the reader may wish to consult the books of Gardner [3], Gruber [7], and Schneider [14].

The setting for this paper is the n -dimensional Euclidean space \mathbb{R}^n . Let \mathcal{K}^n denote the set that consists of all convex bodies (compact, convex subsets with non-empty interiors) in \mathbb{R}^n , and let \mathcal{K}_0^n denote the set of convex bodies that contain the origin in their interiors. The unit ball and its surface in \mathbb{R}^n are denoted by B and S^{n-1} , respectively. We write $V(K)$ for the volume of the compact set K in \mathbb{R}^n . As usual, $V(B) = \omega_n$. The support function of $K \in \mathcal{K}^n$, $h_K(\cdot)$, is defined on S^{n-1} by

$$h_K(u) = \max\{u \cdot x : x \in K\}. \tag{2.1}$$

If $K \in \mathcal{K}_0^n$, then the polar body of K , K^* , is defined by

$$K^* := \{x \in \mathbb{R}^n : x \cdot y \leq 1, \forall y \in K\}. \tag{2.2}$$

The radial function $\rho_L : S^{n-1} \rightarrow [0, \infty)$ of a compact star-shaped set about the origin, $L \in \mathbb{R}^n$, is defined, for $u \in S^{n-1}$, by

$$\rho_L(u) = \max\{\lambda \geq 0 : \lambda u \in L\}. \tag{2.3}$$

If $\rho_L(\cdot)$ is positive and continuous, then L is called a star body about the origin. The set of star bodies about the origin in \mathbb{R}^n is denoted by \mathcal{S}^n .

Obviously, for $K, L \in \mathcal{S}^n$,

$$K \subseteq L \Leftrightarrow \rho_K(u) \leq \rho_L(u), \quad \forall u \in S^{n-1}. \tag{2.4}$$

If $\frac{\rho_K(u)}{\rho_L(u)}$ is independent of $u \in S^{n-1}$, then we say that star bodies K and L are dilates.

If $s > 0$, we have

$$\rho_{sK}(u) = s\rho_K(u), \quad \text{for all } u \in S^{n-1}. \tag{2.5}$$

If $K \in \mathcal{K}_0^n$, then, for $\forall u \in S^{n-1}$, we have

$$\rho_K(u)^{-1} = h_{K^*}(u). \tag{2.6}$$

Let $K_1, K_2 \in \mathcal{K}_0^n$. For $0 \leq i \leq n - 1$, we write $W_i(K_1, K_2)$ for the mixed volume $V(K_1, \dots, K_1, K_2, B, \dots, B)$, where K_1 appears $n - i - 1$ times, the Euclidean unit ball B appears i times, and K_2 appears once. The mixed volume $W_i(K_1, K_2)$ has the following integral representation:

$$W_i(K_1, K_2) = \frac{1}{n} \int_{S^{n-1}} h_{K_2} dS_{i, K_1}, \tag{2.7}$$

where dS_{i, K_1} is the i th surface area measure of K . In particular, $dS_{0, K_1} = dS_{K_1}$ is called the surface area measure of K , and $dS_{n-1, K_1} = dS$ is called the Lebesgue measure on S^{n-1} .

The mixed volume $W_i(K_1, K_1)$ will be written as $W_i(K_1)$. It is called the i th quermassintegral of K_1 and has the following integral representation:

$$W_i(K_1) = \frac{1}{n} \int_{S^{n-1}} h_{K_1} dS_{i, K_1}. \tag{2.8}$$

The Minkowski inequality for mixed quermassintegrals states (see [11]): For $K_1, K_2 \in \mathcal{K}^n$ and $0 \leq i < n - 1$,

$$W_i(K_1, K_2)^{n-i} \geq W_i(K_1)^{n-i-1} W_i(K_2), \tag{2.9}$$

with equality if and only if K_1 and K_2 are homothetic.

