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COMPLETE MOMENT CONVERGENCE FOR ARRAYS OF ROWWISE EXTENDED NEGATIVELY DEPENDENT RANDOM VARIABLES

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Abstract. In this paper, a general result on complete moment convergence for arrays of rowwise extended negatively dependent (END, in short) random variables is established. As applications, we obtain some results on complete moment convergence for weighted sums of END random variables. The results obtained in the paper generalize and improve some corresponding ones for negatively dependent random variables.

1. Introduction

Firstly, let us recall the concepts of complete convergence and complete moment convergence.

The concept of complete convergence was introduced by Hsu and Robbins [1] as follows: a sequence $\{X_n, n \geqslant 1\}$ of random variables converges completely to the constant C if for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P(|X_n - C| > \varepsilon) < \infty.$$

By the Borel-Cantelli lemma, this implies that $X_n \to C$ a.s., so complete convergence is a stronger result than a.s. convergence.

Let $\{X_n, n \geqslant 1\}$ be a sequence of random variables and $a_n > 0$, $b_n > 0$, $q > 0$. If

$$\sum_{n=1}^{\infty} a_n E\{b_n{}^{-1}|X_n| - \varepsilon\}_+^q < \infty \text{ for all } \varepsilon > 0,$$

then the result was defined as the complete moment convergence by Chow [2]. It is easy to check that complete moment convergence implies complete convergence, thus, complete moment convergence is much stronger than complete convergence.

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Next, we will recall the concept of extended negative dependence, which was introduced by Liu [3] as follows.

DEFINITION 1.1. A finite collection of random variables $X_1, X_2, \cdots, X_n$ is said to be extended negatively dependent (END, in short) if there exists a constant $M > 0$ such that both

$$P(X_1 > x_1, X_2 > x_2, \cdots, X_n > x_n) \leqslant M \prod_{i=1}^{n} P(X_i > x_i)$$

and

$$P(X_1 \leqslant x_1, X_2 \leqslant x_2, \cdots, X_n \leqslant x_n) \leqslant M \prod_{i=1}^{n} P(X_i \leqslant x_i)$$

hold for all real numbers $x_1, x_2, \cdots, x_n$, $n \geqslant 2$. An infinite sequence $\{X_n, n \geqslant 1\}$ is said to be END if every finite subcollection is END.

An array of random variables $\{X_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ is called rowwise END random variables if for every $n \geqslant 1$, $\{X_{ni}, 1 \leqslant i \leqslant n\}$ are END random variables.

Obviously the END structure is more comprehensive than negatively dependent (ND) structure which was introduced by Lehmann [4] (cf. also Joag-Dev and Proschan [5]). The ND structure is a special case of END structure with $M = 1$. The END structure can reflect not only a negatively structure but also a positive one to some extent. Liu [3] pointed out that the END random variables can be taken as negatively or positively dependent and provided some interesting examples to support this idea. Joag-Dev and Proschan [5] also pointed out that negatively associated (NA) random variables must be ND and ND is not necessarily NA, thus NA random variables are END.

Some probability limit properties and applications for END sequence have been obtained. See for example, Liu [6] studied the sufficient and necessary conditions of moderate deviations for END random variables with heavy tails; Chen et al. [7] established the strong law of large numbers for END random variables and showed applications to risk theory and renewal theory; Shen [8, 9] presented some probability inequalities for END random variables and gave some applications; Wang and Wang [10] investigated the extended precise large deviations of random sums in the presence of END structure and consistent variation; Wu and Guan [11] presented some convergence properties for the partial sums of END random variables; Wang and Wang [12] investigated a more general precise large deviation result for random sums of END real-valued random variables in the presence of consistent variation; Qiu et al. [13] and Wang et al. [14–16] provided some results on complete convergence for END random variables, and so forth.

Let $\{k_n, n \geqslant 1\}$ be a sequence of positive integers. Recently, Sung et al. [17] established the following complete convergence for ND random variables.

THEOREM A. *Let $\{X_{nk}, 1 \leqslant k \leqslant k_n, n \geqslant 1\}$ be an array of rowwise ND random variables, $\{c_n, n \geqslant 1\}$ be a sequence of positive constants, and $\{b_n, n \geqslant 1\}$ be a sequence of positive constants such that $\lim_{n\to\infty} b_n = \infty$. Suppose that*

(i) $\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} P(|X_{nk}| > \varepsilon) < \infty$ for all $\varepsilon > 0$;

(ii) $\sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} P(|X_{nk}| > \frac{1}{b_n})\right)^{\xi} < \infty$ *for some* $\xi > 0$*;*

(iii) $b_n \sum_{k=1}^{k_n} EX_{nk}^2 I(|X_{nk}| \leqslant \frac{1}{b_n}) \to 0$ *as* $n \to \infty$*;*

(iv) $\sum_{n=1}^{\infty} c_n \exp\{-\eta_0 b_n\} < \infty$ *for some* $\eta_0 > 0$*.*

Then for all $\varepsilon > 0$*,*

$$\sum_{n=1}^{\infty} c_n P\left(\left|\sum_{k=1}^{k_n} (X_{nk} - EX_{nk} I(|X_{nk}| \leqslant 1/b_n))\right| > \varepsilon\right) < \infty.$$

The main purpose of this work is to generalize the result of Theorem A for ND random variables to the case of END random variables and complete convergence is improved to complete moment convergence.

Throughout this paper, C represents a positive constant which may vary in different places. For $x \geqslant 0$, the symbol $\lfloor x \rfloor$ denotes the integer part of x. Denote $\log x = \ln \max(x, e)$, where $\ln x$ denotes the natural logarithm. $I(A)$ will indicate the indicator function of the set A.

This work is organized as follows: some preliminary lemmas are provided in Section 2. Main result and its proof are stated in Section 3. Some applications of the main result are presented in Section 4.

2. Preliminary lemmas

In this section, we give some lemmas which will be used to prove our main results. The first one is a basic property for END random variables, which can be found in Liu [6].

LEMMA 2.1. *Let random variables* $X_1, X_2, \cdots, X_n$ *be END.*

(i) If $f_1, f_2, \cdots, f_n$ *are all nondecreasing (or nonincreasing) functions, then random variables* $f_1(X_1), f_2(X_2), \cdots, f_n(X_n)$ *are END.*

(ii) For each $n \geqslant 1$*, there exists a constant* $M > 0$ *such that*

$$E\left(\prod_{j=1}^{n} X_j^+\right) \leqslant M \prod_{j=1}^{n} EX_j^+.$$

The following one is a generalized version of Lemma 3 in Sung et al. [17]. Here the details of the proof are omitted.

LEMMA 2.2. *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of END random variables with* $EX_k = 0$ *and* $|X_k| \leqslant d_k, k \geqslant 1$*, where* $\{d_k, k \geqslant 1\}$ *is a sequence of positive constants. Then for any* $t \in \mathbb{R}$*, there exists a positive constant* M *such that*

$$E\exp\left\{t\sum_{k=1}^{n} X_k\right\} \leqslant M\exp\left\{\frac{t^2}{2}\sum_{k=1}^{n} e^{td_k} EX_k^2\right\}.$$

The next one is the Kolmogorov exponential inequality for END random variables, which was obtained by Wu and Guan [11].

LEMMA 2.3. *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of END random variables with* $EX_n = 0$ *and* $0 < B_n \doteq \sum_{k=1}^{n} EX_k^2 < \infty$. *Denote* $S_n = \sum_{k=1}^{n} X_k$ *for* $n \geqslant 1$. *Then there exists a constant* $M > 0$ *such that*

$$P(|S_n| \geqslant x) \leqslant P\left(\max_{1 \leqslant k \leqslant n} |X_k| \geqslant y\right) + 2M \exp\left(\frac{x}{y} - \frac{x}{y}\log\left(1 + \frac{xy}{B_n}\right)\right)$$

for any $x > 0$, $y > 0$.

