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ON THE RESTRICTED SUMMABILITY OF THE MULTI–DIMENSIONAL VILENKIN–CESÀRO MEANS

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Abstract. The properties of the maximal operator of the (C,α)-means ($\alpha=(\alpha_1,\dots,\alpha_d)$) of the multi-dimensional Vilenkin-Fourier series are discussed, where the set of indices is inside a cone-like set. Weisz proved that the maximal operator is bounded from martingale Hardy space H_p^γ to the space L_p for $p_0<p$ ($p_0=\max\{1/(1+\alpha_k);k=1,\dots,d\}$) [21]. The next question arise. Is the boundary point p_0 essential or not? In the present paper we show that the maximal operator $\sigma_L^{\alpha,*}$ is not bounded from the Hardy space $H_{p_0}^\gamma$ to the space L_{p_0}.

1. Definitions and notation

Now, we give a brief introduction to the theory of dyadic analysis (for more details see [1, 16]).

Let us denote the set of positive integers by $\mathbb{N}_+$, $\mathbb{N}:=\mathbb{N}_+\cup\{0\}$. Let $m:=(m_0,m_1,\dots)$ be a sequence of the positive integers not less than 2. Denote by $Z_{m_n}:=\{0,1,\dots,m_n-1\}$ the additive group of integers modulo m_n. Define the group G_m as the complete direct product of the groups Z_{m_n} with the product of the discrete topologies of Z_{m_n}‘s.

In this paper we discuss bounded Vilenkin groups, i.e. the case when $\sup_n m_n<\infty$.

The direct product μ of the measures

$$\mu_n(\{j\}):=1/m_n,\ (j\in Z_{m_n})$$

is a Haar measure on G_m with $\mu(G_m)=1$. The elements of G_m are represented by sequences

$$x:=(x_0,x_1,\dots,x_n,\dots),\ (x_n\in Z_{m_n}).$$

It is easy to give a base for the neighbourhood of G_m:

$$I_0(x):=G_m,$$

$$I_n(x):=\{y\in G_m \mid y_0=x_0,\dots,y_{n-1}=x_{n-1}\},\ \ (x\in G_m,n\in\mathbb{N}).$$

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Let us denote $I_n := I_n(0)$, for $n \in \mathbb{N}$. For a set $S \subseteq G_m$, we use the standard notation $\overline{S} := G_m \backslash S$.

If we define the so-called generalized number system based on m in the following way:

$$M_0 := 1,\ M_{n+1} := m_n M_n, \quad (n \in \mathbb{N}),$$

then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{k=0}^{\infty} n_k M_k$, where $n_k \in Z_{m_k}$ $(k \in \mathbb{N})$ and only a finite number of n_k's differ from zero. The order $|n|$ of a positive natural number n is defined by $|n| := \max\{i \in \mathbb{N} : n_i \neq 0\}$.

Next, we introduce on G_m an orthonormal system which is called the Vilenkin system. At first we define the complex-valued function $r_k(x) : G_m \to \mathbb{C}$, the generalized Rademacher functions, by

$$r_k(x) := \exp(2\pi \imath x_k / m_k),\ \left(\imath^2 = -1,\ x \in G_m,\ k \in \mathbb{N}\right).$$

The Vilenkin system $\psi := (\psi_n : n \in \mathbb{N})$ is defined on G_m as:

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x), \quad (n \in \mathbb{N}).$$

Specifically, we call this system the Walsh-Paley system, when $m \equiv 2$.

The Vilenkin systems are orthonormal and complete in $L_2(G_m)$ (see [26]).

Let $0 < \alpha \leqslant 1$ and

$$A_j^{\alpha} := \binom{j+\alpha}{j} = \frac{(\alpha+1)(\alpha+2)\dots(\alpha+j)}{j!}, \quad (j \in \mathbb{N};\ \alpha \neq -1, -2, \dots).$$

It is known that

$$A_j^{\alpha} \sim j^{\alpha}, \quad A_n^{\alpha} - A_{n-1}^{\alpha} = A_n^{\alpha-1}, \quad \sum_{k=0}^{n} A_k^{\alpha-1} = A_n^{\alpha}. \tag{1}$$

(see Zygmund [27, page 42.]). The one-dimensional Dirichlet kernels and Cesàro kernels are defined by

$$D_n := \sum_{k=0}^{n-1} \psi_k, \quad K_n^{\alpha}(x) := \frac{1}{A_n^{\alpha}} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} D_k(x),$$

