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A GENERALIZED GRONWALL–BELLMAN TYPE DELAY INTEGRAL INEQUALITY WITH TWO INDEPENDENT VARIABLES ON TIME SCALES

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Abstract. Using a technique of monotonization, this paper investigates a generalized Gronwall-Bellman type delay integral inequality with two independent variables on time scales. The result not only unifies some continuous inequalities and their discrete analogues but also extends some known integral inequalities on time scales. An application to the estimation of solutions of delay dynamic integral equations on time scales is also given.

1. Introduction

To unify and extend continuous and discrete analysis, the theory of time scales was introduced by Hilger [1] in his Ph. D. Thesis in 1988. Since then, the theory has been evolving, and has been applied to various fields of Mathematics (see [2, 3, 4, 5]) and references given therein.

It is well known that Gronwall type integral inequalities involving functions of one and more than one independent variables play important roles in the study of existence, uniqueness, boundness, stability, invariant manifolds and other qualitative properties of solutions of the theory of differential and integral equations. A lot of contributions to its generalization have been archived by many researchers (see [6, 7, 8, 9, 10, 11, 12, 13, 14]).

Recently, more attention has been paid to generalizations of Gronwall's inequalities on time scales (see [15, 16, 17, 18, 19, 20, 21] and the references therein). A lot of integral inequalities on time scales have been established, which have been designed to unify continuous and discrete analysis. One of the important things is that Bohner [22] studied the following inequality on time scales

$$u(t) \leqslant a(t) + p(t)\int_{t_0}^{t} k(t,\tau)[b(\tau)u(\tau) + q(\tau)]\Delta\tau. \tag{1.1}$$

In 2006, Pachpatte [23] discussed Gronwall-Bellman inequality with nonlinearity on time scales

$$u^{\alpha}(t) \leqslant u_0 + \int_{t_0}^{t}\int_{t_0}^{s}\{[f(s)g(u(s)) + h(s,\tau)g(u(\tau))]\Delta\tau\}\Delta s. \tag{1.2}$$

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2010, Li [24] investigated Gronwall-Bellman type inequality with delay on time scales

$$u^p(t) \leqslant a(t) + c(t)\int_{t_0}^{t} [f(s)u(\tau(s)) + g(s)]\Delta s. \tag{1.3}$$

Later, Zheng [25] studied some delay integral inequalities in two independent variables on time scales

$$u^p(x,y) \leqslant a(x,y) + b(x,y)\int_{x_0}^{x}\int_{y_0}^{y} [f(s,t)u(\tau_1(s),\tau_2(t))]\Delta s\Delta t, \tag{1.4}$$

$$u^p(x,y) \leqslant a(x,y) + b(x,y)\int_{x_0}^{x}\int_{y_0}^{y} [f(s,t)u^q(\tau_1(s),\tau_2(t)) + g(s,t)u^r(\tau_1(s),\tau_2(t))]\Delta s\Delta t, \tag{1.5}$$

$$u^p(x,y) \leqslant c + \int_{x_0}^{x}\int_{y_0}^{y} [f(s,t)\omega(u(\tau_1(s),\tau_2(t))) + g(s,t)u(\tau_1(s),\tau_2(t))]\Delta s\Delta t, \tag{1.6}$$

where p is a constant and $p \geqslant 1$, $a(x,y), b(x,y)$ and $\omega(u)$ are nondecreasing and $\omega(u)$ is submultiplicative.

In this paper, motivated by the work above, we will establish the following much more general Gronwall-Bellman type delay integral inequality with two independent variables on time scales

$$\begin{aligned} u^p(x,y) \leqslant a(x,y) &+ \int_{x_0}^{x}\int_{y_0}^{y} f_1(x,y,s,t)g_1(u(\sigma_1(s),\tau_1(t)))\Delta s\Delta t \\ &+ \int_{x_0}^{x}\int_{y_0}^{y} f_2(x,y,s,t)g_2(u(\sigma_2(s),\tau_2(t)))\Delta s\Delta t, \end{aligned} \tag{1.7}$$

which has two nonlinear terms $g_1(u)$ and $g_2(u)$ where we do not require that g_1 and g_2 are nondecreasing. Moreover, $f_i(x,y,s,t)(i=1,2)$ has a more general form. We also show that many integral inequalities on time scales such as (1.4)-(1.6) can be reduced to the form of (1.7). Finally, our main result is applied to an estimation of the bounds of delay dynamic solutions of integral equations on time scales.