For $L_1, L_2 \in \mathcal{S}^n$ and $i = 0, \dots, n - 1$. The dual mixed quermassintegral $\tilde{W}_i(L_1, L_2)$ has the following integral representation:

$$\tilde{W}_i(L_1, L_2) = \frac{1}{n} \int_{S^{n-1}} \rho_{L_1}^{n-i-1} \rho_{L_2} dS. \tag{2.10}$$

The dual quermassintegral $\tilde{W}_i(L_1)$ has the following integral representation:

$$\tilde{W}_i(L_1) = \frac{1}{n} \int_{S^{n-1}} \rho_{L_1}^{n-i} dS. \tag{2.11}$$

If $i = 0$, then $\tilde{W}_0(L_1) = V(L_1)$.

By using Minkowski’s integral inequality, we can obtain the dual Minkowski inequality for dual mixed quermassintegrals (see [10]): For $L_1, L_2 \in \mathcal{S}^n$ and $0 \leq i < n - 1$,

$$\tilde{W}_i(L_1, L_2)^{n-i} \leq \tilde{W}_i(L_1)^{n-i-1} \tilde{W}_i(L_2), \tag{2.12}$$

with equality if and only if L_1 and L_2 are dilates.

The dual Minkowski inequality for dual quermassintegrals states that (see [10]): For $L \in \mathcal{S}^n$ and $0 < i < n - 1$,

$$\tilde{W}_i(L)^n \leq V(L)^{n-i} \omega_n^i, \tag{2.13}$$

with equality if and only if L is an origin-symmetric ball.

Applying the above dual Minkowski inequality (2.13) and Blaschke-Santaló inequality, we can obtain the following dual Blaschke-Santaló inequality: For $L \in \mathcal{K}_0^n$ and $0 \leq i \leq n - 1$,

$$\tilde{W}_i(L) \tilde{W}_i(L^*) \leq \omega_n^2, \tag{2.14}$$

with equality if and only if L is an origin-symmetric ball.

Suppose that μ is a probability measure on a space X and $g : X \rightarrow I \subset \mathbb{R}$ is a μ -integrable function, where I is a possibly infinite interval. Jessen’s inequality (see [9]) states that if $\phi : X \rightarrow I \subset \mathbb{R}$ is a strictly convex function, then

$$\int_X \phi(g(x)) d\mu(x) \geq \phi \left(\int_X g(x) d\mu(x) \right), \tag{2.15}$$

with equality if and only if $g(x)$ is a constant for μ -almost all $x \in X$. If ϕ is a strictly concave function, then the inequality (2.15) is reversed.

3. Main results

For $K, L \in \mathcal{K}_0^n$ and $i = 0, \dots, n - 1$, since the mixed quermassintegral $W_i(L, K) = \frac{1}{n} \int_{S^{n-1}} h_K dS_{i,L}$, we denote the mixed quermassintegral measure by

$$dw_{i,L} = \frac{1}{n} h_L dS_{i,L}. \tag{3.1}$$

We write the mixed quermassintegral probability measure on S^{n-1} by

$$d\bar{w}_{i,L} = \frac{1}{W_i(L, K)} dw_{i,L}. \tag{3.2}$$

If $K = L$, then we write $dv_{i,L}$ for $d\bar{w}_{i,L}$, and call it the mixed cone-volume probability measure of L . It is easy to check that

$$dv_{i,L} = \frac{h_L}{h_K} dw_{i,L}. \tag{3.3}$$

In particular, the measure $dv_{i,B}$ is just the Lebesgue measure on S^{n-1} . We write the quermassintegral probability measure on S^{n-1} by

$$d\bar{v}_{i,L} = \frac{1}{W_i(L)} dv_{i,L}. \tag{3.4}$$

The measure $d\bar{v}_{0,L}$ will be called the cone-volume probability measure, and it will be written simply as $d\bar{v}_L$.

PROPOSITION 3.1. *Let $K, L \in \mathcal{K}_0^n$ and $i = 0, \dots, n - 1$. Then*

$$\int_{S^{n-1}} \ln \frac{h_K}{h_L} d\bar{w}_{i,L} \geq \ln \frac{W_i(L, K)}{W_i(L)} \geq \frac{1}{n - i} \ln \frac{W_i(K)}{W_i(L)},$$

with equality if and only if K is homothetic to L .

