ON TWO PROBLEMS FOR GAUSS COMPOUND MEAN

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Dedicated to my father

(*Communicated by T. Burić*)

Abstract. In this paper, we give answers to two open problems concerning Gauss Compound mean posed in the paper: [ZH. H. YANG, *A New Proof of Inequalities for Gauss Compound Mean*, Int. Journal of Math. Analysis, Vol. 4, (2010), no. 21, 1013–1018.] One of the problems is equivalent to the Vamanamurthy problem.

1. Introduction

Within the past years, means have been the subject of intensive research. Many interesting results and inequalities for means can be found in the literature [1]–[9]. The well known Gauss compound mean of positive numbers a, b with $a \neq b$ is the limit

$$AGM(a,b) = \lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n$$

where

$$a_1 = a, \quad b_1 = b, \quad a_{n+1} = \frac{a_n + b_n}{2} = A(a_n, b_n), \quad b_{n+1} = \sqrt{a_n b_n} = G(a_n, b_n). \quad (1)$$

The power mean of positive a and b is defined by

$$M_t(a,b) = M^{1/t}(a^t, b^t) \quad \text{if} \quad t \neq 0 \quad \text{and} \quad M_0(a,b) = \sqrt{ab}. \quad (2)$$

The t-order logarithmic mean of $a, b > 0$ is defined by

$$L_t(a,b) = \begin{cases} \left(\frac{b^t - a^t}{t(\ln b - \ln a)} \right)^{\frac{1}{t}}, & t \neq 0 \\ \sqrt{ab}, & t = 0, \end{cases} \quad (3)$$

where $L_1(a,b) = L(a,b) = \frac{b-a}{\ln b - \ln a}$ is the logarithmic mean.

We note, that AGM is connected with elliptic integrals by Gauss's fantastic result:

$$AGM(1,r') = \frac{\pi}{2k(r)}$$

where $k(r) = \int\limits_{0}^{\pi/2} (1 - r^2 \sin^2 x)^{-1/2} dx$, $r' = \sqrt{1 - r^2}$, $0 \leqslant r \leqslant 1$ (see [2]).

In the paper [8], the author presented new sophisticated proofs for famous inequalities

$$L(a,b) < AGM(a,b) < L^{2/3}(a^{3/2}, b^{3/2})$$

where the first inequality was proved by B. C. Carlson and M. Vuorinen [4], while the second one is due to J. and P. Borwein [2].

Denote by

$$D_n(t) = b_{n+1}^{-t} \left(\frac{b_n^t - a_n^t}{\ln b_n - \ln a_n} - \frac{b_{n+1}^t - a_{n+1}^t}{\ln b_{n+1} - \ln a_{n+1}} \right), \tag{4}$$

where a_n, b_n is defined by (1). It was showed in [8] that $t \mapsto D_n(t)$ is continuous on $\mathbb{R}$ and increasing on $(1, \infty)$ with $D_n(1) < 0$, $D_n(3/2) > 0$. So $D_n(t)$ has an unique zero $t_0^{(n)}(a,b)$ on $(1, 3/2)$.

Denote by $t_{0i}(a,b) = \inf\limits_{n \in N} \{t_0^{(n)}(a,b)\}$ and $t_{0s}(a,b) = \sup\limits_{n \in N} \{t_0^{(n)}(a,b)\}$. It is evident that $1 \leqslant t_{0i}(a,b) < 3/2$, $1 < t_{0s}(a,b) \leqslant 3/2$.

At the end in the same paper [8], the author posed two open problems and remarks as follows:

PROBLEM 1. Is $t_{0i}(a,b)$ equal to 1?

REMARK 1. If the answer is "no", then there must exist $t_1 \in (1, t_{0i}(a,b))$ such that $D_n(t_1) < 0$ for all n, whence

$$L(a_n^{t_1}, b_n^{t_1}) < L(a_{n+1}^{t_1}, b_{n+1}^{t_1}) \text{ for all } n,$$

which implies

$$L_{t_1}(a,b) < AGM(a,b), \tag{5}$$

where $L_t(a,b)$ is defined by (3).

Furthermore, are there some concrete constants $t_1 \in (1, t_{0i}(a,b))$ such that (5) holds?

PROBLEM 2. Is $t_{0s}(a,b)$ equal to $3/2$?