The last one is a generalized version of Theorem A for END random variables, which is indispensable in proving our main result. There is no essential difference between the proof of Lemma 2.4 and Theorem A. The only difference is the positive constant M in the definition of END random variables, which has no influence in proving the finiteness. For convenience of the reader, we present the proof of Lemma 2.4 in Appendix.

LEMMA 2.4. *Let* $\{X_{nk}, 1 \leqslant k \leqslant k_n, n \geqslant 1\}$ *be an array of rowwise END random variables,* $\{c_n, n \geqslant 1\}$ *be a sequence of positive constants, and* $\{b_n, n \geqslant 1\}$ *be a sequence of positive constants such that* $\lim_{n\to\infty} b_n = \infty$. *Suppose that*

(i) $\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} P(|X_{nk}| > \varepsilon) < \infty$ *for all* $\varepsilon > 0$;

(ii) $\sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} P(|X_{nk}| > \frac{1}{b_n})\right)^{\xi} < \infty$ *for some* $\xi > 0$;

(iii) $b_n \sum_{k=1}^{k_n} EX_{nk}^2 I(|X_{nk}| \leqslant \frac{1}{b_n}) \to 0$ *as* $n \to \infty$;

(iv) $\sum_{n=1}^{\infty} c_n \exp\{-\eta_0 b_n\} < \infty$ *for some* $\eta_0 > 0$.

Then for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} c_n P\left(\left|\sum_{k=1}^{k_n}\left(X_{nk} - EX_{nk}I\left(|X_{nk}| \leqslant \frac{1}{b_n}\right)\right)\right| > \varepsilon\right) < \infty.$$

3. Main results

THEOREM 3.1. *Let* $q > 0$ *and* $\{X_{nk}, 1 \leqslant k \leqslant k_n, n \geqslant 1\}$ *be an array of rowwise END random variables,* $\{c_n, n \geqslant 1\}$ *be a sequence of positive constants, and* $\{b_n, n \geqslant 1\}$ *be a sequence of positive constants such that* $\lim_{n\to\infty} b_n = \infty$. *Suppose that the following conditions hold:*

(a) $\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} E|X_{nk}|^q I(|X_{nk}| > \varepsilon) < \infty$ *for all* $\varepsilon > 0$;

(b) there exists some $\eta > q$, *as* $n \to \infty$,

$$b_n \sum_{k=1}^{k_n} E|X_{nk}| I\left(|X_{nk}| > \frac{1}{16\eta^* b_n}\right) \to 0, \textit{where } \eta^* = \max\left\{\eta, \frac{1}{16}\right\};$$

(c) $\sum_{n=1}^{\infty} c_n b_n^{-\eta} < \infty$;

(d) $\sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} P(|X_{nk}| > \frac{1}{b_n})\right)^{\xi} < \infty$ *for some* $\xi > 0$;

(e) $b_n \sum_{k=1}^{k_n} EX_{nk}^2 I(|X_{nk}| \leqslant \frac{1}{b_n}) \to 0$ *as* $n \to \infty$.

Then for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} c_n E\left\{\left|\sum_{k=1}^{k_n}\left(X_{nk} - EX_{nk}I\left(|X_{nk}| \leqslant \frac{1}{b_n}\right)\right)\right| - \varepsilon\right\}_+^q < \infty. \tag{3.1}$$

Proof. First we state that the conditions of Lemma 2.4 hold. For all $\varepsilon > 0$, it follows by condition (a) that

$$\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} P(|X_{nk}| > \varepsilon) \leqslant C \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} E|X_{nk}|^q I(|X_{nk}| > \varepsilon) < \infty,$$

which implies that condition (i) of Lemma 2.4 holds.

Noting that $\exp\{-\eta_0 b_n\} = o(b_n^{-\eta})$ for any constants $\eta_0 > 0$ and $\eta > 0$, we have by condition (c) that

$$\sum_{n=1}^{\infty} c_n \exp\{-\eta_0 b_n\} \leqslant C \sum_{n=1}^{\infty} c_n b_n^{-\eta} < \infty,$$

which yields that condition (iv) of Lemma 2.4 holds.

Conditions (ii) and (iii) follows by (d) and (e), respectively. Thus, all the conditions of Lemma 2.4 are satisfied.

Denote $S_n = \sum_{k=1}^{k_n}\left(X_{nk} - EX_{nk}I\left(|X_{nk}| \leqslant \frac{1}{b_n}\right)\right)$, we can see that

$$\begin{aligned}
&\sum_{n=1}^{\infty} c_n E\left\{\left|\sum_{k=1}^{k_n}\left(X_{nk} - EX_{nk}I\left(|X_{nk}| \leqslant \frac{1}{b_n}\right)\right)\right| - \varepsilon\right\}_+^q \\
&= \sum_{n=1}^{\infty} c_n \int_0^{\varepsilon^q} P\left(|S_n| > \varepsilon + t^{\frac{1}{q}}\right) dt + \sum_{n=1}^{\infty} c_n \int_{\varepsilon^q}^{\infty} P\left(|S_n| > \varepsilon + t^{\frac{1}{q}}\right) dt \\
&\triangleq I_1 + I_2.
\end{aligned} \tag{3.2}$$

By Lemma 2.4, we can easily obtain that

$$I_1 \leqslant \varepsilon^q \sum_{n=1}^{\infty} c_n P(|S_n| > \varepsilon) < \infty. \tag{3.3}$$

Hence, to prove (3.1), we only need to show $I_2 < \infty$. It is easily see that

$$\begin{aligned}
I_2 \leqslant &\sum_{n=1}^{\infty} c_n \int_{\varepsilon^q}^{\infty} P\left(|S_n| > t^{\frac{1}{q}}, \bigcup_{k=1}^{k_n}\{|X_{nk}| > t^{\frac{1}{q}}\}\right) dt \\
&+ \sum_{n=1}^{\infty} c_n \int_{\varepsilon^q}^{\infty} P\left(|S_n| > t^{\frac{1}{q}}, \bigcap_{k=1}^{k_n}\{|X_{nk}| \leqslant t^{\frac{1}{q}}\}\right) dt
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \sum_{n=1}^{\infty} c_n \int_{\varepsilon^q}^{\infty} P\left(\bigcup_{k=1}^{k_n}\{|X_{nk}|>t^{\frac{1}{q}}\}\right) dt \\
&\quad + \sum_{n=1}^{\infty} c_n \int_{\varepsilon^q}^{\infty} P\left(\left|\sum_{k=1}^{k_n}\left(X_{nk}I\Big(|X_{nk}|\leqslant t^{\frac{1}{q}}\Big)-EX_{nk}I\Big(|X_{nk}|\leqslant \frac{1}{b_n}\Big)\right)\right|>t^{\frac{1}{q}}\right) dt \\
&\triangleq I_3+I_4. \qquad (3.4)
\end{aligned}
$$

It follows by condition (a) that

$$
I_3 \leqslant \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} E|X_{nk}|^q I(|X_{nk}|>\varepsilon)<\infty. \qquad (3.5)
$$

