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# FURTHER IMPROVED YOUNG INEQUALITIES FOR OPERATORS AND MATRICES

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*Abstract.* In this paper, we show some improvement of Young inequalities for operators and matrix versions for the Hilbert-Schmidt norm. On the basis of an operator equality, we prove intrinsic conclusion by means of a different method with others' researches. Besides, we present some reverse Young inequalities for positive operators.

## 1. Introduction

In what follows, $B(H)$ denotes all bounded linear operators on a complex Hilbert space $H$, and $B^+(H)$ denotes all positive operators in $B(H)$. We use the following notations to define the weighted arithmetic and geometric mean for operators:

$$A\nabla_v B = (1-v)A + vB, \quad A\sharp_v B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^v A^{\frac{1}{2}},$$

where $v \in [0,1]$. When $v = \frac{1}{2}$, we write $A\nabla B$ and $A\sharp B$ for brevity, respectively, see Kubo and Ando [9].

While, $M_n$ denotes $n\times n$ complex matrices. For $C = (c_{ij}) \in M_n$, the Hilbert-Schmidt norm is defined as $\|C\|_2 = \sqrt{\sum_{i,j=1}^n |c_{ij}|^2}$. It's known that $\|\cdot\|_2$ has the unitarily invariant property: $\|UCV\|_2 = \|C\|_2$ for all $C \in M_n$ and all unitary matrices $U$, $V \in M_n$. Besides, $A\circ B$ is the Hadamard product of two matrices $A$, $B \in M_n$.

An operator version of the Young inequality in [3] says that

$$\text{If } A,B \in B^+(H) \text{ and } v\in[0,1], \text{ then } A\nabla_v B \geqslant A\sharp_v B.$$

The first difference-type improvement of the matrix Young inequality is due to Kittaneh and Manasrah [6]:

$$2r(A\nabla B - A\sharp B) \leqslant A\nabla_v B - A\sharp_v B \leqslant 2s(A\nabla B - A\sharp B) \tag{1.1}$$

holds for positive definite matrices $A,B\in M_n$ and $v\in[0,1]$, where $r = \min\{v,1-v\}$ and $s = \max\{v,1-v\}$, which remain of course valid for Hilbert space operators by a standard approximation argument.

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Note that Furuichi [2] independently established the first inequality in (1.1) for two positive operators and Kittaneh et al. [7] also proved (1.1) by taking a different approach. In [8] the authors provided the general refinement and reverse of the Jensen's operator inequality and the relation (1.1) appears as a special case of their results. Recently some researchers obtained the further improvements of Young inequalities, for example [1, 11].

In this paper, we are concerned with all these topics. In section 2, we show an operator mean equality, by means of which as well as induction employed on the operator Young inequality, we obtain the refined Young inequality for positive operators. In section 3, we introduce the homologous matrix inequalities for the Hilbert-Schmidt norm. In section 4, we present some reverse Young inequalities under different conditions.

## 2. Improved Young inequalities for operators

In this section, we mainly show an important lemma and the improvement of the second inequality in (1.1).

LEMMA 2.1. *If $A$ and $B$ are positive operators on Hilbert space, $0 \leqslant \mu, \nu \leqslant 1$, then*

$$A\nabla_{\mu}(A\sharp_{\nu}B) = A\nabla_{\mu\nu}B - \mu(A\nabla_{\nu}B - A\sharp_{\nu}B). \tag{2.1}$$

*Proof.*

$$\begin{aligned} A\nabla_{\mu}(A\sharp_{\nu}B) &= (1-\mu)A + \mu A\sharp_{\nu}B \\ &= A - \mu A + \mu\nu A - \mu\nu A + \mu\nu B - \mu\nu B + \mu A\sharp_{\nu}B \\ &= \mu\nu B + (1-\mu\nu)A - \mu[(1-\nu)A + \nu B - A\sharp_{\nu}B] \\ &= A\nabla_{\mu\nu}B - \mu(A\nabla_{\nu}B - A\sharp_{\nu}B). \quad \square \end{aligned}$$

THEOREM 2.2. *Let $A, B \in B(H)$ be two positive operators,*
(I) *If $0 \leqslant \mu \leqslant \frac{1}{2}$, then*

$$A\nabla_{\mu}B - A\sharp_{\mu}B \leqslant 2R(A\nabla_{\frac{1}{4}}B - A\sharp_{\frac{1}{4}}B) + (2r-R)(A\nabla B - A\sharp B), \tag{2.2}$$