2. Some preliminaries on time scales

In what follows, $\mathbf{R}$ denots the set of real numbers, $\mathbf{R}^+ = [0,\infty)$. A time scale is an arbitrary nonempty closed subset of the real numbers. In this paper, $\mathbf{T}$ denotes an arbitrary time scale.

DEFINITION 2.1. [2] Let $\mathbf{T}$ be a time scale. For $t \in \mathbf{T}$ we define the forward jump operator $\sigma : \mathbf{T} \to \mathbf{T}$ by $\sigma(t) = \inf\{s \in \mathbf{T}, s > t\}$, while the backward jump operator $\rho : \mathbf{T} \to \mathbf{T}$ is defined by $\rho(t) = \sup\{s \in \mathbf{T}, s < t\}$.

DEFINITION 2.2. [2] The graininess function $\mu : \mathbf{T} \to \mathbf{R}^+$ is defined by $\mu(t) = \sigma(t) - t$.

DEFINITION 2.3. [2] A point $t \in \mathbf{T}$ is said to be left-dense if $\rho(t) = t$ and $t \neq \inf\mathbf{T}$, right-dense if $\sigma(t) = t$ and $t \neq \sup\mathbf{T}$, left-scattered if $\rho(t) < t$, and right-scattered if $\sigma(t) > t$.

DEFINITION 2.4. [2] The set $\mathbf{T}^k$ is defined to be $\mathbf{T}$ if $\mathbf{T}$ does not have a left-scattered maximum, otherwise it is $\mathbf{T}$ without the left-scattered maximum.

DEFINITION 2.5. [2] A function $f:\mathbf{T}\to\mathbf{R}$ is called rd-continuous if it is continuous in right-dense points and if the left-sided limits exist in left-dense points. C_{rd} denotes the set of rd-continuous functions.

DEFINITION 2.6. [2] Assume $f:\mathbf{T}\times\mathbf{T}\to\mathbf{R}$ is a function and let $t\in\mathbf{T}^k$. Then the partial delta derivative of $f(x,y)$ with respect to x is denoted by $(f(x,y))_x^\Delta$ and satisfies

$$|f(\sigma(x),y)-f(s,y)-(f(x,y))_x^\Delta(\sigma(x)-s)|\leqslant\varepsilon|\sigma(x)-s|,\qquad\forall\varepsilon>0,$$

where $\varepsilon\in\mho$, and $\mho$ is a neighborhood of t. The function $f(x,y)$ is called partial delta differentiable to x on $\mathbf{T}^k$.

DEFINITION 2.7. [2] If exists $U:\mathbf{T}\times\mathbf{T}\to\mathbf{R}$, $U_y^\Delta(x,y)=u(x,y)$, $U(x,y)$ is called pre-differentiable, and

$$\int_c^d u(x,v)\Delta v=U(x,d)-U(x,c),\forall c,d\in\mathbf{T}.$$

For more details about the calculus of time scales, see[2]. In the rest of this paper, we always assume that $\mathbf{T}_{x_0}=\{x\in\mathbf{T}|x\geqslant x_0\}$, $\mathbf{T}_{[x_0,X]}=\{x\in\mathbf{T}|x_0\leqslant x\leqslant X\}$, $\mathbf{T}_{y_0}=\{y\in\mathbf{T}|y\geqslant y_0\},\Omega=\mathbf{T}_{x_0}\times\mathbf{T}_{y_0}$ where $x_0,y_0\in\mathbf{T}$, and furthermore assume $\mathbf{T}_{x_0}\subset\mathbf{T}^k,\mathbf{T}_{y_0}\subset\mathbf{T}^k$.