It is known [25], that

$$\int_{G_m} |K_N^{\alpha}(x)| d\mu(x) \leqslant c \quad (N \in \mathbb{N}_+). \tag{2}$$

Choosing $\alpha = 1$ we defined the nth Fejér kernel, as special case. It is well-known that the M_nth Dirichlet kernels have a closed form (see e.g. [16])

$$D_{M_n}(x) = \begin{cases} 0, & \text{if } x \notin I_n, \\ M_n, & \text{if } x \in I_n. \end{cases}$$

For one-dimensional Walsh-Fourier series Fine [3] showed that the (C,α) means $\sigma_n^\alpha f$ of an integrable function f converge almost everywhere to f as $n \to \infty$. The maximal operator $\sigma_*^\alpha f := \sup_n |\sigma_n^\alpha f|$ $(0 < \alpha < 1)$ was investigated by Weisz [22]. He proved that the maximal operator σ_*^α is bounded from the Hardy space H_p to the space L_p when $p > 1/(1+\alpha)$. In the endpoint $p = 1/(1+\alpha)$ Goginava constructed a counterexample martingale which shows that the assumption $p > 1/(1+\alpha)$ is essential [8]. Similar result for (C,1) means on bounded Vilenkin groups are due to Simon [17]. Namely the maximal operator of Fejér means is bounded from the Hardy space H_1 to the space L_1. Recently, new aspects of the maximal operator of Vilenkin-Fejér means are showed by Persson and Tephnadze [14, 15].

Let us set $G_m^d := G_m \times \ldots \times G_m$, where the same Vilenkin group G_m appears d times in the direct product. The Kronecker product $\left(\psi_n : n \in \mathbb{N}^d\right)$ of d Vilenkin system is said to be the d-dimensional (or multi-dimensional) Vilenkin system. That is,

$$\psi_n(x) := \psi_{n_1}\left(x^1\right) \ldots \psi_{n_d}\left(x^d\right),$$

where $x := (x^1, \ldots, x^d) \in G_m^d$ and $n := (n_1, \ldots, n_d)$.

If $f \in L_1\left(G_m^d\right)$, then the number $\widehat{f}(n) := \int_{G_m^d} f \overline{\psi_n}$ $\left(n \in \mathbb{N}^d\right)$ is said to be the nth (d-dimensional) Vilenkin-Fourier coefficient of f. We can extend this definition to martingales in the usual way (see Weisz [19, 20]).

The d-dimensional Fourier partial sums are the following:

$$S_n(f;x) := \sum_{i_1=0}^{n_1} \ldots \sum_{i_d=0}^{n_d} \widehat{f}(i) \psi_i(x)$$

where $x := (x_1, \ldots, x_d)$ and $n := (n_1, \ldots, n_d)$.

The d-dimensional (C,α) $(\alpha = (\alpha_1, \ldots, \alpha_d))$ or Cesàro means of a martingale is defined by

$$\sigma_n^\alpha f(x) := \frac{1}{\prod_{i=1}^d A_{n_i}^{\alpha_i}} \sum_{k_1=0}^{n_1} \ldots \sum_{k_d=0}^{n_d} \prod_{i=1}^d A_{n_i-k_i}^{\alpha_i-1} S_k(f;x),$$

where $k := (k_1, \ldots, k_d)$. It is known that

$$K_n^\alpha(x) = K_{n_1}^{\alpha_1}(x^1) \ldots K_{n_d}^{\alpha_d}(x^d), \quad (x := (x^1, \ldots, x^d),\ n := (n_1, \ldots, n_d)).$$

For $x = (x^1, x^2, \ldots, x^d) \in G_m^d$ and $n = (n_1, n_2, \ldots, n_d) \in \mathbb{N}^d$ the d-dimensional rectangles are defined by $I_n(x) := I_{n_1}(x^1) \times \ldots \times I_{n_d}(x^d)$ For $n \in \mathbb{N}^d$ the σ-algebra generated by the rectangles $\{I_n(x), x \in G_m^d\}$ is denoted by $\mathscr{F}_n$.