(II) *If $\frac{1}{2} < \mu \leqslant 1$, then*

$$A\nabla_{\mu}B - A\sharp_{\mu}B \leqslant 2R(A\nabla_{\frac{3}{4}}B - A\sharp_{\frac{3}{4}}B) + (2r-R)(A\nabla B - A\sharp B). \tag{2.3}$$

*where $r = \min\{\mu, 1-\mu\}$, $R = \max\{2r, 1-2r\}$.*

*Proof.* If $0 \leqslant \mu \leqslant \frac{1}{2}$, then $0 \leqslant 2\mu \leqslant 1$. By substituting $B$ by $A\sharp B$ and $\mu$ by $2\mu$ in the second inequality of (1.1) respectively, it follows that

$$A\nabla_{2\mu}(A\sharp B) - A\sharp_{2\mu}(A\sharp B) \leqslant 2\max\{2\mu, 1-2\mu\}[A\nabla(A\sharp B) - A\sharp(A\sharp B)].$$

Lemma 2.1 admits the following equalities:

$$A\nabla_{2\mu}(A\sharp B) - A\sharp_{2\mu}(A\sharp B) = A\nabla_{\mu}B - A\sharp_{\mu}B - 2\mu(A\nabla B - A\sharp B);$$
$$A\nabla(A\sharp B) - A\sharp(A\sharp B) = A\nabla_{\frac{1}{4}}B - A\sharp_{\frac{1}{4}}B - \frac{1}{2}(A\nabla B - A\sharp B).$$

Therefore, we have

$$A\nabla_\mu B - A\sharp_\mu B - 2\mu(A\nabla B - A\sharp B) \leqslant 2\max\{2\mu, 1-2\mu\}(A\nabla_{\frac{1}{4}} B - A\sharp_{\frac{1}{4}} B - \frac{1}{2}(A\nabla B - A\sharp B)).$$

i.e.

$$A\nabla_\mu B - A\sharp_\mu B \leqslant 2R(A\nabla_{\frac{1}{4}} B - A\sharp_{\frac{1}{4}} B) + (2r-R)(A\nabla B - A\sharp B).$$

If $\frac{1}{2} \leqslant \mu \leqslant 1$, then $0 \leqslant 1-\mu \leqslant \frac{1}{2}$. Then by the inequality above we have

$$\begin{aligned} B\nabla_{1-\mu} A - B\sharp_{1-\mu} A &\leqslant 2(1-\mu)(B\nabla A - B\sharp A) \\ &\quad + 2\max\{2(1-\mu), 1-2(1-\mu)\}(B\nabla_{\frac{1}{4}} A - B\sharp_{\frac{1}{4}} A - \frac{1}{2}(B\nabla A - B\sharp A)). \end{aligned}$$

Then we have

$$A\nabla_\mu B - A\sharp_\mu B \leqslant 2R(A\nabla_{\frac{3}{4}} B - A\sharp_{\frac{3}{4}} B) + (2r-R)(A\nabla B - A\sharp B).$$

Therefore, for $0 \leqslant \mu \leqslant \frac{1}{2}$, we have

$$A\nabla_\mu B - A\sharp_\mu B \leqslant 2R(A\nabla_{\frac{1}{4}} B - A\sharp_{\frac{1}{4}} B) + (2r-R)(A\nabla B - A\sharp B);$$

for $\frac{1}{2} \leqslant \mu \leqslant 1$, we have

$$A\nabla_\mu B - A\sharp_\mu B \leqslant 2R(A\nabla_{\frac{3}{4}} B - A\sharp_{\frac{3}{4}} B) + (2r-R)(A\nabla B - A\sharp B).$$

where $r = \min\{\mu, 1-\mu\}$, $R = \max\{2r, 1-2r\}$. $\square$

REMARK. The inequality (2.2) is stronger than the second inequality of (1.1) when $0 \leqslant \mu \leqslant \frac{1}{4}$. In fact, it follows by (1.1) and Lemma 2.1 that

$$A\nabla_{\frac{1}{4}} B - A\sharp_{\frac{1}{4}} B \leqslant 2\max\left\{\frac{1}{4}, \frac{3}{4}\right\}(A\nabla B - A\sharp B) = \frac{3}{2}(A\nabla B - A\sharp B),$$

$$\begin{aligned} A\nabla_\mu B - A\sharp_\mu B &\leqslant 2R(A\nabla_{\frac{1}{4}} B - A\sharp_{\frac{1}{4}} B) + (2r-R)(A\nabla B - A\sharp B) \\ &\leqslant 3(1-2\mu)\mu(A\nabla B - A\sharp B) + (4\mu-1)(A\nabla B - A\sharp B) \\ &= 2(1-\mu)(A\nabla B - A\sharp B). \end{aligned}$$