For I_4, it follows by $\lim\limits_{n\to\infty} b_n=\infty$ that for any $\varepsilon>0$, when n is large enough, $\frac{1}{b_n}<\varepsilon$. For $t\geqslant \varepsilon^q$, denote

$$
\begin{aligned}
Y_{nk} &= -t^{\frac{1}{q}} I(X_{nk}<-t^{\frac{1}{q}})+X_{nk}I(|X_{nk}|\leqslant t^{\frac{1}{q}})+t^{\frac{1}{q}}I(X_{nk}>t^{\frac{1}{q}}),\\
Z_{nk} &= -t^{\frac{1}{q}} I(X_{nk}<-t^{\frac{1}{q}})+t^{\frac{1}{q}}I(X_{nk}>t^{\frac{1}{q}}).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&P\left(\left|\sum_{k=1}^{k_n}\left(X_{nk}I\Big(|X_{nk}|\leqslant t^{\frac{1}{q}}\Big)-EX_{nk}I\Big(|X_{nk}|\leqslant \frac{1}{b_n}\Big)\right)\right|>t^{\frac{1}{q}}\right)\\
&\leqslant P\left(\left|\sum_{k=1}^{k_n}(Y_{nk}-EY_{nk}-Z_{nk}+EZ_{nk})\right|+\sum_{k=1}^{k_n}E|X_{nk}|I\Big(\frac{1}{b_n}<|X_{nk}|\leqslant t^{\frac{1}{q}}\Big)>t^{\frac{1}{q}}\right). \qquad (3.6)
\end{aligned}
$$

By condition (b), we get that

$$
\max_{t\geqslant b_n^{-q}} t^{-\frac{1}{q}}\sum_{k=1}^{k_n}E|X_{nk}|I\Big(\frac{1}{b_n}<|X_{nk}|\leqslant t^{\frac{1}{q}}\Big)\leqslant b_n\sum_{k=1}^{k_n}E|X_{nk}|I\Big(|X_{nk}|>\frac{1}{b_n}\Big)\to 0, \text{ as } n\to\infty,
$$

which implies that for all n large enough,

$$
\sum_{k=1}^{k_n}E|X_{nk}|I\Big(\frac{1}{b_n}<|X_{nk}|\leqslant t^{\frac{1}{q}}\Big)\leqslant \frac{1}{2}t^{\frac{1}{q}},\ t\geqslant \varepsilon^q.
$$

Hence, by (3.6) and the inequality above, we have that for all n large enough,

$$
\begin{aligned}
&P\left(\left|\sum_{k=1}^{k_n}\left(X_{nk}I\Big(|X_{nk}|\leqslant t^{\frac{1}{q}}\Big)-EX_{nk}I\Big(|X_{nk}|\leqslant \frac{1}{b_n}\Big)\right)\right|>t^{\frac{1}{q}}\right)\\
&\leqslant P\left(\left|\sum_{k=1}^{k_n}(Y_{nk}-EY_{nk}-Z_{nk}+EZ_{nk})\right|>\frac{1}{2}t^{\frac{1}{q}}\right)\\
&\leqslant P\left(\left|\sum_{k=1}^{k_n}(Z_{nk}-EZ_{nk})\right|>\frac{1}{4}t^{\frac{1}{q}}\right)+P\left(\left|\sum_{k=1}^{k_n}(Y_{nk}-EY_{nk})\right|>\frac{1}{4}t^{\frac{1}{q}}\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
I_4 &\leqslant C\sum_{n=1}^{\infty} c_n \int_{\varepsilon^q}^{\infty} P\left(\left|\sum_{k=1}^{k_n}(Z_{nk}-EZ_{nk})\right| > \frac{1}{4}t^{\frac{1}{q}}\right)dt \\
&\quad + C\sum_{n=1}^{\infty} c_n \int_{\varepsilon^q}^{\infty} P\left(\left|\sum_{k=1}^{k_n}(Y_{nk}-EY_{nk})\right| > \frac{1}{4}t^{\frac{1}{q}}\right)dt \\
&\triangleq I_5 + I_6. \qquad (3.7)
\end{aligned}
$$

Noting that $|Z_{nk}| = t^{\frac{1}{q}} I(|X_{nk}| > t^{\frac{1}{q}})$, we have by condition (a) that

$$
\begin{aligned}
I_5 &\leqslant C\sum_{n=1}^{\infty} c_n \int_{\varepsilon^q}^{\infty} t^{-\frac{1}{q}} \sum_{k=1}^{k_n} E|Z_{nk}|dt \\
&\leqslant C\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} \int_{\varepsilon^q}^{\infty} P(|X_{nk}| > t^{\frac{1}{q}})dt \\
&\leqslant C\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} E|X_{nk}|^q I(|X_{nk}| > \varepsilon) < \infty. \qquad (3.8)
\end{aligned}
$$

For I_6, applying Lemma 2.3 with $x = \frac{1}{4}t^{\frac{1}{q}}$, $y = \frac{1}{4\eta}t^{\frac{1}{q}}$, $B_n = \sum_{k=1}^{k_n} E(Y_{nk}-EY_{nk})^2$ and $\eta > q$, we obtain

$$
\begin{aligned}
I_6 &\leqslant C\sum_{n=1}^{\infty} c_n \int_{\varepsilon^q}^{\infty} P\left(\max_{1\leqslant k\leqslant k_n} |Y_{nk}-EY_{nk}| \geqslant \frac{1}{4\eta}t^{\frac{1}{q}}\right)dt \\
&\quad + C\sum_{n=1}^{\infty} c_n \int_{\varepsilon^q}^{\infty} \left(\frac{eB_n}{B_n + t^{\frac{2}{q}}/(16\eta)}\right)^{\eta} dt \\
&\triangleq I_7 + I_8. \qquad (3.9)
\end{aligned}
$$

By condition (b), we can see that $0 \leqslant \sum_{k=1}^{k_n} P(|X_{nk}| > \frac{1}{16\eta b_n}) \to 0$ as $n \to \infty$, which implies that for all n large enough, $\sum_{k=1}^{k_n} P(|X_{nk}| > \frac{1}{16\eta b_n}) \leqslant \frac{1}{32\eta}$ and $\frac{1}{b_n} < \varepsilon$. Thus,

$$
\begin{aligned}
\max_{t\geqslant\varepsilon^q}\max_{1\leqslant k\leqslant k_n} t^{-\frac{1}{q}}|EY_{nk}| &\leqslant \max_{t\geqslant\varepsilon^q}\max_{1\leqslant k\leqslant k_n}\Big[t^{-\frac{1}{q}}E|X_{nk}|I\Big(|X_{nk}| \leqslant \frac{1}{16\eta b_n}\Big) \\
&\quad + t^{-\frac{1}{q}}E|X_{nk}|I\Big(\frac{1}{16\eta b_n} < |X_{nk}| \leqslant t^{\frac{1}{q}}\Big) + P(|X_{nk}| > t^{\frac{1}{q}})\Big] \\
&\leqslant \frac{1}{\varepsilon}\cdot\frac{1}{16\eta b_n} + 2\sum_{k=1}^{k_n} P\Big(|X_{nk}| > \frac{1}{16\eta b_n}\Big) \\
&\leqslant \frac{1}{16\eta} + 2\cdot\frac{1}{32\eta} = \frac{1}{8\eta},
\end{aligned}
$$

which implies that

$$\begin{aligned} I_7 &\leqslant C\sum_{n=1}^{\infty}c_n\int_{\varepsilon^q}^{\infty}P\left(\max_{1\leqslant k\leqslant k_n}|X_{nk}|\geqslant\frac{1}{8\eta}t^{\frac{1}{q}}\right)dt \quad (\text{since } |Y_{nk}|\leqslant|X_{nk}|)\\ &\leqslant C\sum_{n=1}^{\infty}c_n\sum_{k=1}^{k_n}\int_{\varepsilon^q}^{\infty}P\left(|X_{nk}|\geqslant\frac{1}{8\eta}t^{\frac{1}{q}}\right)dt\\ &\leqslant C\sum_{n=1}^{\infty}c_n\sum_{k=1}^{k_n}E|X_{nk}|^qI\Big(|X_{nk}|>\frac{\varepsilon}{8\eta}\Big)<\infty. \end{aligned} \tag{3.10}$$