3. Main results

For convenience, we give some remarks. Let

$$\omega_1(s)=\max_{0\leqslant\tau\leqslant s}\{g_1(\tau)\},\tag{3.1}$$

$$\omega_2(s)=\max_{0\leqslant\tau\leqslant s}\{g_2(\tau)/\omega_1(\tau)\}\omega_1(s),\tag{3.2}$$

where $s\in\mathbf{R}^+$. According to (3.1) and (3.2), we define

$$W_1(u)=\int_{u_1}^u\frac{1}{\omega_1(r^{1/p})}\Delta r,\tag{3.3}$$

$$W_2(u)=\int_{u_2}^u\frac{\omega_1((W_1^{-1}(r))^{1/p})}{\omega_2((W_1^{-1}(r))^{1/p})}\Delta r,\tag{3.4}$$

where $u_1,u_2\geqslant 0$ and $u>0$.

Consider (1.7) and assume that

(H_1) $a\in C_{rd}(\Omega,\mathbf{R}^+)$ and $a(x_0,y_0)\neq 0$;

(H_2) $u\in C_{rd}(\Omega,\mathbf{R}^+),f_1,f_2\in C_{rd}(\Omega\times\Omega,R^+)$;

(H_3) $g_1(u)$ and $g_2(u)$ are continuous and nonnegative functions on R^+;

(H_4) $\sigma_i:\mathbf{T}_{x_0}\to\mathbf{T}$ with $\sigma_i(x)\leqslant x, i=1,2$ and $-\infty<\alpha=\inf\{\min\{\sigma_i(x),i=1,2\},x\in\mathbf{T}_{x_0}\}\leqslant x_0$. $\tau_i:\mathbf{T}_{y_0}\to\mathbf{T}$ with $\tau_i(y)\leqslant y, i=1,2$ and $-\infty<\beta=\inf\{\min\{\tau_i(y), i=1,2\},y\in\mathbf{T}_{y_0}\}\leqslant y_0$, $\phi\in C_{rd}([\alpha,x_0]\times[\beta,y_0]\cap\mathbf{T}^2,\mathbf{R}^+)$.

THEOREM 3.1. *Suppose that* (H_1)-(H_4) *hold and for* $(x,y)\in\Omega$, $u(x,y)$ *satisfies* *(1.7) and with the initial condition*

$$\begin{aligned}&u(x,y)=\phi(x,y) \quad if \quad x\in[\alpha,x_0]\cap\mathbf{T} \quad or \quad y\in[\beta,y_0]\cap\mathbf{T},\\&\phi(\sigma_i(x),\tau_i(y))\leqslant a^{1/p}(x,y),\forall(x,y)\in\Omega \quad if \quad \sigma_i(x)\leqslant x_0 \quad or \quad \tau_i(y)\leqslant y_0.\end{aligned} \tag{3.5}$$

Then

$$\begin{aligned}u(x,y)\leqslant\Big\{W_1^{-1}\Big[W_2^{-1}\big(W_2(W_1(r(x,y))+\int_{x_0}^{x}\int_{y_0}^{y}\widetilde{f}_1(x,y,s,t)\Delta s\Delta t\big)\\+\int_{x_0}^{x}\int_{y_0}^{y}\widetilde{f}_2(x,y,s,t)\Delta s\Delta t)\Big]\Big\}^{1/p},\end{aligned} \tag{3.6}$$

where

$$r(x,y)=\max_{x_0\leqslant\xi\leqslant x}\max_{y_0\leqslant\eta\leqslant y}a(\xi,\eta),\quad \widetilde{f}_i(x,y,s,t)=\max_{x_0\leqslant\xi\leqslant x}\max_{y_0\leqslant\eta\leqslant y}f_i(\xi,\eta,s,t), \tag{3.7}$$

provided that

$$\begin{aligned}&W_2\Big[W_1((r(x,y))+\int_{x_0}^{x}\int_{y_0}^{y}\widetilde{f}_1(x,y,s,t)\Delta s\Delta t)+\int_{x_0}^{x}\int_{y_0}^{y}\widetilde{f}_2(x,y,s,t)\Delta s\Delta t\Big]\\&\qquad\leqslant\int_{u_2}^{\infty}\frac{\omega_1((W_1^{-1}(r))^{1/p})}{\omega_2((W_1^{-1}(r))^{1/p})}\Delta r,\\&W_2^{-1}\Big[W_2(W_1(r(x,y))+\int_{x_0}^{x}\int_{y_0}^{y}\widetilde{f}_1(x,y,s,t)\Delta s\Delta t)\\&\qquad+\int_{x_0}^{x}\int_{y_0}^{y}\widetilde{f}_2(x,y,s,t)\Delta s\Delta t\Big]\leqslant\int_{u_1}^{\infty}\frac{1}{\omega_1(r^{1/p})}\Delta r.\end{aligned} \tag{3.8}$$