Proof. By Lebesgue’s dominated convergence theorem, as $p \rightarrow \infty$,

$$\int_{S^{n-1}} \left(\frac{h_K}{h_L} \right)^{\frac{p}{n+p}} dv_{i,L} \rightarrow W_i(L, K) \quad \text{and}$$

$$\int_{S^{n-1}} \left(\frac{h_K}{h_L} \right)^{\frac{p}{n+p}} \ln \frac{h_K}{h_L} dv_{i,L} \rightarrow \int_{S^{n-1}} \frac{h_K}{h_L} \ln \frac{h_K}{h_L} dv_{i,L} = \int_{S^{n-1}} \ln \frac{h_K}{h_L} dw_{i,L}$$

We define the function $f_{K,L} : [1, \infty) \rightarrow \mathbb{R}$ by

$$f_{K,L}(p) = \frac{1}{W_i(L, K)} \int_{S^{n-1}} \left(\frac{h_K}{h_L} \right)^{\frac{p}{n+p}} dv_{i,L}. \tag{3.5}$$

Note that

$$\begin{aligned} \lim_{p \rightarrow \infty} \ln(f_{K,L}(p))^{p+n} &= \lim_{p \rightarrow \infty} \frac{\frac{nW_i(L,K)}{(p+n)^2} \int_{S^{n-1}} \left(\frac{h_K}{h_L} \right)^{\frac{p}{n+p}} \ln \frac{h_K}{h_L} dv_{i,L}}{-\frac{f_{K,L}(p)}{(p+n)^2}} \\ &= -\frac{n}{W_i(L, K)} \int_{S^{n-1}} \frac{h_K}{h_L} \ln \frac{h_K}{h_L} dv_{i,L}. \end{aligned} \tag{3.6}$$

By (3.2), (3.3), (3.6), and (3.5), we have

$$\begin{aligned} \exp \left[-n \int_{S^{n-1}} \ln \frac{h_K}{h_L} d\bar{w}_{i,L} \right] &= \exp \left[-\frac{n}{W_i(L, K)} \int_{S^{n-1}} \ln \frac{h_K}{h_L} dw_{i,L} \right] \\ &= \exp \left[-\frac{n}{W_i(L, K)} \int_{S^{n-1}} \frac{h_K}{h_L} \ln \frac{h_K}{h_L} dv_{i,L} \right] \\ &= \lim_{p \rightarrow \infty} \left[\frac{1}{W_i(L, K)} \int_{S^{n-1}} \left(\frac{h_K}{h_L} \right)^{\frac{p}{n+p}} dv_{i,L} \right]^{p+n}. \end{aligned}$$

Then the first claim follows from Hölder’s inequality

$$\left(\int_{S^{n-1}} \left(\frac{h_K}{h_L} \right)^{\frac{p}{n+p}} dv_{i,L} \right)^{\frac{p+n}{p}} \cdot \left(\int_{S^{n-1}} dv_{i,L} \right)^{-\frac{n}{p}} \leq \int_{S^{n-1}} \frac{h_K}{h_L} dv_{i,L} = W_i(L, K),$$

as $\int_{S^{n-1}} dv_{i,L} = W_i(L)$.

Applying Minkowski’s inequality (2.9) to the first inequality of Proposition 3.1, we obtain

$$\int_{S^{n-1}} \ln \frac{h_K}{h_L} d\bar{w}_{i,L} \geq \frac{1}{n-i} \ln \frac{W_i(K)}{W_i(L)}. \tag{3.7}$$

By the equality condition of Minkowski’s inequality (2.9), we know that equality in (3.7) holds if and only if K is homothetic to L . \square

REMARK 3.1. The case $i = 0$ of Proposition 3.1 was obtained by Stancu [15]. An immediate consequence of Proposition 3.1 is:

COROLLARY 3.1. Let $K, L \in \mathcal{K}_0^n$ and $i = 0, \dots, n - 1$. If $L \subseteq K$, then

$$\int_{S^{n-1}} \frac{h_K}{h_L} \ln \frac{h_K}{h_L} d\bar{v}_{i,L} \geq \frac{1}{n-i} \left(\frac{W_i(K)}{W_i(L)} \right)^{\frac{1}{n-i}} \ln \frac{W_i(K)}{W_i(L)},$$

with equality if and only if K is homothetic to L .

