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ON HAYAJNEH AND KITTANEH'S CONJECTURE ON UNITARILY INVARIANT NORM

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Abstract. In this short note, we give an affirmative answer to a conjecture posed by Hayajneh and Kittaneh on a unitarily invariant norm inequality.

1. Introduction

Let $\mathbb{M}_n$ be the set of $n \times n$ complex matrices. In [4], Bourin posed the following questions. Let $A, B \in \mathbb{M}_n$ be positive semidefinite and $p, q > 0$. Then

$$\|A^{p+q} + B^{p+q}\| \leqslant \|(A^p + B^p)(A^q + B^q)\| \tag{1}$$

holds or not for any unitarily invariant norm. Further whether

$$\|A^{p+q} + B^{p+q}\| \leqslant \|(A^p + B^p)^{1/2}(A^q + B^q)(A^p + B^p)^{1/2}\| \tag{2}$$

holds or not for any unitarily invariant norm.

Hayajneh and Kittaneh proved that the above inequalities (1) and (2) hold for the trace norm and the Hilbert-Schmidt norm. Meanwhile, they posed the following conjectures (see [5]).

CONJECTURE 1. *Let $A_1, A_2, B_1, B_2 \in \mathbb{M}_n$ be positive semidefinite with $A_1A_2 = A_2A_1$ and $B_1B_2 = B_2B_1$. Then*

$$\|A_1A_2 + B_1B_2\| \leqslant \|(A_1 + B_1)(A_2 + B_2)\|$$

holds for any unitarily invariant norm.

CONJECTURE 2. *Let $A_1, A_2, B_1, B_2 \in \mathbb{M}_n$ be positive semidefinite with $A_1A_2 = A_2A_1$ and $B_1B_2 = B_2B_1$. Then*

$$\|A_1A_2 + B_1B_2\| \leqslant \|(A_1 + B_1)^{1/2}(A_2 + B_2)(A_1 + B_1)^{1/2}\|$$

holds for any unitarily invariant norm.

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In [1], Audenaert proved that if $A_i, B_i \in \mathbb{M}_n, (i = 1, 2, \dots, k)$ be positive semidefinite such that A_i commutes with B_i for each i, then

$$\left\| \sum_{i=1}^{k} A_i B_i \right\| \leqslant \left\| \left(\sum_{i=1}^{k} A_i^{1/2} B_i^{1/2} \right)^2 \right\| \leqslant \left\| \left(\sum_{i=1}^{k} A_i \right) \left(\sum_{i=1}^{k} B_i \right) \right\| \tag{3}$$

holds for any unitarily invariant norm.

Particulary, this result confirms the Conjecture 1 and answers a question of Bourin. Recently Lin [7] gave another proof of the above inequalities (3).

Recall that a norm $\|\cdot\|$ on $\mathbb{M}_n$ is unitarily invariant if $\|UAV\| = \|A\|$ for any $A \in \mathbb{M}_n$ and unitary matrices $U, V \in \mathbb{M}_n$. In the sequel, let $\|\cdot\|$ denote any unitarily invariant norm.

In this short paper, We shall give an affirmative answer to the above Conjecture 2 by proving the following theorem.

THEOREM 1. *For $i = 1, 2, \dots, k$, let $A_i, B_i \in \mathbb{M}_n$ be positive semidefinite such that A_i commutes with B_i for each i. Then*

$$\left\| \sum_{i=1}^{k} A_i B_i \right\| \leqslant \left\| \left(\sum_{i=1}^{k} A_i^{1/2} B_i^{1/2} \right)^2 \right\| \leqslant \left\| \left(\sum_{i=1}^{k} A_i \right)^{1/2} \left(\sum_{i=1}^{k} B_i \right) \left(\sum_{i=1}^{k} A_i \right)^{1/2} \right\|.$$

It is readily seen that the Conjecture 2 is a special case (i.e., $k = 2$) of Theorem 1. Thus, Theorem 1 confirms the Conjecture 2 and answers another question of Bourin.