Suppose that for all $j = 2, \ldots, d$ the functions $\gamma_j\colon [1,\infty) \to [1,\infty)$ are strictly monotone increasing continuous functions with properties $\lim_\infty \gamma_j = \infty$ and $\gamma_j(1) = 1$ $(j = 2, \ldots, d)$. Moreover, suppose that there exist $\zeta, c_{j,1}, c_{j,2} > 1$ such that the inequality

$$c_{j,1}\gamma_j(x) \leqslant \gamma_j(\zeta x) \leqslant c_{j,2}\gamma_j(x) \tag{3}$$

holds for each $x \geqslant 1$. In this case the functions γ_j are called CRF (cone-like restriction functions). Let $\gamma := (\gamma_2, \ldots, \gamma_d)$ and $\beta_j \geqslant 1$ be fixed $(j = 2, \ldots, d)$. Weisz investigated the maximal operator of the multi-dimensional (C,α) means and the convergence over a cone-like set L (with respect to the first dimension), where

$$L := \{n \in \mathbb{N}^d : \beta_j^{-1}\gamma_j(n_1) \leqslant n_j \leqslant \beta_j\gamma_j(n_1),\ j = 2,\dots,d\}.$$

If each γ_j is the identical function then we get a cone. The cone-like sets were introduced by Gát in dimension two [5]. The condition (3) on the function γ is natural, because Gát [5] proved that to each cone-like set with respect to the first dimension there exists a larger cone-like set with respect to the second dimension and reversely, if and only if the inequality (3) holds.

Weisz defined a new type martingale Hardy space depending on the function γ (see [21]). For a given $n_1 \in \mathbb{N}$ set $n_j := |\gamma_j(M_{n_1})|$ $(j = 2,\dots,d)$, that is, n_j is the order of $\gamma_j(M_{n_1})$ (this means that $M_{n_j} \leqslant \gamma_j(M_{n_1}) < M_{n_j+1}$ for $j = 2,\dots,d$). Let $\overline{n}_1 := (n_1,\dots,n_d)$. Since, the functions γ_j are increasing, the sequence $(\overline{n}_1,\ n_1 \in \mathbb{N})$ is increasing, too. It is given a class of one-parameter martingales $f = (f_{\overline{n}_1},\ n_1 \in \mathbb{N})$ with respect to the σ-algebras $(\mathscr{F}_{\overline{n_1}},\ n_1 \in \mathbb{N})$. The maximal function of a martingale f is defined by $f^* := \sup_{n_1\in\mathbb{N}} |f_{\overline{n}_1}|$. For $0 < p \leqslant \infty$ the martingale Hardy space $H_p^\gamma(G_m^d)$ consists of all martingales for which $\|f\|_{H_p^\gamma} := \|f^*\|_p < \infty$, where $\|.\|_p$ is the usual L_p norm. It is known (see [20]) that $H_p^\gamma \sim L_p$ for $1 < p \leqslant \infty$, where $\sim$ denotes the equivalence of a norm and a space.

If $f \in L_1(G_m^d)$, then it is easy to show that the sequence $(S_{M_{n_1},\dots,M_{n_d}}(f) : \overline{n_1} = (n_1,\dots,n_d),\ n_1 \in \mathbb{N})$ is a one-parameter martingale with respect to the σ-algebras $(\mathscr{F}_{\overline{n_1}},\ n_1 \in \mathbb{N})$. In this case the maximal function can also be given by

$$f^*(x) = \sup_{n_1\in\mathbb{N}} \frac{1}{\mu(I_{\overline{n_1}}(x))} \left| \int_{I_{\overline{n_1}}(x)} f(u)\,d\mu(u) \right| = \sup_{n_1\in\mathbb{N}} |S_{M_{n_1},\dots,M_{n_d}}(f;x)|$$

for $x \in G_m^d$.

We define the maximal operator $\sigma_L^{\alpha,*}$ by

$$\sigma_L^{\alpha,*} f(x) := \sup_{n\in L} |\sigma_n^\alpha f(x)|.$$

For double Walsh-Fourier series, Móricz, Schipp and Wade [10] proved that $\sigma_n f$ converge to f a.e. in the Pringsheim sense (that is, no restriction on the indices other than $\min(n_1,n_2) \to \infty$) for all functions $f \in L\log^+ L$. In the paper [4] Gát proved that the theorem of Móricz, Schipp and Wade can not be sharpened.