Noting that $Y_{nk}^2=t^{\frac{2}{q}}I(|X_{nk}|>t^{\frac{1}{q}})+X_{nk}^2I(|X_{nk}|\leqslant t^{\frac{1}{q}})$, we have by C_r inequality that

$$\begin{aligned} I_8 &\leqslant C\sum_{n=1}^{\infty}c_n\int_{\varepsilon^q}^{\infty}t^{-\frac{2\eta}{q}}\left(\sum_{k=1}^{k_n}EY_{nk}^2\right)^{\eta}dt\\ &\leqslant C\sum_{n=1}^{\infty}c_n\int_{\varepsilon^q}^{\infty}t^{-\frac{2\eta}{q}}\left(\sum_{k=1}^{k_n}EX_{nk}^2I\Big(|X_{nk}|\leqslant\frac{1}{b_n}\Big)\right)^{\eta}dt\\ &\quad+C\sum_{n=1}^{\infty}c_n\int_{\varepsilon^q}^{\infty}t^{-\frac{2\eta}{q}}\left(\sum_{k=1}^{k_n}EX_{nk}^2I\Big(\frac{1}{b_n}<|X_{nk}|\leqslant t^{\frac{1}{q}}\Big)\right)^{\eta}dt\\ &\quad+C\sum_{n=1}^{\infty}c_n\int_{\varepsilon^q}^{\infty}\left(\sum_{k=1}^{k_n}P\Big(|X_{nk}|>t^{\frac{1}{q}}\Big)\right)^{\eta}dt\\ &\leqslant C\sum_{n=1}^{\infty}c_n\int_{\varepsilon^q}^{\infty}t^{-\frac{2\eta}{q}}\left(\sum_{k=1}^{k_n}EX_{nk}^2I\Big(|X_{nk}|\leqslant\frac{1}{b_n}\Big)\right)^{\eta}dt\\ &\quad+C\sum_{n=1}^{\infty}c_n\int_{\varepsilon^q}^{\infty}t^{-\frac{\eta}{q}}\left(\sum_{k=1}^{k_n}E|X_{nk}|I\Big(|X_{nk}|>\frac{1}{b_n}\Big)\right)^{\eta}dt\\ &\triangleq I_9+I_{10}. \end{aligned} \tag{3.11}$$

From the condition (e), we know that for all n large enough, $b_n\sum_{k=1}^{k_n}EX_{n_k}^2I(|X_{n_k}|\leqslant\frac{1}{b_n})<1$. Hence, we have by condition (c) that

$$I_9=C\sum_{n=1}^{\infty}c_nb_n{}^{-\eta}\left(b_n\sum_{k=1}^{k_n}EX_{nk}^2I\Big(|X_{nk}|\leqslant\frac{1}{b_n}\Big)\right)^{\eta}\int_{\varepsilon^q}^{\infty}t^{-\frac{2\eta}{q}}dt\leqslant C\sum_{n=1}^{\infty}c_nb_n^{-\eta}<\infty. \tag{3.12}$$

Finally, we will prove $I_{10}<\infty$.

From condition (b), we know that for all n large enough, $b_n\sum_{k=1}^{k_n}E|X_{nk}|I(|X_{nk}|>\frac{1}{b_n})\leqslant 1$. Hence, we have by condition (c) again and $\eta>q$ that

$$I_{10}\leqslant C\sum_{n=1}^{\infty}c_n\int_{\varepsilon^q}^{\infty}t^{-\frac{\eta}{q}}b_n^{-\eta}\left(b_n\sum_{k=1}^{k_n}E|X_{nk}|I\Big(|X_{nk}|>\frac{1}{b_n}\Big)\right)^{\eta}dt\leqslant C\sum_{n=1}^{\infty}c_nb_n^{-\eta}<\infty. \tag{3.13}$$

Hence, the desired result (3.1) follows by (3.2)–(3.13) immediately. This completes the proof of the theorem. □

4. Corollaries

In this section, we will give some applications of Theorem 3.1. Firstly, we will present the concept of stochastic domination, which will be used in this section.

DEFINITION 4.1. A sequence $\{X_n, n \geqslant 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a positive constant C such that

$$P(|X_n| > x) \leqslant CP(|X| > x)$$

for all $x \geqslant 0$ and $n \geqslant 1$.

An array $\{X_{ni}, i \geqslant 1, n \geqslant 1\}$ of rowwise random variables is said to be stochastically dominated by a random variable X if there exists a positive constant C such that

$$P(|X_{ni}| > x) \leqslant CP(|X| > x)$$

for all $x \geqslant 0$, $i \geqslant 1$ and $n \geqslant 1$.

Using the integration by parts, we can get the following important property for stochastic domination. For the proof, one can refer to Wu [18], Shen [19], or Shen et al. [20].

LEMMA 4.1. *Let $\{X_{ni}, i \geqslant 1, n \geqslant 1\}$ be an array of rowwise random variables which is stochastically dominated by a random variable X. For any $\alpha > 0$ and $b > 0$,*

$$E|X_{ni}|^{\alpha} I(|X_{ni}| \leqslant b) \leqslant C_1 \left[E|X|^{\alpha} I(|X| \leqslant b) + b^{\alpha} P(|X| > b)\right],$$

$$E|X_{ni}|^{\alpha} I(|X_{ni}| > b) \leqslant C_2 E|X|^{\alpha} I(|X| > b),$$

where C_1 and C_2 are positive constants.

With Lemma 4.1 accounted for, we can get the following corollaries by Theorem 3.1.

COROLLARY 4.1. *Let $\alpha > \frac{1}{2}$, $p > 0$, $\alpha p > 1$ and $\{X_n, n \geqslant 1\}$ be a sequence of END random variables with $EX_k = 0$ which are stochastically dominated by a random variable X. Suppose that $E|X|^p < \infty$ if $p \geqslant 1$ and $E|X| < \infty$ if $0 < p < 1$. Then for any $\varepsilon > 0$ and $0 < q < p$,*

$$\sum_{n=1}^{\infty} n^{\alpha p - \alpha q - 2} E\left\{\left|\sum_{k=1}^{n} X_k\right| - \varepsilon n^{\alpha}\right\}_+^q < \infty. \tag{4.1}$$

Proof. In order to apply Theorem 3.1, let $X_{nk} = n^{-\alpha} X_k$, $c_n = n^{\alpha p - 2}$, $k_n = n$, $b_n = n^r$ with $0 < r < \min\{\alpha - \frac{1}{p}, 2\alpha - 1\}$. We will check the conditions of Theorem 3.1 one by one.

By Lemma 4.1, we can get that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2} \sum_{k=1}^{n} n^{-\alpha q} E|X_k|^q I(|X_k| > n^{\alpha}\varepsilon) \\
&\leqslant C \sum_{n=1}^{\infty} n^{\alpha(p-q)-1} E|X|^q I(|X| > n^{\alpha}\varepsilon) \\
&= C \sum_{m=1}^{\infty} E|X|^q I(m^{\alpha}\varepsilon < |X| \leqslant (m+1)^{\alpha}\varepsilon) \sum_{n=1}^{m} n^{\alpha(p-q)-1} \\
&\leqslant C \sum_{m=1}^{\infty} m^{\alpha(p-q)} E|X|^q I(m^{\alpha}\varepsilon < |X| \leqslant (m+1)^{\alpha}\varepsilon) \\
&\leqslant C \sum_{m=1}^{\infty} E|X|^p I(m^{\alpha}\varepsilon < |X| \leqslant (m+1)^{\alpha}\varepsilon) \leqslant CE|X|^p < \infty,
\end{aligned}$$

which yields the condition (a) of Theorem 3.1.