Proof. From Proposition 3.1 and Minkowski’s inequality (2.9), it follows that

$$\begin{aligned} \int_{S^{n-1}} \frac{h_K}{h_L} \ln \frac{h_K}{h_L} d\bar{v}_{i,L} &= \frac{1}{W_i(L)} \int_{S^{n-1}} \frac{h_K}{h_L} \ln \frac{h_K}{h_L} dv_{i,L} \\ &= \frac{W_i(L, K)}{W_i(L)} \int_{S^{n-1}} \ln \frac{h_K}{h_L} d\bar{w}_{i,L} \\ &\geq \frac{W_i(L, K)}{W_i(L)} \ln \frac{W_i(L, K)}{W_i(L)} \\ &\geq \frac{1}{n-i} \left(\frac{W_i(K)}{W_i(L)} \right)^{\frac{1}{n-i}} \ln \frac{W_i(K)}{W_i(L)}, \end{aligned}$$

with equality if and only if K is homothetic to L . \square

REMARK 3.2. The case $i = 0$ of Corollary 3.1 was firstly obtained by Gardner, Hug, and Weil [4].

LEMMA 3.1. [2] If p, q are probability density functions on a measure space (X, ν) , then

$$\int p \ln p d\nu \geq \int p \ln q d\nu.$$

PROPOSITION 3.2. *Let $K, L \in \mathcal{K}_0^n$ and $i = 0, \dots, n - 1$. Then*

$$\int_{S^{n-1}} \ln \frac{h_K}{h_L} d\bar{v}_{i,L} \leq \ln \frac{W_i(L, K)}{W_i(L)} \leq \int_{S^{n-1}} \ln \frac{h_K}{h_L} d\bar{w}_{i,L}.$$

Proof. By taking $pdv = \frac{h_L}{h_K} \frac{1}{W_i(L)} dw_{i,L}$ and $qdv = \frac{1}{W_i(L, K)} dw_{i,L}$ in Lemma 3.1, we obtain that

$$\begin{aligned} \int_{S^{n-1}} \frac{h_L}{h_K} \frac{1}{W_i(L)} \ln \left(\frac{h_L}{h_K} \frac{1}{W_i(L)} \right) dw_{i,L} &= \int_{S^{n-1}} \frac{h_L}{h_K} \frac{1}{W_i(L)} \left(\ln \frac{h_L}{h_K} + \ln \frac{1}{W_i(L)} \right) dw_{i,L} \\ &= \int_{S^{n-1}} \frac{1}{W_i(L)} \left(\ln \frac{h_L}{h_K} + \ln \frac{1}{W_i(L)} \right) dv_{i,L} \\ &= \int_{S^{n-1}} \left(\ln \frac{h_L}{h_K} + \ln \frac{1}{W_i(L)} \right) d\bar{v}_{i,L} \\ &= \int_{S^{n-1}} \ln \frac{h_L}{h_K} d\bar{v}_{i,L} + \ln \frac{1}{W_i(L)} \\ &\geq \int_{S^{n-1}} \frac{h_L}{h_K} \frac{1}{W_i(L)} \ln \frac{1}{W_i(L, K)} dw_{i,L} \\ &= \ln \frac{1}{W_i(L, K)}. \end{aligned}$$

Thus,

$$\int_{S^{n-1}} \ln \frac{h_K}{h_L} d\bar{v}_{i,L} \leq \ln \frac{W_i(L, K)}{W_i(L)}.$$

On the other hand, by taking $pdv = \frac{1}{W_i(L, K)} dw_{i,L}$ and $qdv = \frac{h_L}{h_K} \frac{1}{W_i(L)} dw_{i,L}$ in Lemma 3.1, we obtain the second inequality of Proposition 3.2. \square

REMARK 3.3. The case $i = 0$ of Proposition 3.2 was firstly obtained by Stancu [15].