If $\frac{1}{4} \leqslant \mu \leqslant \frac{1}{2}$, then we have

$$\begin{aligned} A\nabla_\mu B - A\sharp_\mu B &\leqslant 2R(A\nabla_{\frac{1}{4}} B - A\sharp_{\frac{1}{4}} B) + (2r-R)(A\nabla B - A\sharp B) \\ &= 4\mu(A\nabla_{\frac{1}{4}} B - A\sharp_{\frac{1}{4}} B). \end{aligned}$$

Neither $4\mu(A\nabla_{\frac{1}{4}} B - A\sharp_{\frac{1}{4}} B)$ nor $2(1-\mu)(A\nabla B - A\sharp B)$ is uniformly better than the other when $\frac{1}{4} \leqslant \mu \leqslant \frac{1}{2}$, although both of them are bigger than $2\mu(A\nabla B - A\sharp B)$.

We also obtain the same operator inequalities by means of different method in [11].

COROLLARY 2.3. ([11], Lemma 2.5) *Let $A,B\in B(H)$ be two positive operators,*
(I) *If $0\leqslant\mu\leqslant\frac{1}{2}$, then*

$$A\nabla_\mu B - A\sharp_\mu B \leqslant 2s(A\nabla B - A\sharp B) - 2r_1(A\nabla_{\frac{3}{4}}B - A\sharp_{\frac{3}{4}}B - \frac{1}{2}(A\nabla B - A\sharp B)),$$

(II) *If $\frac{1}{2}<\mu\leqslant 1$, then*

$$A\nabla_\mu B - A\sharp_\mu B \leqslant 2s(A\nabla B - A\sharp B) - 2r_1(A\nabla_{\frac{1}{4}}B - A\sharp_{\frac{1}{4}}B - \frac{1}{2}(A\nabla B - A\sharp B)).$$

*where $r=\min\{\mu,1-\mu\}$, $s=\max\{\mu,1-\mu\}$, $r_1=\min\{2r,1-2r\}$.*

*Proof.* If $0\leqslant\mu\leqslant\frac{1}{2}$, then $0\leqslant 2\mu\leqslant 1$. By Lemma 2.1 and (1.1) it follows that

$$\begin{aligned}
&2(1-\mu)(A\nabla B - A\sharp B) - (A\nabla_\mu B - A\sharp_\mu B)\\
&= 2(1-\mu)A\nabla B - A\nabla_\mu B + 2\mu A\sharp B - 2A\sharp B + A\sharp_\mu B\\
&= B\nabla_{2\mu}(A\sharp B) - 2A\sharp B + A\sharp_\mu B\\
&\geqslant B\sharp_{2\mu}(B\sharp A) + 2\min\{2\mu,1-2\mu\}[B\nabla(B\sharp A) - B\sharp(B\sharp A)] + A\sharp_\mu B - 2A\sharp B\\
&= 2\min\{2\mu,1-2\mu\}[B\nabla(B\sharp A) - B\sharp(B\sharp A)] + B\sharp_\mu A + A\sharp_\mu B - 2A\sharp B\\
&\geqslant 2r(A\nabla_{\frac{3}{4}}B - A\sharp_{\frac{3}{4}}B - \frac{1}{2}(A\nabla B - A\sharp B)).
\end{aligned}$$

The first sign of inequality is used by the first inequality of (1.1).

