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OPTIMAL INEQUALITIES INVOLVING POWER–EXPONENTIAL MEAN, ARITHMETIC MEAN AND GEOMETRIC MEAN

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Abstract. For $a,b>0$ with $a\neq b$, the power-exponential mean is defined by

$$Z\equiv Z(a,b)=\exp\left(\frac{a\ln a+b\ln b}{a+b}\right)=\sqrt{ab}e^{t\tanh t},$$

where $t=\ln\sqrt{a/b}$. In this paper, we prove the double inequality

$$\left(\frac{Z^p+G^p}{2}\right)^{1/p}<A<\left(\frac{Z^q+G^q}{2}\right)^{1/q}$$

holds for $a,b>0$, $a\neq b$ with the best constants $p=2/3$ and $q=1$, where $A=(a+b)/2$, $G=\sqrt{ab}$. We also establish the sharp bounds for $e^{t\tanh t}$ as follows:

$$1<\frac{e^{t\tanh t}}{2\cosh t-1}<1.055,$$
$$\frac{1}{\sqrt{2}}<\frac{e^{t\tanh t}}{2(\cosh t)^{2/3}-1}<1$$

for $t>0$. These improve some known results.

1. Introduction and main results

Let $p,q\in\mathbb{R}$ and $a,b\in\mathbb{R}_+:=(0,\infty)$. The Gini means [4] are defined as

$$G_{p,q}(a,b)=\begin{cases}\left(\dfrac{a^p+b^p}{a^q+b^q}\right)^{1/(p-q)} & \text{if } p\neq q,\\ \exp\left(\dfrac{a^p\ln a+b^p\ln b}{a^p+b^p}\right) & \text{if } p=q.\end{cases}\tag{1.1}$$

The basic properties of Gini means, as well as their comparison theorems, monotonicity, and log-convexity can be found in [3, 4, 8, 12, 18, 19, 20].

As special cases of Gini means $G_{p,q}(a,b)$, we see that

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(i) $A(a,b) = G_{1,0}(a,b) = (a+b)/2$ is the arithmetic mean;
(ii) $G(a,b) = G_{0,0}(a,b) = \sqrt{ab}$ is the geometric mean;
(iii) $Z(a,b) = G_{1,1}(a,b) = a^{a/(a+b)}b^{b/(a+b)}$ is the power-exponential mean;
(iv) $A_p(a,b) = G_{p,0}(a,b)$ is the power mean of order p.

We remark that the power-exponential mean is also called "symmetric geometric mean" ([15]), "special Gini mean" ([14]) and "weighted geometric mean" ([13]). In this paper, we adopt the term "power-exponential mean" to name the mean $Z(a,b)$ (see e.g. [21], [22],). Although $Z(a,b)$ is a weighted geometric mean with weights $a/(a+b)$ and $b/(a+b)$, it is not widely known. So we hope to establish some new inequalities for this mean.

Sándor [10, (30)] showed that for $a,b>0$ with $a \neq b$

$$A < \sqrt{I(a^2,b^2)} < Z(a,b), \tag{1.2}$$

where

$$I(a,b) = e^{-1}\left(\frac{b^b}{a^a}\right)^{1/(b-a)}, \text{ if } a \neq b \text{ and } I(a,a) = a \tag{1.3}$$

is the identric (exponential) mean of a and b. Yang [18] proved that

$$\sqrt{ab} < I(a,b) < Z^2\left(\sqrt{a},\sqrt{b}\right) < I(a,b)\exp(1-\frac{G^2(a,b)}{L^2(a,b)}) < Z(a,b) \tag{1.4}$$

hold for $a,b>0$ with $a \neq b$, where

$$L(a,b) = \frac{b-a}{\ln b - \ln a}, \text{ if } a \neq b \text{ and } L(a,a) = a \tag{1.5}$$

is the logarithmic mean of a and b. Using comparison theorem of Páles's [9] it is obtained that

$$Z(a,b) > A_2(a,b), \tag{1.6}$$

and its companion inequality

$$Z(a,b) < \sqrt{2}A_2(a,b) \tag{1.7}$$

is due to Neuman and Sándor [5, Theorem 4]. Yang [19, (5.15)], [22] showed that

$$\sqrt{Z(a,b)G(a,b)} < A(a,b) < \frac{Z(a,b)+G(a,b)}{2}. \tag{1.8}$$

In [17, (4.16)], Yang also presented a double inequality

$$1 < \frac{Z(a,b)}{2A(a,b)-G(a,b)} < \frac{3}{e} \approx 1.1036. \tag{1.9}$$

More inequalities for mean Z can be found in [13, 14, 15, 19, 21, 22, 17].