COROLLARY 1. *Let $A, B \in \mathbb{M}_n$ be positive semidefinite and $p, q > 0$. Then*

$$\|A^{p+q} + B^{p+q}\| \leqslant \|(A^p + B^p)^{1/2}(A^q + B^q)(A^p + B^p)^{1/2}\|.$$

2. Proof of Theorem 1

For $X \in \mathbb{M}_n$, the conjugate transpose of X is denoted by X^*. Let the singular values of $X \in \mathbb{M}_n$ denote by $\sigma_1(X) \geqslant \sigma_2(X) \geqslant \cdots \geqslant \sigma_n(X)$. The block matrix $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$, where $A, B, X \in \mathbb{M}_n$, is positive partial transpose (i.e., PPT) if $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$ and $\begin{bmatrix} A & X^* \\ X & B \end{bmatrix}$ are both positive semidefinite.

In order to prove Theorem 1, we need to show the following lemma, which is a variation of Theorem 4.1 in [6].

LEMMA 1. *Let $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$ be PPT, where $A, B, X \in \mathbb{M}_n$. Then*

$$\|X^* X\| \leqslant \|A^{1/2} B A^{1/2}\|.$$

Proof. Since $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$ is positive semidefinite, we have $X = A^{1/2}CB^{1/2}$ for some contraction $C \in \mathbb{M}_n$ (see [2, p. 13]). Similarly, $X^* = A^{1/2}DB^{1/2}$ for some contraction $D \in \mathbb{M}_n$. Hence, for $k = 1, \ldots, n$,

$$\begin{aligned}\prod_{i=1}^{k} \sigma_i(X^*X) &= \prod_{i=1}^{k} \sigma_i(A^{1/2}DB^{1/2}A^{1/2}CB^{1/2}) \\ &\leqslant \prod_{i=1}^{k} \sigma_i(DB^{1/2}A^{1/2}CB^{1/2}A^{1/2}) \\ &\leqslant \prod_{i=1}^{k} \sigma_i(B^{1/2}A^{1/2})\sigma_i(B^{1/2}A^{1/2}) \\ &= \prod_{i=1}^{k} \sigma_i(A^{1/2}BA^{1/2}),\end{aligned}$$

where the first inequality is by the fact (see [3, p. 253]) that $\prod_{j=1}^{k} \sigma_j(XY) \leqslant \prod_{j=1}^{k} \sigma_j(YX), k = 1, 2, \ldots,$ whenever XY is normal; the second one is due to the fact that $\sigma(AB) \prec_{log} \sigma(A) \circ \sigma(B)$ (see [8, p. 353]). Therefore, we have $\sigma(X^*X) \prec_{wlog} \sigma(A^{1/2}BA^{1/2})$, then $\sigma(X^*X) \prec_w \sigma(A^{1/2}BA^{1/2})$. Consequently, the assertion follows. □

Now, we are in a position to present the proof of Theorem 1.

Proof of Theorem 1. The first inequality in Theorem 1 follows from the result of Audenaert, we next prove the second inequality in Theorem 1. Since A_i commutes with B_i, then $A_i^{1/2}$ also commutes with $B_i^{1/2}$, hence, A_iB_i and $A_i^{1/2}B_i^{1/2}$ are both positive semidefinite. It is easy to see that

$$\begin{bmatrix} A_i & A_i^{1/2}B_i^{1/2} \\ A_i^{1/2}B_i^{1/2} & B_i \end{bmatrix}, \quad i = 1, \ldots, k,$$

are positive semidefinite, by using the Schur complement, so they are all PPT. Consequently

$$\begin{bmatrix} \sum_{i=1}^{k} A_i & \sum_{i=1}^{k} A_i^{1/2}B_i^{1/2} \\ \sum_{i=1}^{k} A_i^{1/2}B_i^{1/2} & \sum_{i=1}^{k} B_i \end{bmatrix}$$

is also PPT. By Lemma 1, we get

$$\left\| \left(\sum_{i=1}^{k} A_i^{1/2}B_i^{1/2} \right)^2 \right\| \leqslant \left\| \left(\sum_{i=1}^{k} A_i \right)^{1/2} \left(\sum_{i=1}^{k} B_i \right) \left(\sum_{i=1}^{k} A_i \right)^{1/2} \right\|,$$

hence, the second inequality follows in Theorem 1. □

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