If $\frac{1}{2}<\mu\leqslant 1$, by the similar proof as that in Theorem 2.2 we complete the proof. □

## 3. Matrix Young inequalities for the Hilbert-Schmidt norm

Firstly we present some scalar inequalities according to inequalities (2.2) and (2.3) as follows:

LEMMA 3.1. *Let $a$, $b$ be two nonnegative real numbers and $\mu\in[0,1]$.*
(I) *If $0\leqslant\mu\leqslant\frac{1}{2}$, then*

$$(1-\mu)a+\mu b\leqslant a^{1-\mu}b^\mu + \mu(\sqrt{a}-\sqrt{b})^2 + R(\sqrt[4]{ab}-\sqrt{a})^2, \tag{3.1}$$

(II) *If $\frac{1}{2}<\mu\leqslant 1$, then*

$$(1-\mu)a+\mu b\leqslant a^{1-\mu}b^\mu + (1-\mu)(\sqrt{a}-\sqrt{b})^2 + R(\sqrt[4]{ab}-\sqrt{b})^2. \tag{3.2}$$

*where $r=\min\{\mu,1-\mu\}$ and $R=\max\{2r,1-2r\}$.*

Replacing $a$ and $b$ by their squares in (3.1) and (3.2), respectively, then

$$(1-\mu)a^2+\mu b^2\leqslant (a^{1-\mu}b^\mu)^2 + \mu(a-b)^2 + R(\sqrt{ab}-a)^2, \tag{3.3}$$

and

$$(1-\mu)a^2+\mu b^2 \leqslant (a^{1-\mu}b^{\mu})^2+(1-\mu)(a-b)^2+R(\sqrt{ab}-b)^2. \tag{3.4}$$

Based on which and the equality

$$((1-\mu)a+\mu b)^2-\mu^2(a-b)^2=(1-\mu)a^2+\mu b^2-\mu(a-b)^2,$$

we obtain the following lemma.

LEMMA 3.2. *Let $a$, $b$ be two nonnegative real numbers and $\mu \in [0,1]$.*
(I) *If $0 \leqslant \mu \leqslant \frac{1}{2}$, then*

$$((1-\mu)a+\mu b)^2 \leqslant (a^{1-\mu}b^{\mu})^2+\mu^2(a-b)^2+R(\sqrt{ab}-a)^2, \tag{3.5}$$

(II) *If $\frac{1}{2}<\mu \leqslant 1$, then*

$$((1-\mu)a+\mu b)^2 \leqslant (a^{1-\mu}b^{\mu})^2+(1-\mu)^2(a-b)^2+R(\sqrt{ab}-b)^2. \tag{3.6}$$

*where $r=\min\{\mu,1-\mu\}$ and $R=\max\{2r,1-2r\}$.*

Now we present some inequalities for the Hilbert-Schmidt norm. It is known that every positive semidefinite matrix is unitarily diagnalizable. Thus, for two positive semidefinite matrices $A$ and $B$, there exist two unitary matrices $U$, $V$ such that $A=Udiag(\alpha_1,\alpha_2,\cdots,\alpha_n)U^*$ and $B=Vdiag(\beta_1,\beta_2,\cdots,\beta_n)V^*$, ($\alpha_i$, $\beta_i \geqslant 0$, $i=1,2,\cdots,n$). Then, Lemma 3.2 is equivalent to the following theorem (See [4]).

THEOREM 3.3. *Let $A$, $B$, $X \in M_n$ such that $A$ and $B$ are two positive semidefinite matrices,*
(I) *If $0 \leqslant \mu \leqslant \frac{1}{2}$, then*

$$\|(1-\mu)AX+\mu XB\|_2^2 \leqslant r^2\|AX-XB\|_2^2+\|A^{1-\mu}XB^{\mu}\|_2^2+R\|A^{\frac{1}{2}}XB^{\frac{1}{2}}-AX\|_2^2;$$

(II) *If $\frac{1}{2}<\mu \leqslant 1$, then*

$$\|(1-\mu)AX+\mu XB\|_2^2 \leqslant r^2\|AX-XB\|_2^2+\|A^{1-\mu}XB^{\mu}\|_2^2+R\|A^{\frac{1}{2}}XB^{\frac{1}{2}}-XB\|_2^2.$$

*where $r=\min\{\mu,1-\mu\}$, $R=\max\{2r,1-2r\}$.*

## 4. Refined reverse Young inequalities

In this section, we show some reverse inequalities under the different conditions.