An identity related to means Z and I

$$Z(a,b) = \frac{I\left(a^2,b^2\right)}{I(a,b)}$$

is due to Sándor in [11] (see also [21]). Another one among means Z, L, A and G

$$Z^2(\sqrt{a},\sqrt{b}) = G(a,b)\exp\frac{A(a,b)-G(a,b)}{L(a,b)} \tag{1.10}$$

was obtained by Neuman and Sándor [6]. (see also [21]).

The main aim of this paper is to present the sharp bounds for arithmetic mean A in terms of power-exponential mean Z and geometric mean G, that is, to determine the best p such that

$$A > \left(\frac{Z^p+G^p}{2}\right)^{1/p} \tag{1.11}$$

and its reverse hold for all $a,b>0$ with $a \neq b$.

Our main results are contained in the following theorems.

THEOREM 1. *Let $a,b>0$ with $a \neq b$.*
(i) When $p \geqslant 2$, the double inequality

$$\left(\frac{Z^p+(2^p-1)G^p}{2^p}\right)^{1/p} < A < \left(\frac{Z^p+G^p}{2}\right)^{1/p} \tag{1.12}$$

holds with the best weights $1/2$ and

$$\beta_p = \begin{cases} 2^{-p} & \text{if } p>0, \\ 1 & \text{if } p \leqslant 0. \end{cases} \tag{1.13}$$

(ii) When $p \leqslant 2/3$, the double inequality

$$\left(\frac{Z^p+G^p}{2}\right)^{1/p} < A < \left(\beta_p Z^p+(1-\beta_p)G^p\right)^{1/p} \tag{1.14}$$

holds with the best weights $1/2$ and β_p, where the lower and upper bounds in the case of $p=0$ are defined by their limits as $p \to 0$.

(iii) When $2/3<p<2$, the double inequality

$$\left(\gamma_p Z^p+(1-\gamma_p)G^p\right)^{1/p} \leqslant A < \left(\delta_p Z^p+(1-\delta_p)G^p\right)^{1/p} \tag{1.15}$$

holds, where $\delta_p = \max_{p\in(2/3,2)}\left(2^{-1},2^{-p}\right)$ and γ_p are the best constants, and here

$$\gamma_p = \frac{(\cosh t_0)^p-1}{e^{pt_0\tanh t_0}-1},$$

t_0 is the unique solution of the equation

$$\frac{d}{dt}\frac{(\cosh t)^p - 1}{e^{pt\tanh t} - 1} = 0$$

on $(0,\infty)$.

In particular, taking $p=1$, we have

$$\gamma_1 Z + (1-\gamma_1)G \leqslant A < \frac{Z+G}{2}, \tag{1.16}$$

where $\gamma_1 \approx 0.46814$ and $1/2$ are the best.

THEOREM 2. *Let $a,b>0$ with $a \neq b$. Then the double inequality*

$$\left(\frac{Z^p+G^p}{2}\right)^{1/p} < A < \left(\frac{Z^q+G^q}{2}\right)^{1/q} \tag{1.17}$$

holds with the best constants $p=2/3$ and $q=1$.

Let $M(x,y)$ be a homogeneous mean of positive arguments x and y. Then $M(x,y)$ can be expressed as

$$M(x,y) = \sqrt{xy}M\left(e^t, e^{-t}\right),$$

where $t = (1/2)\ln(x/y)$.

By symmetry, we assume that $b>a>0$. Then $t=(1/2)\ln(a/b)>0$. Therefore we have

$$Z(a,b) = \sqrt{ab}e^{t\tanh t}, \quad A(a,b) = \sqrt{ab}\cosh t, \quad G(a,b) = 1. \tag{1.18}$$

Thus Theorems 1 and 2 can be restated in the form of hyperbolic functions.

THEOREM 3. *Let $t>0$. (i) When $p \geqslant 2$, the double inequality*

$$\left(\beta_p e^{pt\tanh t} + 1 - \beta_p\right)^{1/p} < \cosh t < \left(\frac{1}{2}e^{pt\tanh t} + \frac{1}{2}\right)^{1/p} \tag{1.19}$$

holds with the best weights $1/2$ and β_p given in (1.13).

(ii) When $p \leqslant 2/3$, the double inequality (1.19) is reversed, where the lower and upper bounds in the case of $p=0$ are defined by their limits as $p \to 0$.