For condition (b), note that $r < \alpha - \frac{1}{p}$. Denote $p^* = \max\{p, 1\}$. Hence, if $\eta \geqslant \frac{1}{16}$, then we have by Lemma 4.1 that

$$\begin{aligned}
n^r \sum_{k=1}^{n} n^{-\alpha} E|X_k| I\left(|X_k| > \frac{1}{16\eta} n^{\alpha-r}\right) &\leqslant C n^{r-\alpha+1} n^{(\alpha-r)(1-p^*)} E|X|^{p^*} \\
&\leqslant CE|X|^{p^*} n^{1-(\alpha-r)p^*} \to 0, \text{ as } n \to \infty;
\end{aligned}$$

if $\eta < \frac{1}{16}$, then we have by Lemma 4.1 again that

$$\begin{aligned}
n^r \sum_{k=1}^{n} n^{-\alpha} E|X_k| I(|X_k| > n^{\alpha-r}) &\leqslant C n^{r-\alpha+1} n^{(\alpha-r)(1-p^*)} E|X|^{p^*} \\
&\leqslant CE|X|^{p^*} n^{1-(\alpha-r)p^*} \to 0, \text{ as } n \to \infty.
\end{aligned}$$

For condition (c), by taking $\eta > \frac{\alpha p-1}{r}$, we have

$$\sum_{n=1}^{\infty} n^{\alpha p-2} \cdot n^{-\eta r} = \sum_{n=1}^{\infty} n^{-1+(\alpha p-1)-\eta r} < \infty.$$

For condition (d), by taking $\xi > \frac{\alpha p-1}{(\alpha-r)p-1} > 0$, we have by Lemma 4.1 and Markov's inequality that

$$\begin{aligned}
\sum_{n=1}^{\infty} n^{\alpha p-2} \left(\sum_{k=1}^{n} P(|X_k| > n^{\alpha-r})\right)^{\xi} &\leqslant C \sum_{n=1}^{\infty} n^{\alpha p-2} \left(\sum_{k=1}^{n} P(|X| > n^{\alpha-r})\right)^{\xi} \\
&\leqslant C \sum_{n=1}^{\infty} n^{\alpha p-2+(r-\alpha)p\xi} (nE|X|^p)^{\xi} \\
&\leqslant C(E|X|^p)^{\xi} \sum_{n=1}^{\infty} n^{-1+\alpha p-1-[(\alpha-r)p-1]\xi} < \infty.
\end{aligned}$$

For condition (e), if $p \geqslant 2$, then

$$n^r \sum_{k=1}^{n} n^{-2\alpha} EX_k^2 I(|X_k| \leqslant n^{\alpha-r}) \leqslant Cn^{r-2\alpha+1} \to 0, \text{ as } n \to \infty;$$

if $p < 2$, then by $0 < r < \alpha - \frac{1}{p}$, we get $1 - \alpha p + (p-1)r < \frac{1}{p} - \alpha < 0$, and thus,

$$\begin{aligned} n^r \sum_{k=1}^{n} n^{-2\alpha} EX_k^2 I(|X_k| \leqslant n^{\alpha-r}) &\leqslant n^{r-2\alpha} \sum_{k=1}^{n} n^{(\alpha-r)(2-p)} E|X_k|^p I(|X_k| \leqslant n^{\alpha-r}) \\ &\leqslant Cn^{1-\alpha p+(p-1)r} \to 0, \text{ as } n \to \infty. \end{aligned}$$

Hence, we have by Theorem 3.1 that

$$\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2} E\left\{\left|\sum_{k=1}^{n} (X_k - EX_k I(|X_k| \leqslant n^{\alpha-r}))\right| - \varepsilon n^{\alpha}\right\}_+^q < \infty. \tag{4.2}$$

To prove (4.1), it remains to show that

$$\left|\sum_{k=1}^{n} n^{-\alpha} EX_k I(|X_k| \leqslant n^{\alpha-r})\right| \to 0, \text{ as } n \to \infty.$$

If $p \geqslant 1$, we have by $EX_k = 0$ and $r < \alpha - \frac{1}{p}$ that

$$\begin{aligned} \left|\sum_{k=1}^{n} n^{-\alpha} EX_k I(|X_k| \leqslant n^{\alpha-r})\right| &\leqslant n^{-\alpha} \sum_{k=1}^{n} E|X_k| I(|X_k| > n^{\alpha-r}) \\ &\leqslant n^{-\alpha} \sum_{k=1}^{n} n^{(\alpha-r)(1-p)} E|X_k|^p I(|X_k| > n^{\alpha-r})| \\ &\leqslant CE|X|^p n^{1-\alpha p+(p-1)r} \leqslant CE|X|^p n^{\frac{1}{p}-\alpha} \to 0, \text{ as } n \to \infty; \end{aligned}$$

if $0 < p < 1$, it follows by $\alpha p > 1$ that $\alpha > 1$, and then

$$\begin{aligned} \left|\sum_{k=1}^{n} n^{-\alpha} EX_k I(|X_k| \leqslant n^{\alpha-r})\right| &\leqslant n^{-\alpha} \sum_{k=1}^{n} E|X_k| I(|X_k| > n^{\alpha-r}) \\ &\leqslant Cn^{1-\alpha} E|X| \to 0, \text{ as } n \to \infty. \end{aligned}$$

This completes the proof of the corollary. □

COROLLARY 4.2. *Let* $\{X_{nk}, 1 \leqslant k \leqslant n, n \geqslant 1\}$ *be an array of mean zero rowwise END random variables which are stochastically dominated by a random variable* X *with* $E|X|^{2p} < \infty$ *for some* $p \geqslant 1$. *Let* $\{a_{nk}, 1 \leqslant k \leqslant n, n \geqslant 1\}$ *be a array of real numbers and* $\{b_n, n \geqslant 1\}$ *be a sequence of positive constants such that* $\lim_{n\to\infty} b_n = \infty$. *Suppose that the following conditions are satisfied:*

(1) $b_n = O(n^r)$ *for some* $0 < r < \frac{1}{2p}$*;*

(2) $\sum_{n=1}^{\infty} b_n^{-\eta} < \infty$ *for some* $\eta > 2p$*;*

(3) $b_n \sum_{k=1}^{n} a_{nk}^2 = o(1)$*;*

(4) $\max_{1\leqslant k\leqslant n} |a_{nk}| = O(n^{-\frac{1}{p}})$.

Then for any $0 < q \leqslant p$,

$$\sum_{n=1}^{\infty} E\left\{\left|\sum_{k=1}^{n} a_{nk}X_{nk}\right| - \varepsilon\right\}_+^q < \infty. \tag{4.3}$$

Proof. Without loss of generality, we may assume that $a_{nk} \geqslant 0$ for $1 \leqslant k \leqslant n, n \geqslant 1$, otherwise we can prove the result for two arrays of END random variables $\{a_{nk}^+ X_{nk}, 1 \leqslant k \leqslant k_n, n \geqslant 1\}$ and $\{a_{nk}^- X_{nk}, 1 \leqslant k \leqslant n, n \geqslant 1\}$ separately, where $a_{nk}^+ = \max\{a_{nk}, 0\}$ and $a_{nk}^- = \max\{-a_{nk}, 0\}$. Then we can also assume that $\max_{1\leqslant k\leqslant n} a_{nk} \leqslant n^{-\frac{1}{p}}$ and $b_n \leqslant n^r$.