LEMMA 4.1. ([10]) *If $0 \leqslant a \leqslant A,B \leqslant b$ and $0 \leqslant \mu \leqslant 1$, then*

$$A\nabla_{\mu}B \leqslant S(h)A\sharp_{\mu}B \tag{4.1}$$

*where $S(t)=\frac{t^{\frac{1}{t-1}}}{e\log t^{\frac{1}{t-1}}}$ ($t>0$, $t \neq 1$) and $S(1)=\lim\limits_{t\to 1}S(t)=1$; $h=\frac{b}{a}$.*

These inequalities have recently been improved by Furuichi as follows:

THEOREM F. [2] *If* $0 < aI_H \leqslant A, B \leqslant bI_H$, *then*

$$A\nabla_\mu B - 2r(A\nabla B - A\sharp B) \leqslant S(\sqrt{h})A\sharp_\mu B,$$

*where* $r = \min\{\mu, 1-\mu\}$, $h = \frac{b}{a}$.

See also [12] for another improvement of the reverse weighted arithmetic-geometric operator mean inequalities. Their proof is independent of [2].

THEOREM ZF. [12] *If* $0 < aA \leqslant B \leqslant bA$ *with* $a < 1 < b$, *then*

$$A\nabla_\mu B - 2r(A\nabla B - A\sharp B) \leqslant \max\{S(\sqrt{a}), S(\sqrt{b})\}A\sharp_\mu B,$$

*where* $r = \min\{\mu, 1-\mu\}$.

THEOREM 4.2. *If* $0 < \frac{1}{h}A \leqslant B \leqslant hA$ *and* $0 \leqslant \mu \leqslant 1$, *then*
(i) *If* $0 \leqslant \mu \leqslant \frac{1}{2}$, *then*

$$A\nabla_\mu B - (2r - r')(A\nabla B - A\sharp B) - 2r'(A\nabla_{\frac{1}{4}} B - A\sharp_{\frac{1}{4}} B) \leqslant S(\sqrt[4]{h})A\sharp_\mu B, \tag{4.2}$$

(ii) *If* $\frac{1}{2} \leqslant \mu \leqslant 1$, *then*

$$A\nabla_\mu B - (2r - r')(A\nabla B - A\sharp B) - 2r'(A\nabla_{\frac{3}{4}} B - A\sharp_{\frac{3}{4}} B) \leqslant S(\sqrt[4]{h})A\sharp_\mu B, \tag{4.3}$$

*where* $r = \min\{\mu, 1-\mu\}$, $r' = \min\{|1-2\mu|, 1-|1-2\mu|\}$.

*Proof.* Since $0 < \frac{1}{h}A \leqslant B \leqslant hA$ admits that $\frac{1}{\sqrt{h}}A \leqslant A\sharp B \leqslant \sqrt{h}A$, we also substitute $B$ by $A\sharp B$, $\mu$ by $2\mu$ when $0 \leqslant \mu \leqslant \frac{1}{2}$ and $1-\mu$ by $2(1-\mu)$ when $\frac{1}{2} \leqslant \mu \leqslant 1$ in (i) of Theorem ZF, respectively, then by similar work as in the proof of Theorem 2.2 and the equality $S(\frac{1}{\sqrt[4]{h}}) = S(\sqrt[4]{h})$, we can get the required conclusion. □

Since $0 < aI_H \leqslant A, B \leqslant bI_H$ admits that $\sqrt{\frac{a}{b}}A \leqslant A\sharp B \leqslant \sqrt{\frac{b}{a}}A$, by similar work as in the above we have

COROLLARY 4.3. *If* $0 < aI_H \leqslant A, B \leqslant bI_H$ *and* $0 \leqslant \mu \leqslant 1$, *then*
(i) *If* $0 \leqslant \mu \leqslant \frac{1}{2}$, *then*

$$A\nabla_\mu B - (2r - r')(A\nabla B - A\sharp B) - 2r'(A\nabla_{\frac{1}{4}} B - A\sharp_{\frac{1}{4}} B) \leqslant S(\sqrt[4]{h})A\sharp_\mu B, \tag{4.4}$$

(ii) *If* $\frac{1}{2} \leqslant \mu \leqslant 1$, *then*

$$A\nabla_\mu B - (2r - r')(A\nabla B - A\sharp B) - 2r'(A\nabla_{\frac{3}{4}} B - A\sharp_{\frac{3}{4}} B) \leqslant S(\sqrt[4]{h})A\sharp_\mu B, \tag{4.5}$$

*where* $r = \min\{\mu, 1-\mu\}$, $r' = \min\{|1-2\mu|, 1-|1-2\mu|\}$, $h = \frac{b}{a}$.

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