Proof. Obviously, for any $(x,y)\in\Omega$, $r(x,y)$ is positive and nondecreasing with respect to x and y, $\widetilde{f}_i(x,y,s,t)$ $(i=1,2)$ is nonnegative and nondecreasing with respect to x and y for each fixed s and t. They satisfy $a(x,y)\leqslant r(x,y)$ and $f_i(x,y,s,t)\leqslant\widetilde{f}_i(x,y,s,t)$ for $i=1,2$. By (3.1) and (3.2), we have $g_1(u)\leqslant\omega_1(u), g_2(u)\leqslant\omega_2(u)$.

Take any fixed $X\in\mathbf{T}_{x_0}$ and for arbitrary $x\in\mathbf{T}_{[x_0,X]}, y\in\mathbf{T}_{y_0}$, we get

$$\begin{aligned}u^p(x,y)&\leqslant r(x,y)+\int_{x_0}^{x}\int_{y_0}^{y}\widetilde{f}_1(x,y,s,t)\omega_1(u(\sigma_1(s),\tau_1(t)))\Delta s\Delta t\\&\quad+\int_{x_0}^{x}\int_{y_0}^{y}\widetilde{f}_2(x,y,s,t)\omega_2(u(\sigma_2(s),\tau_2(t)))\Delta s\Delta t\\&\leqslant r(X,y)+\int_{x_0}^{x}\int_{y_0}^{y}\widetilde{f}_1(X,y,s,t)\omega_1(u(\sigma_1(s),\tau_1(t)))\Delta s\Delta t\\&\quad+\int_{x_0}^{x}\int_{y_0}^{y}\widetilde{f}_2(X,y,s,t)\omega_2(u(\sigma_2(s),\tau_2(t)))\Delta s\Delta t.\end{aligned} \tag{3.9}$$

Let

$$z(x,y) = r(X,y) + \int_{x_0}^{x}\int_{y_0}^{y} \widetilde{f}_1(X,y,s,t)\omega_1(u(\sigma_1(s),\tau_1(t)))\Delta s\Delta t + \int_{x_0}^{x}\int_{y_0}^{y} \widetilde{f}_2(X,y,s,t)\omega_2(u(\sigma_2(s),\tau_2(t)))\Delta s\Delta t. \tag{3.10}$$

Hence, $u(x,y) \leqslant z^{1/p}(x,y)$ and $z(x_0,y) = r(X,y)$. Clearly, $z(x,y)$ is a nonnegative and nondecreasing function for $x \in \mathbf{T}_{[x_0,X]}$ and $y \in \mathbf{T}_{y_0}$.

If $\sigma_i(x) \geqslant x_0$ and $\tau_i(y) \geqslant y_0$, then $\sigma_i(x) \in \mathbf{T}_{[x_0,X]}$ and $\tau_i(y) \in \mathbf{T}_{y_0}$, and

$$u(\sigma_i(x),\tau_i(y)) \leqslant z^{1/p}(\sigma_i(x),\tau_i(y)) \leqslant z^{1/p}(x,y), \qquad i = 1,2. \tag{3.11}$$

If $\sigma_i(x) \leqslant x_0$ or $\tau_i(y) \leqslant y_0$, then from (3.5), we have

$$u(\sigma_i(x),\tau_i(y)) = \phi(\sigma_i(x),\tau_i(y)) \leqslant a^{1/p}(x,y) \leqslant z^{1/p}(x,y), \qquad i = 1,2. \tag{3.12}$$