The convergence almost everywhere of double Walsh-Fejér means $\sigma_n f$ of integrable functions, where the set of indices is inside a positive cone around the identical function, that is $\beta^{-1} \leqslant n_1/n_2 \leqslant \beta$ is provided with some fixed parameter $\beta \geqslant 1$, was proved by Gát [6] and Weisz [23]. Analogical results for Vilenkin-Fejér means are presented by Gát and Blahota [2], for multidimensional Vilenkin-Cesàro means by Weisz [24].

A common generalization of results of Móricz, Schipp, Wade [10] and Gát [6], Weisz [23] for cone-like set was given by the second author and Gát in [7]. That is, a necessary and sufficient condition for cone-like sets in order to preserve the convergence property, was given. Recently, the properties of the maximal operator of the (C,α) means of a multi-dimensional Vilenkin-Fourier series provided that the supremum in the maximal operator is taken over a cone-like set, was discussed by Weisz [21]. Namely, it was proved that the maximal operator is bounded from H_p^γ to L_p for $p_0 < p \leqslant \infty$ (with $p_0 := \max\{1/(1+\alpha_i); i = 1,\dots,d\}$) and is of weak type $(1,1)$. Consequently, the (C,α) means of multi-dimensional Vilenkin-Fourier series of an integrable function f converge almost everywhere to f. Weak type $(1,1)$ inequality are showed for more general systems by the second author [12, 13], but only in dimension 2.

At the endpoint $p = p_0$, we show that the maximal operator $\sigma_L^{\alpha,*}$ is not bounded from the Hardy space $H_{p_0}^\gamma$ to the space L_{p_0}. That is, we construct a counterexample martingale in the Hardy space $H_{p_0}^\gamma$ which shows that the boundary point p_0 is essential for the boundedness of the maximal operator $\sigma_L^{\alpha,*}$.

We mention that in dimension 2 and for Fejér means a counterexample martingale is presented by the second author [11], earlier. Unfortunately, that counterexample martingale and method do not work for the maximal operator of (C,α) means $(0 < \alpha_1,\dots,\alpha_d < 1)$. This fact motivated us to search a suitable martingale and method for the original question.

2. Auxiliary propositions and main results

THEOREM W. (Weisz [21]) *Let γ be CRF. The maximal operator $\sigma_L^{\alpha,*}$ is bounded from the Hardy space H_p^γ to the space L_p for $p_0 < p \leqslant 1$ ($p_0 := \max\{1/(1+\alpha_i); i = 1,\dots,d\}$).*

Our main theorem shows that the boundary point p_0 is essential.

THEOREM 1. *Let γ be CRF and $0 < \alpha_1 \leqslant \alpha_2,\dots,\alpha_d \leqslant 1$. The maximal operator $\sigma_L^{\alpha,*}$ is not bounded from the Hardy space $H_{p_0}^\gamma$ to the space L_{p_0} (where $p_0 := 1/(1+\alpha_1)$).*

To prove our theorem we need the following Lemma.

LEMMA 1. *Let $n \in \mathbb{N}$ and $0 < \alpha \leqslant 1$. Then*

$$\int_{G_m} \max_{1 \leqslant N \leqslant M_n} (A_N^\alpha |K_N^\alpha(x)|)^{1/(\alpha+1)} d\mu(x) \geqslant c(\alpha) \frac{n}{\log n}.$$

We note that analogical result for Walsh system was proved by Goginava in [8]. In this paper we follow his method. But, we write only a few lines about the proof (for more details see [8]).

Proof. Using equality

$$\int_{G_m} D_i(x)\overline{D_j(x)} d\mu(x) = \min(i,j)$$

and (1) we obtain

$$\int_{G_m}\left|\sum_{i=1}^{N}A_{N-i}^{\alpha-1}D_i(x)\right|^2 d\mu(x)=\sum_{i=1}^{N}\sum_{j=1}^{N}A_{N-i}^{\alpha-1}A_{N-j}^{\alpha-1}\min(i,j)$$
$$=\sum_{i=1}^{N}\sum_{j=1}^{i}A_{N-i}^{\alpha-1}A_{N-j}^{\alpha-1}j+\sum_{i=1}^{N}\sum_{j=i+1}^{N}A_{N-i}^{\alpha-1}A_{N-j}^{\alpha-1}i \qquad (4)$$
$$\geqslant c_1(\alpha)N^{2\alpha+1}.$$

Let us denote

$$E_{N_i}:=\{x\in G_m:\ |K_{N_i}^{\alpha}(x)|\leqslant c_2(\alpha)N_i\},\quad \Omega_{N_i}:=\overline{E}_{N_i}\setminus\bigcup_{j=1}^{i-1}\overline{E}_{N_j},$$

where $N_i:=\left[\frac{M_n}{n^i}\right]$, $i=1,2,\dots,\left[\frac{n}{\log_2 n}\right]$, $n\geqslant 2$ and $c_2(\alpha)$ is some positive constant (depends only on α) discussed later. We note that the sets Ω_{N_i} and Ω_{N_j} are disjoint sets for different i and j.