We apply Theorem 3.1 with $c_n = 1$, $n \geqslant 1$, and replace X_{nk} by $a_{nk}X_{nk}$, $1 \leqslant k \leqslant n$, $n \geqslant 1$. In the following, we will check the conditions of Theorem 3.1.

For condition (a), it follows by Lemma 4.1 and conditions (1) and (4) that for any $\varepsilon > 0$,

$$\begin{aligned}
\sum_{n=1}^{\infty}\sum_{k=1}^{n} E|a_{nk}X_{nk}|^q I(|a_{nk}X_{nk}| > \varepsilon) &\leqslant \sum_{n=1}^{\infty} n^{-\frac{q}{p}} \sum_{k=1}^{n} E|X_{nk}|^q I(|X_{nk}| > n^{\frac{1}{p}}\varepsilon) \\
&\leqslant C\sum_{n=1}^{\infty} n^{1-\frac{q}{p}} E|X|^q I(|X| > n^{\frac{1}{p}}\varepsilon) \\
&= C\sum_{n=1}^{\infty} n^{1-\frac{q}{p}} E|X|^q \sum_{m=n}^{\infty} I(m^{\frac{1}{p}}\varepsilon < |X| \leqslant (m+1)^{\frac{1}{p}}\varepsilon) \\
&\leqslant C\sum_{m=1}^{\infty} m^{2-\frac{q}{p}} E|X|^q I(m^{\frac{1}{p}}\varepsilon < |X| \leqslant (m+1)^{\frac{1}{p}}\varepsilon) \\
&\leqslant CE|X|^{2p} < \infty.
\end{aligned}$$

For condition (b), noting that $\eta > 2p \geqslant 2$ and $r < \frac{1}{2p}$, we have by Lemma 4.1 again that

$$\begin{aligned}
n^r \sum_{k=1}^{n} E|a_{nk}X_{nk}| I\Big(|a_{nk}X_{nk}| > \frac{1}{16\eta b_n}\Big) &\leqslant Cn^{r-\frac{1}{p}+1} E|X| I\Big(|X| > \frac{1}{16\eta} n^{\frac{1}{p}-r}\Big) \\
&\leqslant Cn^{r-\frac{1}{p}+1} E|X|^{2p} n^{(\frac{1}{p}-r)(1-2p)} \\
&= CE|X|^{2p} n^{2rp-1} \to 0, \text{ as } n \to \infty.
\end{aligned}$$

Condition (c) of Theorem 3.1 holds by the assumption (2).

For condition (d) of Theorem 3.1, taking $\xi > 1/(1-2pr) > 0$, we have by Markov's inequality that

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}P\Big(|a_{nk}X_{nk}|>\frac{1}{b_n}\Big)\right)^{\xi} &\leqslant C\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}P\Big(|a_{nk}X|>\frac{1}{b_n}\Big)\right)^{\xi}\\ &\leqslant C\sum_{n=1}^{\infty}\left(b_n^{2p}\sum_{k=1}^{n}|a_{nk}|^{2p}E|X|^{2p}\right)^{\xi}\\ &\leqslant C\sum_{n=1}^{\infty}n^{(2pr-1)\xi}<\infty.\end{aligned}$$

For condition (e) of Theorem 3.1, it follows by condition (3) that

$$\begin{aligned}b_n\sum_{k=1}^{n}E(a_{nk}X_{nk})^2I\Big(\,|\,a_{nk}X_{nk}\,|\leqslant\frac{1}{b_n}\Big) &\leqslant b_n\sum_{k=1}^{n}a_{nk}^2EX_{nk}^2\\ &\leqslant Cb_n\sum_{k=1}^{n}a_{nk}^2\to 0,\ \text{as } n\to\infty.\end{aligned}$$

Hence, by Theorem 3.1 we obtain that

$$\sum_{n=1}^{\infty}E\left\{\left|\sum_{k=1}^{n}a_{nk}\Big(X_{nk}-EX_{nk}I\Big(|a_{nk}X_{nk}|\leqslant\frac{1}{b_n}\Big)\Big)\right|-\varepsilon\right\}_{+}^{q}<\infty. \tag{4.4}$$

To prove (4.3), it remains to show

$$\sum_{k=1}^{n}a_{nk}EX_{nk}I\Big(|a_{nk}X_{nk}|\leqslant\frac{1}{b_n}\Big)\to 0.$$

Noting that $EX_{nk}=0$, we have by Lemma 4.1 and $r<\frac{1}{2p}$ that

$$\begin{aligned}\left|\sum_{k=1}^{n}a_{nk}EX_{nk}I\Big(|a_{nk}X_{nk}|\leqslant\frac{1}{b_n}\Big)\right| &= \left|\sum_{k=1}^{n}a_{nk}EX_{nk}I\Big(|a_{nk}X_{nk}|>\frac{1}{b_n}\Big)\right|\\ &\leqslant C\sum_{k=1}^{n}n^{-\frac{1}{p}}E|X_{nk}|I(|X_{nk}|>n^{\frac{1}{p}-r})\\ &\leqslant Cn^{1-\frac{1}{p}}E|X|^{2p}|X|^{1-2p}I(|X|>n^{\frac{1}{p}-r})\\ &\leqslant Cn^{-1+(2p-1)r}E|X|^{2p}\\ &\leqslant Cn^{-\frac{1}{2p}}E|X|^{2p}\to 0,\ \text{as } n\to\infty.\end{aligned}$$

The proof is completed. □

As a special case of Corollary 4.2,we obtain Corollary 4.3.

COROLLARY 4.3. *Let $\{X_{nk}, 1 \leqslant k \leqslant n, n \geqslant 1\}$ be an array of mean zero rowwise END random variables which are stochastically dominated by a random variable X with $E|X|^{2p} < \infty$ for some $1 \leqslant p < 2$. Then for any $\varepsilon > 0$ and $0 < q \leqslant p$,*

$$\sum_{n=1}^{\infty} E\left\{n^{-\frac{1}{p}}\left|\sum_{k=1}^{n} X_{nk}\right| - \varepsilon\right\}_{+}^{q} < \infty.$$

Proof. Let $a_{nk} = n^{-\frac{1}{p}}$ for $1 \leqslant k \leqslant n$ and $n \geqslant 1$. Then conditions of Corollary 4.2 are trivially satisfied with $b_n = n^r$ for some $0 < r < \min\{\frac{1}{2p}, \frac{2}{p} - 1\}$. The desired result follows by Corollary 4.2 immediately. The proof is completed. □

REMARK 4.1. It is deserved to mention that as the same assumptions of Corollary 4.3 for ND random variables, Sung et al. [17] only obtained that $\sum_{k=1}^{n} X_{nk}/n^{1/p} \to 0$ completely. As it is mentioned in Introduction, complete moment convergence is much stronger than complete convergence. Hence, the result of Corollary 4.3 generalizes and improves the corresponding one of Sung et al. [17].