REMARK 2. If the answer is "no", then there must exist $t_2 \in (t_{0s}(a,b), 3/2)$ such that $D_n(t_2) > 0$ for all n, whence

$$L(a_n^{t_2}, b_n^{t_2}) > L(a_{n+1}^{t_2}, b_{n+1}^{t_2}) \text{ for all } n,$$

which implies

$$AGM(a,b) < L_{t_2}(a,b), \tag{6}$$

where $L_t(a,b)$ is defined by (3).

Furthermore, are there some concrete constants $t_2 \in (t_{0s}(a,b), 3/2)$ such that (6) holds?

REMARK 3. If replacing a^t, b^t by a,b in $L_t(a,b)$ and a, b by $a^{1/t}$, $b^{1/t}$ in $AGM(a,b)$, then the inequality (5) is equivalent to

$$L(a,b) < AGM_{1/t_1}(a,b) \text{ for } 1/t_1 \in (1/t_{0i}(a,b), 1). \tag{7}$$

This shows that the open problem presented by Vamanamurthy is equivalent to the problem 1.

In this note, we will prove that

- for $\forall a, b > 0$ with $a \neq b$ we have $t_{0i}(a,b) > 1$, and $\inf\limits_{a,b>0, a\neq b} \{t_{0i}(a,b)\} = 1$;

- for $\forall a, b > 0$ with $a \neq b$ we have $t_{0s}(a,b) = 3/2$.

These solve two open problems as above.

2. Two lemmas

In order to solve two open problems posed in [8], we need two important lemmas. Without loss of generality, we assume that $b < a$. Denote by $s_n = s_n(a,b) = b_n/a_n$, then $0 < s_n = s_n(a,b) < 1$. We have

$$\frac{a_{n+1}}{a_n} = \frac{1 + b_n/a_n}{2} = \frac{1 + s_n}{2},$$

$$\frac{b_{n+1}}{a_{n+1}} = \frac{\sqrt{a_n b_n}}{(a_n + b_n)/2} = 2\frac{\sqrt{b_n/a_n}}{1 + b_n/a_n} = 2\frac{\sqrt{s_n}}{1 + s_n},$$

$$\frac{a_n}{b_{n+1}} = \frac{a_n}{\sqrt{a_n b_n}} = \frac{1}{\sqrt{b_n/a_n}} = \frac{1}{\sqrt{s_n}}.$$

Then $D_n(t)$ and $D'_n(t)$ can be expressed as

$$
\begin{aligned}
D_n(t) &= b_{n+1}^{-t} \left(\frac{b_n^t - a_n^t}{\ln b_n - \ln a_n} - \frac{b_{n+1}^t - a_{n+1}^t}{\ln b_{n+1} - \ln a_{n+1}} \right) \\
&= \left(\frac{a_n}{b_{n+1}} \right)^t \left(\frac{(b_n/a_n)^t - 1}{\ln(b_n/a_n)} \right) - \frac{1 - (a_{n+1}/b_{n+1})^t}{\ln(b_{n+1}/a_{n+1})} \\
&= \frac{s_n^{t/2} - s_n^{-t/2}}{\ln s_n} - \frac{1 - \left(\frac{1+s_n}{2\sqrt{s_n}} \right)^t}{\ln \frac{2\sqrt{s_n}}{1+s_n}},
\end{aligned}
\tag{8}
$$

$$D'_n(t) = \frac{s_n^{-t/2} + s_n^{t/2}}{2} - \left(\frac{s_n^{-1/2} + s_n^{1/2}}{2}\right)^t.$$

We now show that $t \mapsto D_n(t)$, $n \in \mathbb{N}$ is a convex function on $[1,\infty)$.

LEMMA 1. *Let* $D_n(t)$ $n \in N$ *be defined in (2). Then* $D''_n(t) > 0$ *for* $n \in \mathbb{N}$, $t \in [1,\infty)$.

Proof. Differentiation gives

$$s_n^{t/2} D''_n(t) = \frac{s_n^t - 1}{4} \ln s_n - \left(\frac{1 + s_n}{2}\right)^t \ln\left(\frac{1 + s_n}{2\sqrt{s_n}}\right) := g(t),$$

$$g'(t) = \left(\frac{1 + s_n}{2}\right)^t \ln\left(\frac{2\sqrt{s_n}}{1 + s_n}\right) \ln\left(\frac{1 + s_n}{2}\right) + \frac{s_n^t \ln^2 s_n}{4} \geqslant 0,$$

where the inequality holds due to $0 < s_n < 1$.