Inequalities (4) and (2) imply

$$c_1(\alpha)N_i^{2\alpha+1}\leqslant\int_{G_m}(A_{N_i}^{\alpha}|K_{N_i}^{\alpha}(x)|)^2d\mu(x)$$
$$=\int_{E_{N_i}}(A_{N_i}^{\alpha}|K_{N_i}^{\alpha}(x)|)^2d\mu(x)+\int_{\overline{E}_{N_i}}(A_{N_i}^{\alpha}|K_{N_i}^{\alpha}(x)|)^2d\mu(x)$$
$$\leqslant c_2(\alpha)A_{N_i}^{\alpha}N_i\int_{E_{N_i}}A_{N_i}^{\alpha}|K_{N_i}^{\alpha}(x)|d\mu(x)$$
$$+\int_{\overline{E}_{N_i}}(A_{N_i}^{\alpha}|K_{N_i}^{\alpha}(x)|)^{(2\alpha+1)/(\alpha+1)}(A_{N_i}^{\alpha}|K_{N_i}^{\alpha}(x)|)^{1/(\alpha+1)}d\mu(x)$$
$$\leqslant c_2(\alpha)c_3(\alpha)N_i^{2\alpha+1}+c_4(\alpha)N_i^{2\alpha+1}\int_{\overline{E}_{N_i}}(A_{N_i}^{\alpha}|K_{N_i}^{\alpha}(x)|)^{1/(\alpha+1)}d\mu(x).$$

Now, we define $c_2(\alpha):=\frac{c_1(\alpha)}{2c_3(\alpha)}$, then we obtain

$$\int_{\overline{E}_{N_i}}(A_{N_i}^{\alpha}|K_{N_i}^{\alpha}(x)|)^{1/(\alpha+1)}d\mu(x)\geqslant c_5(\alpha)>0. \qquad (5)$$

From the definition of the set $\overline{E}_{N_i}$ follows

$$c(\alpha)N_i\,\mu(\overline{E}_{N_i})<\int_{\overline{E}_{N_i}}|K_{N_i}^{\alpha}(x)|d\mu(x)\leqslant\|K_{N_i}^{\alpha}\|_1\leqslant c_6(\alpha),$$

(see also (2)), so

$$\mu(\overline{E}_{N_i}) \leqslant \frac{c_7(\alpha)}{N_i}. \tag{6}$$

Inequalities (5) and (6) yield

$$\int\limits_{\Omega_{N_i}} (A_{N_i}^{\alpha}|K_{N_i}^{\alpha}(x)|)^{1/(\alpha+1)} d\mu(x) \geqslant c_5(\alpha) - c_8(\alpha) N_i \sum_{j=1}^{i-1} \mu(\overline{E}_{N_j}) \geqslant c_9(\alpha),$$

if n is big enough. (For more details see [8].) This inequality implies

$$\begin{aligned}
\int\limits_{G_m} \max_{1\leqslant N\leqslant M_n} (A_N^{\alpha}|K_N^{\alpha}(x)|)^{1/(\alpha+1)} d\mu(x) &\geqslant \sum_{i=1}^{[n/\log_2 n]} \int\limits_{\Omega_{N_i}} \max_{1\leqslant N\leqslant M_n} (A_N^{\alpha}|K_N^{\alpha}(x)|)^{1/(\alpha+1)} d\mu(x) \\
&\geqslant \sum_{i=1}^{[n/\log_2 n]} \int\limits_{\Omega_{N_i}} (A_{N_i}^{\alpha}|K_{N_i}^{\alpha}(x)|)^{1/(\alpha+1)} d\mu(x) \\
&\geqslant c_9(\alpha) \frac{n}{\log_2 n}.
\end{aligned}$$

This completes the proof of Lemma 1. □

In unrestricted case for double (C,α) means of Walsh-Fourier series Goginava constructed a two-dimensional counterexample martingale at the endpoint p_0 [9]. Unfortunately, his martingale is not suitable in our problem, but his method gave us some idea to solve our original problem.