There exists another proof of the first inequality of Proposition 3.2. Note that for $K, L \in \mathcal{K}_0^n$ and $i = 0, \dots, n - 1$,

$$\lim_{p \rightarrow \infty} \left(\frac{1}{W_i(L)} \int_{S^{n-1}} \left(\frac{h_K}{h_L} \right)^{\frac{1}{p+n}} dv_{i,L} \right)^{p+n} = \exp \left[\int_{S^{n-1}} \ln \frac{h_K}{h_L} d\bar{v}_{i,L} \right]. \tag{3.8}$$

From Hölder’s inequality and (3.8), it follows that

$$\left(\int_{S^{n-1}} \left(\frac{h_K}{h_L} \right)^{\frac{1}{p+n}} dv_{i,L} \right)^{p+n} \cdot \left(\int_{S^{n-1}} dv_{i,L} \right)^{1-(p+n)} \leq \int_{S^{n-1}} \frac{h_K}{h_L} dv_{i,L} = W_i(L, K),$$

thus

$$\int_{S^{n-1}} \ln \frac{h_K}{h_L} d\bar{v}_{i,L} \leq \ln \frac{W_i(L, K)}{W_i(L)}. \tag{3.9}$$

We will denote the support of the mixed cone-volume measure of $v_{i,K}$ and $v_{i,L}$ by

$\text{supp } v_{i,K}$ and $\text{supp } v_{i,L}$, respectively. $\left(\frac{h_K}{h_L}\right)_{i, \text{average}} := \frac{\int_{S^{n-1}} \frac{h_K}{h_L} dv_{i,L}}{\int_{S^{n-1}} dv_{i,L}}$, $\left(\frac{h_K}{h_L}\right)_{i, \text{max}} = \max_{u \in \text{supp } v_{i,L}} \frac{h_K}{h_L}$, $\left(\frac{h_K}{h_L}\right)_{i, \text{min}} = \min_{u \in \text{supp } v_{i,K}} \frac{h_K}{h_L}$. In particular, $\left(\frac{h_K}{h_L}\right)_{0, \text{average}} = \left(\frac{h_K}{h_L}\right)_{\text{average}}$, $\left(\frac{h_K}{h_L}\right)_{0, \text{max}} = \left(\frac{h_K}{h_L}\right)_{\text{max}}$, and $\left(\frac{h_K}{h_L}\right)_{0, \text{min}} = \left(\frac{h_K}{h_L}\right)_{\text{min}}$.

THEOREM 3.1. *Let $K, L \in \mathcal{K}_0^n$ and $i = 0, \dots, n - 1$. If $L \subseteq K$, then*

$$\int_{S^{n-1}} \ln \frac{h_L}{h_K} d\bar{v}_{i,K} \geq \frac{\left(\frac{h_L}{h_K}\right)_{i, \text{average}}}{\left(\frac{h_L}{h_K}\right)_{i, \text{max}}} \frac{1}{n-i} \ln \frac{W_i(L)}{W_i(K)}, \tag{3.10}$$

with equality if and only if $K = L$.

Proof. We define the non-negative function

$$G(q) := \int_{S^{n-1}} \left(\frac{h_K}{h_L}\right)^q \ln \frac{h_K}{h_L} dv_{i,L}.$$

If $u \mapsto \ln \left(\frac{h_K}{h_L}\right)(u)$ is zero on the support of the mixed cone-volume measure $dv_{i,L}$, then G is identically zero. Assume, for now, that this is not the case which also implies that $G(1) \geq G(0) > 0$. If $G(1) = G(0)$, then conclusion is trivial (as using (3.1) to obtain $K = L$), so we can assume $G(1) > G(0)$.