Thus from (3.11) and (3.12), we have

$$z(x,y) \leqslant r(X,y) + \int_{x_0}^{x}\int_{y_0}^{y} \widetilde{f}_1(X,y,s,t)\omega_1(z^{1/p}(s,t))\Delta s\Delta t + \int_{x_0}^{x}\int_{y_0}^{y} \widetilde{f}_2(X,y,s,t)\omega_2(z^{1/p}(s,t))\Delta s\Delta t. \tag{3.13}$$

Furthermore,

$$z_x^{\triangle}(x,y) = \int_{y_0}^{y} \widetilde{f}_1(X,y,x,t)\omega_1(z^{1/p}(x,t))\Delta t + \int_{y_0}^{y} \widetilde{f}_2(X,y,x,t)\omega_2(z^{1/p}(x,t))\Delta t. \tag{3.14}$$

That is,

$$\frac{z_x^{\triangle}(x,y)}{\omega_1(z^{1/p}(x,y))} = \frac{\int_{y_0}^{y} \widetilde{f}_1(X,y,x,t)\omega_1(z^{1/p}(x,t))\Delta t + \int_{y_0}^{y} \widetilde{f}_2(X,y,x,t)\omega_2(z^{1/p}(x,t))\Delta t}{\omega_1(z^{1/p}(x,y))} \leqslant \int_{y_0}^{y} \widetilde{f}_1(X,y,x,t)\Delta t + \int_{y_0}^{y} \widetilde{f}_2(X,y,x,t)\frac{\omega_2(z^{1/p}(x,t))}{\omega_1(z^{1/p}(x,t))}\Delta t. \tag{3.15}$$

First, we prove that

$$[W_1(z(x,y))]_x^{\triangle} \leqslant \frac{z_x^{\triangle}(x,y)}{\omega_1(z^{1/p}(x,y))}, \qquad \forall x \in \mathbf{T}_{[x_0,X]}. \tag{3.16}$$

If $\sigma(x) > x$, then

$$[W_1(z(x,y))]_x^{\triangle} = \frac{W_1(z(\sigma(x),y)) - W_1(z(x,y))}{\sigma(x) - x} = \frac{1}{\sigma(x)-x}\int_{z(x,y)}^{z(\sigma(x),y)} \frac{1}{\omega_1(r^{1/p})}\Delta r \leqslant \frac{z(\sigma(x),y) - z(x,y)}{\sigma(x)-x}\frac{1}{\omega_1(z^{1/p}(x,y))} = \frac{z_x^{\triangle}(x,y)}{\omega_1(z^{1/p}(x,y))}.$$

If $\sigma(x)=x$, then

$$[W_1(z(x,y))]_x^\triangle = \lim_{s\to x}\frac{W_1(z(x,y))-W_1(z(s,y))}{x-s} = \lim_{s\to x}\frac{1}{x-s}\int_{z(s,y)}^{z(x,y)}\frac{1}{\omega_1(r^{1/p})}\Delta r$$
$$= \lim_{s\to x}\frac{z(x,y)-z(s,y)}{x-s}\frac{1}{\omega_1(z^{1/p}(\xi,y))} = \frac{z_x^\triangle(x,y)}{\omega_1(z^{1/p}(x,y))},$$

where ξ satisfies $z^{1/p}(s,y)\leqslant z^{1/p}(\xi,y)\leqslant z^{1/p}(x,y)$. Integrating both sides of (3.15) with respect to x from x_0 to x, we obtain

$$\begin{aligned} W_1(z(x,y)) &\leqslant W_1(z(x_0,y)) + \int_{x_0}^{x}\int_{y_0}^{y}\widetilde{f}_1(X,y,s,t)\Delta s\Delta t \\ &\quad + \int_{x_0}^{x}\int_{y_0}^{y}\widetilde{f}_2(X,y,s,t)\frac{\omega_2(z^{1/p}(s,t))}{\omega_1(z^{1/p}(s,t))}\Delta s\Delta t \\ &\leqslant W_1(r(X,y)) + \int_{x_0}^{X}\int_{y_0}^{y}\widetilde{f}_1(X,y,s,t)\Delta s\Delta t \\ &\quad + \int_{x_0}^{x}\int_{y_0}^{y}\widetilde{f}_2(X,y,s,t)\frac{\omega_2(z^{1/p}(s,t))}{\omega_1(z^{1/p}(s,t))}\Delta s\Delta t. \end{aligned} \tag{3.17}$$