Clearly, the proof will be done if we prove that

$$g(1) = h(s_n) = \left(\frac{1 + s_n}{2}\right) \ln\left(\frac{2\sqrt{s_n}}{1 + s_n}\right) + \frac{s_n - 1}{4} \ln s_n \geqslant 0.$$

Differentiation again yields that for $0 < s_n < 1$,

$$h'(s_n) = \frac{1}{2} \ln\left(\frac{2s_n}{1 + s_n}\right) \leqslant 0,$$

which indicates that $h(s_n) > h(1) = 0$ and the proof of our lemma is complete. $\square$

LEMMA 2. *For* $n \in \mathbb{N}$ *and* $a, b > 0$ *with* $a \neq b$, *let*

$$r_n = 1 - \frac{1}{2} \frac{D_n(1)}{D_n(3/2) - D_n(1)},$$

where $D_n(t)$ *is given by (2). Then we have*

$$\lim_{n \to \infty} r_n = \frac{3}{2}.$$

Proof. To complete the proof, we write r_n as

$$r_n = 1 + \frac{1}{2} \frac{1}{D_n(3/2)/(-D_n(1)) + 1}.$$

By (8) we have

$$
\begin{aligned}
\frac{D_n(3/2)}{-D_n(1)} &= \left[\frac{s_n^{-3/4} - s_n^{3/4}}{\ln s_n} - \frac{\left(\frac{1+s_n}{2\sqrt{s_n}} \right)^{3/2} - 1}{\ln \left(\frac{2\sqrt{s_n}}{1+s_n} \right)} \right] \Bigg/ \left[\frac{\frac{1+s_n}{2\sqrt{s_n}} - 1}{\ln \left(\frac{2\sqrt{s_n}}{1+s_n} \right)} - \frac{\frac{1}{\sqrt{s_n}} - \sqrt{s_n}}{\ln s_n} \right] \\
&= \frac{1}{s_n^{1/4}} \frac{\left(1 - s_n^{3/2} \right) \frac{\ln \left(\frac{2\sqrt{s_n}}{1+s_n} \right)}{\ln s_n} - \left(\left(\frac{1+s_n}{2} \right)^{3/2} - s_n^{3/4} \right)}{\frac{1+s_n}{2} - \sqrt{s_n} - \frac{\ln \left(\frac{2\sqrt{s_n}}{1+s_n} \right)}{\ln s_n} (1 - s_n)} \\
&= \frac{1}{s_n^{1/4}} \frac{\left(1 - s_n^{3/2} \right) \ln \left(\frac{2\sqrt{s_n}}{1+s_n} \right) - \left(\left(\frac{1+s_n}{2} \right)^{3/2} - s_n^{3/4} \right) \ln s_n}{\left(\frac{1+s_n}{2} - \sqrt{s_n} \right) \ln s_n - \ln \left(\frac{2\sqrt{s_n}}{1+s_n} \right) (1 - s_n)} \\
&:= \frac{1}{s_n^{1/4}} \frac{f_1(s_n)}{f_2(s_n)}.
\end{aligned}
$$

From

$$
s_{n+1} = \frac{b_{n+1}}{a_{n+1}} = \frac{2\sqrt{s_n}}{1+s_n},
$$

we see that $1 > s_{n+1}(a,b) > s_n(a,b) > 0$, which is due to $\sqrt{x}(1+x) \leqslant 2$ for $0 < x < 1$. This implies that sequence $\{s_n(a,b)\}_{n=1}^{\infty}$ is convergent, and it is easily deduced that $\lim_{n \to \infty} s_n(a,b) = 1$.

To prove $\lim_{n \to \infty} r_n = \lim_{s_n \to 1} r_n = \frac{3}{2}$, it suffices to show that $\lim_{s_n \to 1} [D_n(3/2)/(-D_n(1))] = 0$. To this end, we use power series expansion to get

$$
\begin{aligned}
f_1(s_n) &= \left(1 - s_n^{3/2} \right) \ln \left(\frac{2\sqrt{s_n}}{1+s_n} \right) - \left(\left(\frac{1+s_n}{2} \right)^{3/2} - s_n^{3/4} \right) \ln s_n \\
&= \frac{21}{5 \times 2^{15}} (s_n - 1)^7 + O\left((s_n - 1)^8 \right), \\
f_2(s_n) &= \left(\frac{1+s_n}{2} - \sqrt{s_n} \right) \ln s_n - \ln \left(\frac{2\sqrt{s_n}}{1+s_n} \right) (1 - s_n) \\
&= \frac{1}{384} (s_n - 1)^5 + O\left((s_n - 1)^6 \right).
\end{aligned}
$$

Then we obtain

$$
\begin{aligned}
\lim_{s_n \to 1} \frac{D_n(3/2)}{-D_n(1)} &= \lim_{s_n \to 1} \frac{1}{s_n^{1/4}} \frac{f_1(s_n)}{f_2(s_n)} \\
&= \lim_{s_n \to 1} \frac{\frac{21}{5 \times 2^{15}} (s_n - 1)^7 + O\left((s_n - 1)^8 \right)}{\frac{1}{384} (s_n - 1)^5 + O\left((s_n - 1)^6 \right)} = 0.
\end{aligned}
$$

This completes the proof. □

3. **Main result**

Now we are in a position to state and prove our main result.