Now, we prove our main Theorem.

Proof of Theorem 1. Let us define a martingale in $H_{p_0}^{\gamma}$

$$f_{\overline{n_1}}(x) := (D_{M_{n_1+1}}(x^1) - D_{M_{n_1}}(x^1)) \prod_{j=2}^{d} \psi_{M_{n_j-1}}(x^j).$$

Now, we show that, it is a one-parameter martingale, where $n_2, \ldots, n_d$ is defined to n_1, earlier. Now, we calculate the kth Fourier coefficients and the jth partial sums $S_j(f_{\overline{n_1}}; x)$ of the Fourier series of $f_{\overline{n_1}}$.

$$\widehat{f_{\overline{n_1}}}(k) = \begin{cases} 1, & \text{if } k_1 = M_{n_1}, \ldots, M_{n_1+1}-1, \text{ and } k_j = M_{n_j-1} \text{ for all } j = 2, \ldots, d, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$
\begin{aligned}
&S_j(f_{\overline{n_1}};x)\\
&=\sum_{\nu=0}^{j_1-1}\widehat{f_{\overline{n_1}}}(\nu,M_{n_2-1},\dots,M_{n_d-1})\psi_\nu(x^1)\prod_{l=2}^{d}\psi_{M_{n_l-1}}(x^l)\\
&=\begin{cases}(D_{j_1}(x^1)-D_{M_{n_1}}(x^1))\prod_{l=2}^d\psi_{M_{n_l-1}}(x^l) & \text{if } j_1=M_{n_1}+1,\dots,M_{n_1+1}-1, \text{ and}\\ & j_l>M_{n_l-1} \text{ for all } l=2,\dots,d,\\ f_{\overline{n_1}}(x) & \text{if } j_1\geqslant M_{n_1+1} \text{ and } j_l>M_{n_l-1}\\ & \text{for all } l=2,\dots,d,\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}
\tag{7}
$$

We immediately have that

$$f_{\overline{n_1}}^*(x)=\sup_{m_1\in\mathbb{N}}|S_{M_{m_1},\dots,M_{m_d}}(f_{\overline{n_1}};x)|=|f_{\overline{n_1}}(x)|.$$

Moreover,

$$\|f_{\overline{n_1}}\|_{H_{p_0}^\gamma}=\|f_{\overline{n_1}}^*\|_{p_0}\leqslant M_{n_1}^{1-1/p_0}<\infty. \tag{8}$$

That is, $f_{\overline{n_1}}\in H_{p_0}^\gamma$. We can write the nth Dirichlet kernel with respect to the Vilenkin system in the following form:

$$D_n(x)=D_{M_{|n|}}(x)+r_{|n|}(x)D_{n-M_{|n|}}(x) \tag{9}$$

Let us set $L_1^N:=M_{n_1}+N$ where $0<N<M_{n_1}$ and $L_j^N:=[\gamma_j(M_{n_1}+N)]$ for $j=2,\dots,d$, (where $[x]$ denotes the integer part of x). In this case $L^N:=(L_1^N,\dots,L_d^N)\in L$. Now, we calculate $\sigma_{L^N}^\alpha f_{\overline{n_1}}$.