We let the right of (3.17) be $z_1(x,y)$. Then $z_1(x,y)$ is nonnegative and nondecreasing with respect to x and y, where $x\in\mathbf{T}_{[x_0,X]}, y\in\mathbf{T}_{y_0}$, and

$$z_1(x_0,y) = W_1(r(X,y)) + \int_{x_0}^{X}\int_{y_0}^{y}\widetilde{f}_1(X,y,s,t)\Delta s\Delta t. \tag{3.18}$$

Then (3.17) becomes to

$$z(x,y)\leqslant W_1^{-1}(z_1(x,y)) \tag{3.19}$$

and

$$\begin{aligned} z_{1x}^\triangle(x,y) &= \int_{y_0}^{y}\widetilde{f}_2(X,y,x,t)\frac{\omega_2(z^{1/p}(x,t))}{\omega_1(z^{1/p}(x,t))}\Delta t \\ &\leqslant \int_{y_0}^{y}\widetilde{f}_2(X,y,x,t)\frac{\omega_2((W_1^{-1}(z_1(x,t)))^{1/p})}{\omega_1((W_1^{-1}(z_1(x,t)))^{1/p})}\Delta t. \end{aligned} \tag{3.20}$$

Thus

$$\frac{\omega_1((W_1^{-1}(z_1(x,y)))^{1/p})z_{1x}^\triangle(x,y)}{\omega_2((W_1^{-1}(z_1(x,y)))^{1/p})} \leqslant \int_{y_0}^{y}\widetilde{f}_2(X,y,x,t)\Delta t. \tag{3.21}$$

Similiarly, we can prove that

$$[W_2(z_1(x,y))]_x^\triangle \leqslant \frac{\omega_1(W_1^{-1}(z_1(x,y))^{1/p})z_{1x}^\triangle(x,y)}{\omega_2(W_1^{-1}(z_1(x,y))^{1/p})}.$$

Integrating both sides of (3.21) with respect to x from x_0 to x, we obtain

$$W_2(z_1(x,y)) \leqslant W_2(z_1(x_0,y)) + \int_{x_0}^{x}\int_{y_0}^{y} \widetilde{f}_2(X,y,s,t)\Delta s\Delta t, \tag{3.22}$$

and

$$z_1(x,y) \leqslant W_2^{-1}\{W_2(z_1(x_0,y)) + \int_{x_0}^{x}\int_{y_0}^{y} \widetilde{f}_2(X,y,s,t)\Delta s\Delta t\}. \tag{3.23}$$

Hence,

$$\begin{aligned} u(x,y) &\leqslant z^{1/p}(x,y) \leqslant (W_1^{-1}(z_1(x,y)))^{1/p} \\ &\leqslant \{W_1^{-1}[W_2^{-1}(W_2(z_1(x_0,y)) + \int_{x_0}^{x}\int_{y_0}^{y} \widetilde{f}_2(X,y,s,t)\Delta s\Delta t)]\}^{1/p} \\ &= \{W_1^{-1}[W_2^{-1}(W_2(W_1(r(X,y)) + \int_{x_0}^{X}\int_{y_0}^{y} \widetilde{f}_1(X,y,s,t)\Delta s\Delta t) \\ &\quad + \int_{x_0}^{x}\int_{y_0}^{y} \widetilde{f}_2(X,y,s,t)\Delta s\Delta t)]\}^{1/p} \end{aligned} \tag{3.24}$$

for $(x,y) \in \mathbf{T}_{[x_0,X]} \times \mathbf{T}_{y_0}$. Let $x = X$, which gives

$$\begin{aligned} u(X,y) \leqslant \{W_1^{-1}[W_2^{-1}(W_2(W_1(r(X,y)) + \int_{x_0}^{X}\int_{y_0}^{y} \widetilde{f}_1(X,y,s,t)\Delta s\Delta t) \\ + \int_{x_0}^{X}\int_{y_0}^{y} \widetilde{f}_2(X,y,s,t)\Delta s\Delta t)]\}^{1/p}. \end{aligned} \tag{3.25}$$