A simple verification shows that $G(q)$ is a log-convex function. By a Hadamard type inequality for positive log-convex function (see [6]), we have that

$$\frac{G(1) - G(0)}{\ln(G(1)/G(0))} \geq \int_0^1 \left[\int_{S^{n-1}} \left(\frac{h_K}{h_L}\right)^q \ln \frac{h_K}{h_L} dv_{i,L} \right] dq$$

and, since $G(1) > G(0)$, by Fubini-Tonelli's theorem,

$$G(0) \geq G(1) \cdot \exp \left[- \frac{G(1) - G(0)}{\int_{S^{n-1}} \left(\frac{h_K}{h_L} - 1\right) dv_{i,L}} \right].$$

Note that

$$\frac{G(1) - G(0)}{\int_{S^{n-1}} \left(\frac{h_K}{h_L} - 1\right) dv_{i,L}} = \frac{\int_{S^{n-1}} \ln \frac{h_K}{h_L} \cdot \left(\frac{h_K}{h_L} - 1\right) dv_{i,L}}{\int_{S^{n-1}} \left(\frac{h_K}{h_L} - 1\right) dv_{i,L}} \leq \ln \left(\frac{h_K}{h_L}\right)_{i, \text{max}},$$

Then

$$\int_{S^{n-1}} \ln \frac{h_K}{h_L} d\bar{v}_{i,L} \geq \exp \left[-\ln \left(\frac{h_K}{h_L} \right)_{i,\max} \right] \cdot \frac{W_i(L, K)}{W_i(L)} \int_{S^{n-1}} \ln \frac{h_K}{h_L} d\bar{v}_{i,L},$$

from which (3.10) follows from (3.7).

Assuming that G is identically zero, then $h_K(u) = h_L(u)$ for all u 's almost everywhere with respect to the mixed cone measure of L , or equivalently with respect to the mixed surface area measure of L . This implies $W_i(L, K) = W_i(L)$, and since $L \subseteq K$ are convex bodies, K and L must coincide. \square

REMARK 3.4. The case $i = 0$ of Theorem 3.1 was firstly obtained by Stancu [15].

Proof of Theorem 1.1. If L is not included in K , there exists a λ , $0 < \lambda < 1$, such that $\tilde{L} := \lambda L \subseteq K$ and apply (3.10) for \tilde{L} and K .

Thus,

$$\int_{S^{n-1}} \ln \frac{h_K}{h_L} d\bar{v}_{i,L} \geq \frac{\left(\frac{h_K}{h_L}\right)_{i,\text{average}}}{\left(\frac{h_K}{h_L}\right)_{i,\text{max}}} \frac{1}{n-i} \ln \frac{W_i(K)}{W_i(L)} + \ln \lambda \cdot \left(1 - \frac{\left(\frac{h_K}{h_L}\right)_{i,\text{average}}}{\left(\frac{h_K}{h_L}\right)_{i,\text{max}}} \right).$$

By taking $\lambda = \min_{u \in \text{supp}_{v_i,K}} \frac{h_K}{h_L}(u)$, we obtain the desired inequality. \square

A direct consequence of Theorem 1.1 is:

COROLLARY 3.2. Let $K, L \in \mathcal{K}_0^n$ such that there exists a positive constant $c > 0$ with $h_K(u) = ch_L(u)$ for each u in the support of the mixed cone volume measure of L . For $i = 0, \dots, n - 1$, then

$$\int_{S^{n-1}} \ln \frac{h_K}{h_L} d\bar{v}_{i,L} \geq \frac{1}{n-i} \ln \frac{W_i(K)}{W_i(L)},$$

with equality if and only if $K = cL$.

COROLLARY 3.3. For any $L \in \mathcal{K}_0^n$ whose support function restricted to $\text{supp } v_{i,L}$ is constant and $i = 0, \dots, n - 1$, we have that

$$\int_{S^{n-1}} \ln \frac{h_{L^*}}{h_L} d\bar{v}_{i,L} \geq \frac{2}{n-i} \ln \frac{\omega_n}{W_i(L)},$$

with equality if and only if K is an origin-symmetric ball.