(iii) When $2/3 < p < 2$, the double inequality

$$\left(\gamma_p e^{pt\tanh t} + (1-\gamma_p)\right)^{1/p} \leqslant \cosh t < \left(\delta_p e^{pt\tanh t} + 1 - \delta_p\right)^{1/p}, \tag{1.20}$$

where $\delta_p = \max_{p\in(2/3,2)}\left(2^{-1}, 2^{-p}\right)$ and γ_p are the best constants, and here

$$\gamma_p = \frac{(\cosh t_0)^p - 1}{e^{pt_0\tanh t_0} - 1},$$

t_0 is the unique solution of the equation

$$\frac{d}{dt}\frac{(\cosh t)^p-1}{e^{pt\tanh t}-1}=0$$

on $(0,\infty)$.

In particular, taking $p=1$, we have

$$\gamma_1 e^{t\tanh t}+(1-\gamma_1)\leqslant \cosh t<\frac{1}{2}e^{t\tanh t}+\frac{1}{2}, \tag{1.21}$$

where $\gamma_1\approx 0.46814$ and $1/2$ are the best.

THEOREM 4. *Let $t>0$. Then the double inequality*

$$\left(\frac{1}{2}e^{pt\tanh t}+\frac{1}{2}\right)^{1/p}<\cosh t<\left(\frac{1}{2}e^{qt\tanh t}+\frac{1}{2}\right)^{1/q} \tag{1.22}$$

holds with the best constants $p=2/3$ and $q=1$.

2. Proofs of main results

To prove our main results, we need the following lemmas.

LEMMA 1. ([7, Proposition 1.2, Corollary 1.3], [1, Theorem 2]) *For $-\infty\leqslant a<b\leqslant\infty$, let f and g be differentiable functions on the interval (a,b). Assume also that the derivative g' is nonzero and does not change sign on (a,b). Suppose that $f(a^+)=g(a^+)=0$ or $f(b^-)=g(b^-)=0$. If f'/g' is increasing (decreasing) on (a,b) then so is f/g.*

LEMMA 2. ([16, Theorem 8]) *Let $-\infty\leqslant a<b\leqslant\infty$. Suppose that (i) f and g are differentiable functions on (a,b); (ii) $g'\neq 0$ on (a,b); (iii) $f(a^+)=g(a^+)=0$; (iv) there is a $c\in(a,b)$ such that f'/g' is increasing (decreasing) on (a,c) and decreasing (increasing) on (c,b). Then*

(1) when $\operatorname{sgn}g'\operatorname{sgn}H_{f,g}(b^-)\geqslant(\leqslant)0$, f/g is increasing (decreasing) on (a,b), where $H_{f,g}=(f'/g')g-f$.

(2) when $\operatorname{sgn}g'\operatorname{sgn}H_{f,g}(b^-)<(>)0$, there is a unique number $x_a\in(a,b)$ such that f/g is increasing (decreasing) on (a,x_a) and decreasing (increasing) on (x_a,b).

LEMMA 3. ([2]) *Let a_n and b_n $(n=0,1,2,\ldots)$ be real numbers and let the power series $A(t)=\sum_{n=1}^{\infty}a_nt^n$ and $B(t)=\sum_{n=1}^{\infty}b_nt^n$ be convergent for $|t|<R$. If $b_n>0$ for $n=0,1,2,\ldots$, and a_n/b_n is strictly increasing (or decreasing) for $n=0,1,2,\ldots$, then the function $t\mapsto A(t)/B(t)$ is strictly increasing (or decreasing) on $(0,R)$.*

LEMMA 4. *Let h be the function defined on $(0,\infty)$ by*

$$h(t)=\frac{t\cosh^2 t+t\sinh^2 t-\cosh t\sinh t}{t+\cosh t\sinh t}\frac{\cosh t}{t\sinh t}. \tag{2.1}$$

Then $h(t)$ is increasing from $(0,\infty)$ onto $(2/3,2)$.

Proof. Write

$$A(t) = \left(t\cosh^2 t + t\sinh^2 t - \cosh t \sinh t\right)\cosh t,$$
$$B(t) = (t + \cosh t \sinh t)\, t \sinh t.$$

Using "product into sum" formulas for hyperbolic functions leads to

$$4A(t) = 2t\cosh t - \sinh t - \sinh 3t + 2t\cosh 3t,$$
$$4B(t) = 4t^2 \sinh t - t\cosh t + t\cosh 3t.$$