Replacing X by x yields

$$\begin{aligned} u(x,y) \leqslant \{W_1^{-1}[W_2^{-1}(W_2(W_1(r(x,y)) + \int_{x_0}^{x}\int_{y_0}^{y} \widetilde{f}_1(x,y,s,t)\Delta s\Delta t) \\ \int_{x_0}^{x}\int_{y_0}^{y} \widetilde{f}_2(x,y,s,t)\Delta s\Delta t)]\}^{1/p}. \end{aligned} \tag{3.26}$$

This completes the proof of Theorem 3.1. □

REMARK 3.1. The result of Theorem 3.1 holds for an arbitrary time scale. If $\mathbf{T} = \mathbf{R}$, then the inequality established in Theorem 3.1 reduces to the the inequality established by [6] (the case of $n = 2$). If $\mathbf{T} = \mathbf{Z}$ and $f_1(x,y,s,t) = b(x,y)f(s,t)$, $f_2(x,y,s,t) = 0$, then from Theorem 3.1, we obtain Theorem 2.1 in [11].

REMARK 3.2. (1) If we take $f_1(x,y,s,t) = b(x,y)f(s,t)$ and $f_2(x,y,s,t) = 0$ then (1.7) reduces to (1.4). In our results, $a(x,y)$ and $b(x,y)$ need not be nondecreasing.

(2) If we take $f_1(x,y,s,t) = b(x,y)f(s,t)$, $f_2(x,y,s,t) = b(x,y)g(s,t)$, $g_1(u(\sigma_1(s),\tau_1(t))) = u^q(\sigma_1(s),\tau_1(t))$, $g_2(u(\sigma_2(s),\tau_2(t)))) = u^r(\sigma_1(s),\tau_1(t))$, then (1.7) reduces to (1.5). In our results, $a(x,y)$ and $b(x,y)$ need not be nondecreasing.

(3) If we take $f_1(x,y,s,t) = f(s,t)$, $f_2(x,y,s,t) = g(s,t)$, $g_2(u(\sigma_1(s),\tau_1(t)))) = u(\sigma_1(s),\tau_1(t))$ then (1.7)reduces to (1.6). In our results, $a(x,y)$, $f(x,y)$, $g(x,y)$ and $\omega(u)$ need not be nondecreasing, and $w(u)$ need not be submultiplicative.

Consider the inequality

$$\varphi(u(x,y)) \leqslant a(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} f_1(x,y,s,t)g_1(u(\sigma_1(s),\tau_1(t)))\Delta s\Delta t + \int_{x_0}^{x}\int_{y_0}^{y} f_2(x,y,s,t)g_2(u(\sigma_2(s),\tau_2(t)))\Delta s\Delta t, \tag{3.27}$$

which looks much more complicated than (1.7).

COROLLARY 3.1. *In addition to the assumptions* (H_1)-(H_4), *suppose that* $\varphi(u)$ *is positive on* $(0,\infty)$ *and* $u(x,y)$ *satisfies (3.27) for* $(x,y) \in \Omega$ *and with the initial condition*

$$\begin{aligned} &u(x,y) = \phi(x,y) \quad if \quad x \in [\alpha, x_0] \cap \mathbf{T} \quad or \quad y \in [\beta, y_0] \cap \mathbf{T}, \\ &\phi(\sigma_i(x),\tau_i(y)) \leqslant a(x,y), \forall (x,y) \in \Omega \quad if \quad \sigma_i(x) \leqslant x_0 \quad or \quad \tau_i(y) \leqslant y_0. \end{aligned} \tag{3.28}$$

Then

$$u(x,y) \leqslant \varphi^{-1}\{W_1^{-1}[W_2^{-1}(W_2(W_1(r(x,y)) + \int_{x_0}^{x}\int_{y_0}^{y} \widetilde{f}_1(x,y,s,t)\Delta s\Delta t) + \int_{x_0}^{x}\int_{y_0}^{y} \widetilde{f}_2(x,y,s,t)\Delta s\Delta t)]\}, \tag{3.29}$$

where $r(x,y), \tilde{f}_i(x,y,s,t)$ *are given in Theorem 3.1,* $W_1(u), W_2(u)$ *are given in (3.3) and (3.4) when the case of* $p = 1$.