By equality (7), (9) and (1) we may write that

$$
\begin{aligned}
&|\sigma_{L^N}^\alpha f_{\overline{n_1}}(x)|\\
&=\frac{1}{\prod_{j=1}^d A_{L_j^N}^{\alpha_j}}\left|\sum_{k_1=0}^{L_1^N}\cdots\sum_{k_d=0}^{L_d^N}\prod_{i=1}^d A_{L_i^N-k_i}^{\alpha_i-1}S_k(f_{\overline{n_1}};x)\right|\\
&=\frac{1}{\prod_{j=1}^d A_{L_j^N}^{\alpha_j}}\left|\sum_{k_1=M_{n_1}+1}^{L_1^N}\cdots\sum_{k_d=M_{n_d-1}+1}^{L_d^N}\prod_{i=1}^d A_{L_i^N-k_i}^{\alpha_i-1}\prod_{l=2}^d\psi_{M_{n_l-1}}(x^l)(D_{k_1}(x^1)-D_{M_{n_1}}(x^1))\right|\\
&=\frac{1}{\prod_{j=1}^d A_{L_j^N}^{\alpha_j}}\left|\sum_{k_2=1}^{L_2^N-M_{n_2-1}}\cdots\sum_{k_d=1}^{L_d^N-M_{n_d-1}}\prod_{i=2}^d A_{L_i^N-M_{n_i-1}-k_i}^{\alpha_i-1}\sum_{k_1=1}^{L_1^N-M_{n_1}}A_{L_1^N-M_{n_1}-k_1}^{\alpha_1-1}D_{k_1}(x^1)\right|\\
&=\frac{1}{\prod_{j=1}^d A_{L_j^N}^{\alpha_j}}\left|\sum_{k_2=1}^{L_2^N-M_{n_2-1}}\cdots\sum_{k_d=1}^{L_d^N-M_{n_d-1}}\prod_{i=2}^d A_{L_i^N-M_{n_i-1}-k_i}^{\alpha_i-1}\right|\left|A_{L_1^N-M_{n_1}}^{\alpha_1}K_{L_1^N-M_{n_1}}^{\alpha_1}(x^1)\right|.
\end{aligned}
$$

Since, it is easily seen that

$$\sum_{k_i=1}^{L_j^N - M_{n_i-1}} A^{\alpha_i-1}_{L_i^N - M_{n_i-1}-k_i} \geqslant c_i A^{\alpha_i}_{L_i^N}$$

for $i = 2, \dots, d$, we have that

$$|\sigma^{\alpha}_{L^N} f_{\overline{n_1}}(x)| \geqslant \frac{c(\alpha)}{M_{n_1}^{\alpha_1}} A_N^{\alpha_1} \left| K_N^{\alpha_1}(x^1) \right|.$$

Moreover, we have that

$$\begin{aligned}
\sigma_L^{\alpha,*} f_{\overline{n_1}}(x) &= \sup_{n\in L} |\sigma_n^{\alpha} f_{\overline{n_1}}(x)| \geqslant \max_{1\leqslant N<M_{n_1}} |\sigma^{\alpha}_{L^N} f_{\overline{n_1}}(x)| \\
&\geqslant \frac{c(\alpha)}{M_{n_1}^{\alpha_1}} \max_{1\leqslant N<M_{n_1}} A_N^{\alpha_1} \left| K_N^{\alpha_1}(x^1) \right|.
\end{aligned}$$

By inequality (8) and Lemma 1 we obtain that

$$\begin{aligned}
\frac{\|\sigma_L^{\alpha,*} f_{\overline{n_1}}\|_{p_0}}{\|f_{\overline{n_1}}\|_{H^{\gamma}_{p_0}}} &\geqslant \frac{1}{M_{n_1}^{1-1/p_0}} \left(\int_{G_m^d} \max_{1\leqslant N<M_{n_1}} |\sigma^{\alpha}_{L^N} f_{\overline{n_1}}(x)|^{p_0} d\mu(x) \right)^{1/p_0} \\
&\geqslant \frac{c(\alpha) M_{n_1}^{\alpha_1}}{M_{n_1}^{\alpha_1}} \left(\int_{G_m} \max_{1\leqslant N<M_{n_1}} (A_N^{\alpha_1} \left| K_N^{\alpha_1}(x^1) \right|)^{p_0} d\mu(x^1) \right)^{1/p_0} \\
&\geqslant c(\alpha) \left(\frac{n_1}{\log n_1} \right)^{1+\alpha_1} \to \infty \quad \text{as } n_1 \to \infty.
\end{aligned}$$

This completes the proof of our Theorem. □

REFERENCES

[1] G. N. AGAJEV, N. YA. VILENKIN, G. M. DZHAFARLI, A. I. RUBINSTEIN, *Multiplicative systems of functions and harmonic analysis on 0-dimensional groups*, "ELM" (Baku, USSR) (1981) (Russian).

[2] I. BLAHOTA, G. GÁT, *Pointwise convergence of double Vilenkin-Fejér means*, Studia Sci. Math. Hungar. **36** (2000), 49–63.

[3] N. J. FINE, *Cesàro summability of Walsh-Fourier series*, Proc. Nat. Acad. Sci. USA **41** (1955), 588–591.

[4] G. GÁT, *On the divergence of the $(C,1)$ means of double Walsh series*, Proc. Amer. Math. Soc. **128**, 6 (1996), 1711–1720.