Proof. Taking $K = B$ in Corollary 3.2, we have

$$\begin{aligned} \int_{S^{n-1}} \ln \frac{h_{L^*}}{h_L} d\bar{v}_{i,L} &= \int_{S^{n-1}} \ln \frac{1}{\rho_L h_L} d\bar{v}_{i,L} \geq \int_{S^{n-1}} \ln \frac{1}{h_L^2} d\bar{v}_{i,L} \\ &= 2 \int_{S^{n-1}} \ln \frac{1}{h_L} d\bar{v}_{i,L} \geq \frac{2}{n-i} \ln \frac{\omega_n}{W_i(L)}, \end{aligned}$$

with equality if and only if L is an origin-symmetric ball. \square

REMARK 3.5. The case $i = 0$ of Corollary 3.2 and Corollary 3.3 were obtained by Stancu [15].

For $K \in \mathcal{S}^n$, we denote the dual mixed cone-volume measure by

$$d\tilde{v}_{i,K} = \frac{1}{n} \rho_K^{n-i} dS. \tag{3.11}$$

Since $\frac{1}{n\tilde{W}_i(K)} \int_{S^{n-1}} \rho_K^{n-i} dS = 1$, we write the dual mixed cone-volume probability measure of K on S^{n-1} by

$$d\tilde{\tilde{v}}_{i,K} = \frac{1}{\tilde{W}_i(K)} d\tilde{v}_{i,K}. \tag{3.12}$$

If $i = 0$, the measure $d\tilde{\tilde{v}}_{0,K}$ will be denoted by the dual cone-volume probability measure, and it will be written simply as $d\tilde{\tilde{v}}_K$.

Recently, Gardner et al. [5] established the dual log-Minkowski inequality (also see [16]). Next, we will establish the following double inequalities which are more general than the dual log-Minkowski inequality.

PROPOSITION 3.3. *Let $K, L \in \mathcal{S}^n$. For $i = 0, \dots, n - 1$, we have*

$$\int_{S^{n-1}} \ln \frac{\rho_K}{\rho_L} d\tilde{\tilde{v}}_{i,K} \geq \frac{1}{n-i} \ln \frac{\tilde{W}_i(K)}{\tilde{W}_i(L)} \geq \int_{S^{n-1}} \ln \frac{\rho_K}{\rho_L} d\tilde{\tilde{v}}_{i,L}.$$

Both equalities hold if and only if K and L are dilates.

Proof. Applying (2.10), (3.12), and Jensen’s inequality, we obtain

$$\begin{aligned} \tilde{W}_i(L, K) &= \frac{1}{n} \int_{S^{n-1}} \rho_L^{n-i-1} \rho_K dS \\ &= \tilde{W}_i(K) \int_{S^{n-1}} \left(\frac{\rho_L}{\rho_K} \right)^{n-i-1} d\tilde{\tilde{v}}_{i,K} \\ &= \tilde{W}_i(K) \int_{S^{n-1}} \exp \left(\ln \left(\frac{\rho_L}{\rho_K} \right)^{n-i-1} \right) d\tilde{\tilde{v}}_{i,K} \\ &\geq \tilde{W}_i(K) \exp \int_{S^{n-1}} \ln \left(\frac{\rho_L}{\rho_K} \right)^{n-i-1} d\tilde{\tilde{v}}_{i,K}. \end{aligned} \tag{3.13}$$

Taking the natural logarithm of both sides of (3.13) and using the dual Minkowski inequality (2.12), one can obtain

$$\int_{S^{n-1}} \ln \frac{\rho_K}{\rho_L} d\tilde{\tilde{v}}_{i,K} \geq \frac{1}{n-i} \ln \frac{\tilde{W}_i(K)}{\tilde{W}_i(L)}. \tag{3.14}$$

By the equality condition of the dual Minkowski inequality (2.12), we know that equality in (3.14) holds if and only if K and L are dilates.