5. Appendix

Proof of Lemma 2.4. Let $N_1 = \left\{n : \sum_{k=1}^{k_n} P(|X_{nk}| > \frac{1}{b_n}) \leqslant 1\right\}$, $N_2 = N - N_1$. Hence,

$$\begin{aligned}
&\sum_{n\in N_2} c_n P\left(\left|\sum_{k=1}^{k_n}\left(X_{nk} - EX_{nk}I\left(|X_{nk}| \leqslant \frac{1}{b_n}\right)\right)\right| > \varepsilon\right) \\
&\leqslant \sum_{n\in N_2} c_n \leqslant \sum_{n\in N_2} c_n \left(\sum_{k=1}^{k_n} p(|X_{nk}| > \frac{1}{b_n})\right)^{\xi} < \infty.
\end{aligned}$$

So it means that we also need to prove

$$\sum_{n\in N_1} c_n P\left(\left|\sum_{k=1}^{k_n}\left(X_{nk} - EX_{nk}I\left(|X_{nk}| \leqslant \frac{1}{b_n}\right)\right)\right| > \varepsilon\right) < \infty \text{ for all } \varepsilon > 0. \tag{5.1}$$

Because $\frac{1}{b_n} \to 0$ as $n \to \infty$, there exists a $M^* > 0$ such that $\frac{1}{b_n} < \frac{\varepsilon}{4(\xi+1)}$ for all $n > M^*$. For fixed $n \geqslant 1$, denote for $1 \leqslant k_n \leqslant n$ that

$$\begin{aligned}
Y_{nk} &= -\frac{1}{b_n}I\left(X_{nk} < -\frac{1}{b_n}\right) + X_{nk}I\left(|X_{nk}| \leqslant \frac{1}{b_n}\right) + \frac{1}{b_n}I\left(X_{nk} > \frac{1}{b_n}\right), \\
U_{nk} &= -\frac{1}{b_n}I\left(X_{nk} < -\frac{1}{b_n}\right), \\
V_{nk} &= \frac{1}{b_n}I\left(X_{nk} > \frac{1}{b_n}\right),
\end{aligned}$$

$$Z_{nk} = X_{nk} I\Big(\frac{1}{b_n} < |X_{nk}| \leqslant \frac{\varepsilon}{4(\xi+1)}\Big).$$

It is easily checked that $\{Y_{nk} - EY_{nk}, 1 \leqslant k \leqslant k_n, n \geqslant 1\}$, $\{U_{nk} - EU_{nk}, 1 \leqslant k \leqslant k_n, n \geqslant 1\}$ and $\{V_{nk} - EV_{nk}, 1 \leqslant k \leqslant k_n, n \geqslant 1\}$ are all arrays of rowwise END random variables by Lemma 2.1 (i). Hence,

$$\begin{aligned}
&\sum_{n\in N_1} c_n P\left(\sum_{k=1}^{k_n}\Big(X_{nk} - EX_{nk} I\Big(|X_{nk}| \leqslant \frac{1}{b_n}\Big)\Big) > \varepsilon\right)\\
&= \sum_{n\in N_1} c_n P\left(\sum_{k=1}^{k_n}\Big(X_{nk} - EX_{nk} I\Big(|X_{nk}| \leqslant \frac{1}{b_n}\Big)\Big) > \varepsilon, \bigcup_{k=1}^{k_n}\{|X_{nk}| > \frac{\varepsilon}{4(\xi+1)}\}\right)\\
&\quad + \sum_{n\in N_1} c_n P\left(\sum_{k=1}^{k_n}\Big(X_{nk} - EX_{nk} I\Big(|X_{nk}| \leqslant \frac{1}{b_n}\Big)\Big) > \varepsilon, \bigcap_{k=1}^{k_n}\{|X_{nk}| \leqslant \frac{\varepsilon}{4(\xi+1)}\}\right)\\
&\leqslant \sum_{n\in N_1} c_n P\left(\bigcup_{k=1}^{k_n}\Big\{|X_{nk}| > \frac{\varepsilon}{4(\xi+1)}\Big\}\right)\\
&\quad + \sum_{n\in N_1} c_n P\left(\sum_{k=1}^{k_n}\Big(X_{nk} I\Big(|X_{nk}| \leqslant \frac{\varepsilon}{4(\xi+1)}\Big) - EX_{nk} I\Big(|X_{nk}| \leqslant \frac{1}{b_n}\Big)\Big) > \varepsilon\right)\\
&\triangleq I_1 + I_2. \qquad (5.2)
\end{aligned}$$

From condition (i), we obtain that

$$I_1 \leqslant \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} P\left(|X_{nk}| > \frac{\varepsilon}{4(\xi+1)}\right) < \infty. \qquad (5.3)$$

For I_2, noting that it is finite when $n \leqslant M^*$, so we only need to consider the case $n > M^*$. It is easily checked that

$$\begin{aligned}
&\sum_{n\in N_1, n>M^*} c_n P\left(\sum_{k=1}^{k_n}\Big(X_{nk} I\Big(|X_{nk}| \leqslant \frac{\varepsilon}{4(\xi+1)}\Big) - EX_{nk} I\Big(|X_{nk}| \leqslant \frac{1}{b_n}\Big)\Big) > \varepsilon\right)\\
&\leqslant \sum_{n\in N_1, n>M^*} c_n P\left(\sum_{k=1}^{k_n}(Y_{nk} - EY_{nk}) > \frac{\varepsilon}{4}\right) + \sum_{n\in N_1, n>M^*} c_n P\left(\sum_{k=1}^{k_n}(-U_{nk} + EU_{nk}) > \frac{\varepsilon}{4}\right)\\
&\quad + \sum_{n\in N_1, n>M^*} c_n P\left(\sum_{k=1}^{k_n}(-V_{nk} + EV_{nk}) > \frac{\varepsilon}{4}\right) + \sum_{n\in N_1, n>M^*} c_n P\left(\sum_{k=1}^{k_n} Z_{nk} > \frac{\varepsilon}{4}\right)\\
&\triangleq I_3 + I_4 + I_5 + I_6. \qquad (5.4)
\end{aligned}$$

Obviously, $|Y_{nk} - EY_{nk}| \leqslant \frac{2}{b_n}$ and $Y_{nk}^2 = \frac{1}{b_n^2} I(|X_{nk}| > \frac{1}{b_n}) + X_{nk}^2 I(|X_{nk}| \leqslant \frac{1}{b_n})$. Applying Lemma 2.2 with $t = \frac{4(\eta_0+1)b_n}{\varepsilon}$, we have by Markov's inequality and condition (iii) that for $n \in N_1$ and $n > M^*$,

$$P\left(\sum_{k=1}^{k_n}(Y_{nk}-EY_{nk}) > \frac{\varepsilon}{4(\xi+1)}\right) \leqslant M\exp\left\{-\frac{\varepsilon}{4}t+\frac{t^2}{2}e^{\frac{2t}{b_n}}\sum_{k=1}^{k_n}E(Y_{nk}-EY_{nk})^2\right\}$$
$$\leqslant M\exp\left\{-\frac{\varepsilon}{4}t+\frac{t^2}{2}e^{\frac{2t}{b_n}}\sum_{k=1}^{k_n}EY_{nk}^2\right\}$$
$$\leqslant M\exp\{-(\eta_0+1)b_n+o(1)b_n\} \leqslant M\exp\{-\eta_0 b_n\},$$

which together with condition (iv) yields that

$$I_3 \leqslant \sum_{n\in N_1, n>M^*} Mc_n\exp\{-\eta_0 b_n\} < \infty. \tag{5.5}$$

For I_4, note that $|U_{nk}-EU_{nk}| \leqslant \frac{2}{b_n}$. Applying Lemma 2.2 again with $t=\frac{4(\eta_0+1)b_n}{\varepsilon}$, we have by Markov's inequality that for $n \in N_1$ and $n > M^*$,

$$P\left(\sum_{k=1}^{k_n}(-U_{nk}+EU_{nk}) > \frac{\varepsilon}{4}\right) \leqslant e^{-\frac{\varepsilon}{4}t}E\exp\left\{t\sum_{k=1}^{k_n}(-U_{nk}+EU_{nk})\right\}$$
$$\leqslant M\exp\left\{-\frac{\varepsilon}{4}t+\frac{t^2}{2}e^{\frac{2t}{b_n}}\sum_{k=1}^{k_n}E(-U_{nk}+EU_{nk})^2\right\}$$
$$\leqslant M\exp\left\{-\frac{\varepsilon}{4}t+\frac{t^2}{2}e^{\frac{2t}{b_n}}\sum_{k=1}^{k_n}\frac{1}{b_n^2}P\Big(|X_{nk}|>\frac{1}{b_n}\Big)\right\}$$
$$\leqslant M\exp\{-(\eta_0+1)b_n+o(1)b_n\} \leqslant M\exp\{-\eta_0 b_n\},$$

which together with condition (iv) again yields that

$$I_4 \leqslant \sum_{n\in N_1, n>M^*} Mc_n\exp\{-\eta_0 b_n\} < \infty. \tag{5.6}$$

Similar to the proof of $I_4<\infty$, we can get $I_5<\infty$.