[5] G. GÁT, *Pointwise convergence of cone-like restricted two-dimensional $(C,1)$ means of trigonometric Fourier series*, J. Approx. Theory **149** (2007), 74–102.

[6] G. GÁT, *Pointwise convergence of the Cesàro means of double Walsh series*, Annales Univ. Sci. Budapest., Sect. Comp. **16** (1996), 173–184.

[7] G. GÁT, K. NAGY, *Pointwise convergence of cone-like restricted two-dimensional Fejér means of Walsh-Fourier series*, Acta Math. Sinica, English Series **26**, 12 (2010), 2295–2304.

[8] U. GOGINAVA, *The maximal operator of the (C,α) means of the Walsh-Fourier series*, Annales Univ. Sci. Budapest., Sect. Comput. **26** (2006), 127–135.

[9] U. GOGINAVA, *Maximal (C,α,β) operators of two-dimensional Walsh-Fourier series*, Acta Math. Acad. Paed. Nyíregyház. **24** (2008), 209–214.

[10] F. MÓRICZ, F. SCHIPP, W. R. WADE, *Cesàro summability of double Walsh-Fourier series*, Trans. Amer. Math. Soc. **329**, 1 (1992), 131–140.

[11] K. NAGY, *On the restricted summability of double Walsh-Fourier series*, Publ. Math. (Debrecen) **85**, 1–2 (2014), 113–122.

[12] K. NAGY, *Almost everywhere convergence of cone-like restricted two-dimensional Fejér means with respect to Vilenkin-like systems*, St. Petersburg Mathematical Journal **25**, 4 (2014), 605–614. Originally published in Algebra i Analiz **25**, 4 (2013), 125–138.

[13] K. NAGY, *Almost everywhere convergence of cone-like restricted double Fejér means on compact totally disconnected groups*, Acta Math. Acad. Paed. Nyíregyház. **27** (2011), 267–277.

[14] L.-E. PERSSON, G. TEPHNADZE, *A sharp boundedness result concerning some maximal operators of Vilenkin-Fejér means*, Mediterr. J. Math. **13**, 4 (2016), 1841–1853.

[15] G. TEPHNADZE, *On the maximal operators of Vilenkin-Fejér means*, Turk. J. Math. **37** (2013), 308–318.

[16] F. SCHIPP, W. R. WADE, P. SIMON, AND J. PÁL, *Walsh Series. An Introduction to Dyadic Harmonic Analysis*, Adam Hilger (Bristol-New York 1990).

[17] P. SIMON, *Investigations with respect to the Vilenkin system*, Annales Univ. Sci. Budapest., Sect. Math. **27** (1985), 87–101.

[18] P. SIMON, *Cesàro summability with respect to two-parameter Walsh system*, Monatsh. Math. **131** (2000), 321–334.

[19] F. WEISZ, *Martingale Hardy spaces and their applications in Fourier analysis*, Springer-Verlang, Berlin, 1994.

[20] F. WEISZ, *Summability of multi-dimensional Fourier series and Hardy space*, Kluwer Academic, Dordrecht, 2002.

[21] F. WEISZ, *Restricted summability of multi-dimensional Vilenkin-Fourier series*, Annales Univ. Sci. Budapest., Sect. Comp. **35** (2011), 305–317.

[22] F. WEISZ, (C,α) *summability of Walsh-Fourier series*, Anal. Math. **27**, 2 (2001), 141–155.

[23] F. WEISZ, *Cesàro summability of two-dimensional Walsh-Fourier series*, Trans. Amer. Math. Soc. **348** (1996), 2169–2181.

[24] F. WEISZ, *Maximal estimates for the* (C,α) *means of* d*-dimensional Walsh-Fourier series*, Proc. Amer. Math. Soc. **128** (1999), 2337–2345.

[25] F. WEISZ, *Summability results of Walsh- and Vilenkin-Fourier series*, Functions, Series, Operators. Alexits Memorial Conference, Budapest, Hungary, 1999, eds. L. Leindler, F. Schipp and J. Szabados (2002), 443–464.

[26] N. YA. VILENKIN, *On a class of complete orthonormal systems*, Amer. Math. Soc. Transl., **28**, 2 (1963), 1–35.

[27] A. ZYGMUND *Trigonometric series*, Cambridge Press, London, 3rd edition, 2002.

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