THEOREM 1. *For $n \in \mathbb{N}$, let $t_0^{(n)} = t_0^{(n)}(a,b)$ denote the unique zero of $D_n(t)$ defined by* (2) *on* $(1,3/2)$. *Then we have*

$$t_{0s} = \sup_{n \in \mathbb{N}}\{t_0^{(n)}(a,b)\} = \frac{3}{2}, \quad t_{0i} = \inf_{n \in \mathbb{N}}\{t_0^{(n)}(a,b)\} > 1$$
$$and \qquad \inf_{a>0,b>0,a\neq b}\{t_{0i}(a,b)\} = 1.$$

Proof. Lemma 1 tells us that $t \mapsto D_n(t)$ is convex on $[1,\infty)$, which implies that for $t_0^{(n)} \in [1,3/2]$ we have

$$\frac{D_n(3/2) - D_n(1)}{3/2 - 1} > \frac{D_n\left(t_0^{(n)}\right) - D_n(1)}{t_0^{(n)} - 1} = \frac{-D_n(1)}{t_0^{(n)} - 1}.$$

This gives

$$r_n = 1 - \frac{1}{2}\frac{D_n(1)}{D_n(3/2) - D_n(1)} < t_0^{(n)} < \frac{3}{2}.$$

It follows from Lemma 2 that $\lim_{n\to\infty} t_0^{(n)}(a,b) = 3/2$, and therefore, $t_{0s} = \sup_{n \in \mathbb{N}}\{t_0^{(n)}(a,b)\} = \frac{3}{2}$.

Denote by

$$F(s_n,t) = \left(\frac{2\sqrt{s_n}}{1+s_n}\right)^t \ln\left(\frac{2\sqrt{s_n}}{1+s_n}\right) \times D_n(t)$$
$$= \left(\ln\left(\frac{2\sqrt{s_n}}{1+s_n}\right)\right)\left(\frac{2}{1+s_n}\right)^t\left(\frac{s_n^t-1}{\ln s_n}\right) - \left(\frac{2\sqrt{s_n}}{1+s_n}\right)^t + 1.$$

Put $t_0 = \inf_{a>0,b>0,a\neq b.} t_{0i}(a,b)$. Let $\{s_k\}_{k=1}^{\infty}$ be a sequence such that $0 < s_{k+1} < s_k < 1$, $\lim_{k\to\infty} s_k = 0$. Let $\{t(s_k)\}_{k=1}^{\infty}$ be a sequence of solutions $F(s_k,t(s_k)) = 0$ in $(1,3/2)$. Then $\{t(s_k)\}_{k=1}^{\infty}$ is a bounded sequence. It implies that the sequence has a convergent subsequence $\{t(s_{k_l})\}_{l=1}^{\infty}$. Denote by $t_0^* = \lim_{l\to\infty} t(s_{k_l})$. It is clear that

$$D_n(t(a,b)) = 0 \quad \Longleftrightarrow \quad F(s_n(a,b),t(a,b)) = 0.$$

So from $\lim_{l\to\infty} F(s_{k_l},t(s_{k_l})) = 0$ we obtain $2^{t_0^*-1} = 1$. It implies $t_0 = 1$. □

CONCLUSSION 1. For all $a,b > 0$ with $a \neq b$ there is $t_1 \in (1,t_{0i}(a,b))$ such that

$$L_{t_1}(a,b) < AGM(a,b),$$

but $\sup\limits_{a>0, b>0, a\neq b.} \{t_1(a,b)\} = 1.$

For all $a, b > 0$ with $a \neq b$ there is no $t_2 \in (1, 3/2)$ such that

$$L_{t_2}(a,b) > AGM(a,b).$$

So $L_1(a,b)$ and $L_{3/2}(a,b)$ are optimal lower and upper bounds for $AGM(a,b)$, respectively.

Competing interests. The author declares that he has no competing interests.

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