On the other hand,

$$\tilde{W}_i(L, K) = \frac{1}{n} \int_{S^{n-1}} \rho_L^{n-i-1} \rho_K dS_n = \tilde{W}_i(L) \int_{S^{n-1}} \frac{\rho_K}{\rho_L} d\tilde{v}_{i,L}.$$

Similarly, we can obtain

$$\int_{S^{n-1}} \ln \frac{\rho_K}{\rho_L} d\tilde{v}_{i,L} \leq \frac{1}{n-i} \ln \frac{\tilde{W}_i(K)}{\tilde{W}_i(L)}. \quad \square$$

REMARK 3.6. The case $i = 0$ of Proposition 3.3 was firstly obtained by Gardner, Hug and Weil [4].

PROPOSITION 3.4. *Let K be an arbitrary convex body in \mathcal{K}_0^n with its Santaló point at the origin and $i = 0, \dots, n - 1$. Then*

$$\int_{S^{n-1}} \ln h_K d\tilde{v}_{i,B} \geq \frac{1}{n-i} \ln \frac{\tilde{W}_i(K)}{\omega_n} \geq \int_{S^{n-1}} \ln \rho_K d\tilde{v}_{i,B},$$

with equality if and only if K is an origin-symmetric ball.

Proof. By (2.11), (2.6), (3.12), and Jessen’s inequality, we have

$$\begin{aligned} \tilde{W}_i(K^*) &= \frac{1}{n} \int_{S^{n-1}} \rho_{K^*}^{n-i} dS = \frac{1}{n} \int_{S^{n-1}} h_K^{-(n-i)} dS \\ &= \omega_n \int_{S^{n-1}} \exp(\ln(h_K^{-(n-i)})) d\tilde{v}_{i,B} \\ &\geq \omega_n \exp \int_{S^{n-1}} (\ln(h_K^{-(n-i)})) d\tilde{v}_{i,B}. \end{aligned}$$

Thus,

$$\int_{S^{n-1}} \ln h_K d\tilde{v}_{i,B} \geq \frac{1}{n-i} \ln \frac{\omega_n}{\tilde{W}_i(K^*)} \geq \frac{1}{n-i} \ln \frac{\tilde{W}_i(K)}{\omega_n},$$

where the last step is due to the dual Blaschke-Santaló’s inequality (2.14).

On the other hand, by taking $L = B$ in the second inequality of Proposition 3.3, we obtain

$$\int_{S^{n-1}} \ln \rho_K d\tilde{v}_{i,B} \leq \frac{1}{n-i} \ln \frac{\tilde{W}_i(K)}{\omega_n},$$

with equality if and only if K is an origin-symmetric ball. \square

REMARK 3.7. The case $i = 0$ of Proposition 3.4 was firstly obtained by Guan and Ni [8].

The dual form of Corollary 3.3 will be established.

PROPOSITION 3.5. For any $L \in \mathcal{K}_0^n$ and $i = 0, \dots, n - 1$, we have that

$$\int_{S^{n-1}} \ln \frac{h_L^*}{h_L} d\bar{v}_{i,L} \leq \frac{2}{n-i} \ln \frac{\omega_n}{\widetilde{W}_i(L)},$$

with equality if and only if K is an origin-symmetric ball.

Proof. By (2.6), note that $h_L \geq \rho_L$ for $\forall L \in \mathcal{K}_0^n$, and take $K = B$ in Proposition 3.3, then we have

$$\begin{aligned} \int_{S^{n-1}} \ln \frac{\rho_L^*}{\rho_L} d\bar{v}_{i,L} &= \int_{S^{n-1}} \ln \frac{1}{\rho_L h_L} d\bar{v}_{i,L} \\ &\leq \int_{S^{n-1}} \ln \frac{1}{\rho_L^2} d\bar{v}_{i,L} \\ &= 2 \int_{S^{n-1}} \ln \frac{1}{\rho_L} d\bar{v}_{i,L} \\ &\leq \frac{2}{n-i} \ln \frac{\omega_n}{\widetilde{W}_i(L)}, \end{aligned}$$

with equality if and only if L is an origin-symmetric ball. \square

REMARK 3.8. The case $i = 0$ of Proposition 3.5 was firstly obtained by Stancu [15].

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