Finally, we will prove $I_6<\infty$. It follows by the subadditivity of probablity and the definition of END that for $n \in N_1$,

$$P\left(\sum_{k=1}^{k_n}X_{nk}I\Big(\frac{1}{b_n}<|X_{nk}|\leqslant\frac{\varepsilon}{4(\xi+1)}\Big) > \frac{\varepsilon}{4}\right)$$
$$\leqslant P\left(\text{at least } \lfloor\xi+1\rfloor's\ X_{nk} \text{ have the property } X_{nk} > \frac{1}{b_n}\right)$$
$$= P\left\{\bigcup_{1\leqslant j_1<\cdots<j_{\lfloor\xi+1\rfloor}\leqslant k_n}\Big(X_{nj_1}>\frac{1}{b_n},\cdots,X_{nj_{\lfloor\xi+1\rfloor}}>\frac{1}{b_n}\Big)\right\}$$

$$\leqslant \sum_{j_1,\cdots,j_{\lfloor \xi+1 \rfloor}} M \prod_{k=1}^{\lfloor \xi+1 \rfloor} P\Big(X_{nj_k} > \frac{1}{b_n}\Big) \leqslant M \left(\sum_{k=1}^{k_n} P\Big(|X_{nk}| > \frac{1}{b_n}\Big) \right)^{\lfloor \xi+1 \rfloor}$$

$$\leqslant M \left(\sum_{k=1}^{k_n} P\Big(|X_{nk}| > \frac{1}{b_n}\Big) \right)^{\xi},$$

which together with condition (ii) yields that

$$I_6 \leqslant \sum_{n\in N_1, n>M^*} Mc_n \left(\sum_{k=1}^{k_n} P\Big(|X_{nk}| > \frac{1}{b_n}\Big) \right)^{\xi} < \infty. \tag{5.7}$$

From the statement above, we have

$$\sum_{n\in N_1} c_n P\left(\sum_{k=1}^{k_n} \Big(X_{nk} - EX_{nk} I\Big(|X_{nk}| \leqslant \frac{1}{b_n}\Big)\Big) > \varepsilon \right) < \infty. \tag{5.8}$$

Noting that $\{-X_{nk}, 1 \leqslant k \leqslant k_n, n \geqslant 1\}$ is also an array of rowwise END random variables, we have by (5.9) that

$$\sum_{n\in N_1} c_n P\left(\sum_{k=1}^{k_n} \Big(-X_{nk} + EX_{nk} I\Big(|X_{nk}| \leqslant \frac{1}{b_n}\Big)\Big) > \varepsilon \right) < \infty. \tag{5.9}$$

By (5.8) and (5.9), we can get (5.1) immediately. This completes the proof of the lemma. □

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REFERENCES

[1] P. L. HSU, AND H. ROBBINS, *Complete convergence and the law of large numbers*, Proceedings of the National Academy of Sciences, **33**, 25–31, 1947.

[2] Y. S. CHOW, *On the rate of moment complete convergence of sample sums and extremes*, Bulletin of the Institute of Mathematics Academia Sinica, **16** (3), 177–201, 1988.

[3] L. LIU, *Precise large deviations for dependent random variables with heavy tails*, Statistics and Probability Letters, **79**, 1290–1298, 2009.

[4] E. LEHMANN, *Some concepts of dependence*, The Annals of Mathematical Statistics, **37**, 1137–1153, 1966.

[5] K. JOAG-DEV, F. PROSCHAN, *Negative association of random variables with applications*, The Annals of Statistics, **11**, 286–295, 1983.

[6] L. LIU, *Necessary and sufficient conditions for moderate deviations of dependent random variables with heavy tails*, Science in China Series A: Mathematics, **53** (6), 1421–1434, 2010.

[7] Y. CHEN, A. CHEN, K. W. NG, *The strong law of large numbers for extend negatively dependent random variables*, Journal of Applied Probability, **47**, 908–922, 2010.

[8] A. T. SHEN, *Probability inequalities for END sequence and their applications*, Journal of Inequalities and Applications, Volume 2011, Article ID 98, 12 pages, 2011.

[9] A. T. SHEN, *Bernstein-type inequality for widely dependent sequence and its application to nonparametric regression models*, Abstract and Applied Analysis, Volume 2013, Article ID 862602, 9 pages, 2013.

[10] S. J. WANG, W. S. WANG, *Extended precise large deviations of random sums in the presence of END structure and consistent variation*, Journal of Applied Mathematics, Volume 2012, Article ID 436531, 12 pages, 2012.

[11] Y. F. WU, M. GUAN, *Convergence properties of the partial sums for sequences of END random variables*, Journal of the Korean Mathematical Society, **49** (6), 1097–1110, 2012.

[12] S. J. WANG, X. J. WANG, *Precise large deviations for random sums of END real-valued random variables with consistent variation*, Journal of Mathematical Analysis and Applications, **402**, 660–667, 2013.

[13] D. H. QIU, P. Y. CHEN, R. G. ANTONINI, A. VOLODIN, *On the complete convergence for arrays of rowwise extended negatively dependent random variables*, Journal of the Korean Mathematical Society, **50** (2), 379–392, 2013.

[14] X. J. WANG, T. C. HU, A. VOLODIN, S. H. HU, *Complete convergence for weighted sums and arrays of rowwise extended negatively dependent random variables*, Communications in Statistics-Theory and Methods, **42**, 2391–2401, 2013.

[15] X. J. WANG, S. J. WANG, S. H. HU, J. M. LING, Y. F. WEI, *On complete convergence of weighted sums for arrays of rowwise extended negatively dependent random variables*, Stochastics: An International Journal of Probability and Stochastic Processes, **85** (6), 1060–1072, 2013.

[16] X. J. WANG, X. Q. LI, S. H. HU, X. H. WANG, *On complete convergence for an extended negatively dependent sequence*, Communications in Statistics-Theory and Methods, **43** (14), 2923–2937, 2014.

[17] S. H. SUNG, K. BUDSABA, A. VOLODIN, *Complete convergence for arrays of negatively dependent random variables*, Informatics and its Applications, **6** (4), 95–102, 2012.

[18] Q. Y. WU, *Probability Limit Theory for Mixing Sequences*, Science Press, Beijing, 2006.

[19] A. T. SHEN, *On the strong convergence rate for weighted sums of arrays of rowwise negatively orthant dependent random variables*, RACSAM, **107** (2), 257–271, 2013.

[20] A. T. SHEN, Y. ZHANG, A. VOLODIN, *Applications of the Rosenthal-type inequality for negatively superadditive dependent random variables*, Metrika, **78**, 295–311, 2015.

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