The proof is similar to one of Theorem 3.1.

4. Applications

Consider the following delay dynamic integral equation on time scale

$$u^p(x,y) = C + \int_{x_0}^{x}\int_{y_0}^{y} F(x,y,s,t,u(\sigma_1(s),\tau_1(t)),u(\sigma_2(s),\tau_2(t)))\Delta s\Delta t, \tag{4.1}$$

with the initial condition

$$\begin{aligned} &u(x,y) = \phi(x,y) \quad if \quad x \in [\alpha, x_0] \cap \mathbf{T} \quad or \quad y \in [\beta, y_0] \cap \mathbf{T}, \\ &\phi(\sigma_i(x),\tau_i(y)) \leqslant |C|^{1/p}, \forall (x,y) \in \mathbf{T}_{x_0} \times \mathbf{T}_{y_0} \quad if \quad \sigma_i(x) \leqslant x_0, \quad or \quad \tau_i(y) \leqslant y_0, \end{aligned} \tag{4.2}$$

where $(x,y) \in \Omega, u \in C_{rd}(\mathbf{T}_{x_0} \times \mathbf{T}_{y_0}, \mathbf{R})$ C is a nonzero constant, $\phi, \alpha, \beta, \sigma_i, \tau_i$ are the same as in Theorem 3.1 and $F \in C_{rd}(\mathbf{T}_{x_0} \times \mathbf{T}_{y_0} \times \mathbf{T}_{x_0} \times \mathbf{T}_{y_0} \times \mathbf{R} \times \mathbf{R}, \mathbf{R})$. We can obtain the following Corollary.

COROLLARY 4.1. *Assume that*

$$|F(x,y,s,t,u(\sigma_1(s),\tau_1(t)),u(\sigma_2(s),\tau_2(t)))| \leqslant f_1(x,y,s,t)g_1(u(\sigma_1(s),\tau_1(t)))+f_2(x,y,s,t)g_2(u(\sigma_2(s),\tau_2(t))) \tag{4.3}$$

where the definition of f_1,f_2,g_1,g_2 *are given in* (H_2) *and* (H_3). *Then the solution of (4.1) has an estimate for* $x\in\mathbf{T}_{x_0}, y\in\mathbf{T}_{y_0}$

$$|u(x,y)| \leqslant \Big\{W_1^{-1}\big[W_2^{-1}(W_2(W_1(|C|)+\int_{x_0}^{x}\int_{y_0}^{y}\widetilde{f}_1(x,y,s,t)\Delta s\Delta t) + \int_{x_0}^{x}\int_{y_0}^{y}\widetilde{f}_2(x,y,s,t)\Delta s\Delta t)\big]\Big\}^{1/p}. \tag{4.4}$$

where $\tilde{f}_i(x,y,s,t),\omega_i(u),W_i(u)$ *are defined in Theorem 3.1.*

Proof. From (4.1) we have

$$\begin{aligned}|u^p(x,y)| &\leqslant |C|+\int_{x_0}^{x}\int_{y_0}^{y}|F(x,y,s,t,u(\sigma_1(s),\tau_1(t)),u(\sigma_2(s),\tau_2(t)))|\Delta s\Delta t,\\ &\leqslant |C|+\int_{x_0}^{x}\int_{y_0}^{y}f_1(x,y,s,t)g_1(u(\sigma_1(s),\tau_1(t)))\Delta s\Delta t\\ &\quad+\int_{x_0}^{x}\int_{y_0}^{y}f_2(x,y,s,t)g_2(u(\sigma_2(s),\tau_2(t)))\Delta s\Delta t\end{aligned} \tag{4.5}$$

Then according to Theorem 3.1, we can obtain (4.4). □

Competing interests. The authors declare that there is no conflict of interests regarding the publication of this article.

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