Expanding in power series gives

$$\begin{aligned} 4A(t) &= 2t\cosh t - \sinh t - \sinh 3t + 2t\cosh 3t \\ &= 2\sum_{n=1}^{\infty} \frac{t^{2n-1}}{(2n-2)!} - \sum_{n=1}^{\infty} \frac{t^{2n-1}}{(2n-1)!} - \sum_{n=1}^{\infty} \frac{3^{2n-1} t^{2n-1}}{(2n-1)!} + 2\sum_{n=1}^{\infty} \frac{3^{2n-2} t^{2n-1}}{(2n-2)!} \\ &= \sum_{n=2}^{\infty} \frac{(4n-5)\, 3^{2n-2} + 4n - 3}{(2n-1)!} t^{2n-1} := \sum_{n=2}^{\infty} a_n t^{2n-1}, \end{aligned}$$

and

$$\begin{aligned} 4B(t) &= 4t^2 \sinh t - t\cosh t + t\cosh 3t. \\ &= 4\sum_{n=2}^{\infty} \frac{t^{2n-1}}{(2n-3)!} - \sum_{n=1}^{\infty} \frac{t^{2n-1}}{(2n-2)!} + \sum_{n=1}^{\infty} \frac{3^{2n-2} t^{2n-1}}{(2n-2)!} \\ &= \sum_{n=2}^{\infty} \frac{3^{2n-2} + 8n - 9}{(2n-2)!} t^{2n-1} := \sum_{n=2}^{\infty} b_n t^{2n-1}. \end{aligned}$$

From Lemma 3, to prove h is increasing, it suffices to prove

$$\frac{a_n}{b_n} = \frac{(4n-5)\, 3^{2n-2} + 4n - 3}{(2n-1)\left(3^{2n-2} + 8n - 9\right)}$$

is increasing for $n \geqslant 2$. A direct computation leads to

$$\begin{aligned} \frac{a_{n+1}}{b_{n+1}} - \frac{a_n}{b_n} &= \frac{(4n-1)\, 3^{2n} + 4n + 1}{(2n+1)\left(3^{2n} + 8n - 1\right)} - \frac{(4n-5)\, 3^{2n-2} + 4n - 3}{(2n-1)\left(3^{2n-2} + 8n - 9\right)} \\ &= \frac{2u_n}{(2n-1)(2n+1)\left(3^{2n} + 8n - 1\right)\left(3^{2n-2} + 8n - 9\right)}, \end{aligned}$$

where

$$u_n = 3^{4n-1} + 2(8n-1)(4n-5)(4n-3)\, 3^{2n-2} - \left(32n^2 - 16n - 3\right).$$

Clearly, $3^{4n-1} \geqslant 3^7 = 2187$ and $3^{2n-2} \geqslant 1$ for $n \geqslant 2$, so we get

$$u_n \geqslant 2187 + 2(8n-1)(4n-5)(4n-3) - \left(32n^2 - 16n - 3\right)$$

$$= 64n(n-1)(4n-5)+2160 > 0.$$

Hence, a_n/b_n is increasing for $n \in \mathbb{N}$ with $n \geqslant 2$, and therefore, the function $t \mapsto A(t)/B(t) = h(t)$ is increasing on $(0,\infty)$.

Simple calculation gives

$$\lim_{t\to 0^+} h(t) = \frac{2}{3} \quad \text{and} \quad \lim_{t\to\infty} h(t) = 2,$$

which proves this lemma. □

Now we are in a position to prove Theorems 3 and 4.

Proof of Theorem 3. For $t > 0$, we define

$$f(t) = \frac{(\cosh t)^p - 1}{p} \text{ if } p \neq 0 \text{ and } f(t) = \ln(\cosh t) \text{ if } p = 0, \tag{2.2}$$

$$g(t) = \frac{e^{pt\tanh t} - 1}{p} \text{ if } p \neq 0 \text{ and } g(t) = t\tanh t \text{ if } p = 0. \tag{2.3}$$

Then

$$\frac{f(t)}{g(t)} = \frac{(\cosh t)^p - 1}{e^{pt\tanh t} - 1} \text{ if } p \neq 0 \text{ and } \frac{f(t)}{g(t)} = \frac{\ln(\cosh t)}{t\tanh t} \text{ if } p = 0.$$

Differentiation gives

$$\frac{f'(t)}{g'(t)} = \frac{\cosh^{p+1} t}{e^{pt\tanh t}} \frac{\sinh t}{t + \sinh t\cosh t}, \tag{2.4}$$

$$\left(\frac{f'(t)}{g'(t)}\right)' = -\frac{\cosh^{p-1} t}{e^{pt\tanh t}} \frac{t\sinh t}{t + \sinh t\cosh t}(p - h(t)), \tag{2.5}$$

where $h(t)$ is defined by (2.1).

(i) When $p \geqslant 2$, by Lemma 4 we see that $(f'/g')' < 0$. It is deduced from Lemma 1 that f/g is decreasing on $(0,\infty)$, and therefore, we have

$$\beta_p = \lim_{t\to\infty} \frac{f(t)}{g(t)} < \frac{f(t)}{g(t)} = \frac{(\cosh t)^p - 1}{e^{pt\tanh t} - 1} < \lim_{t\to 0^+} \frac{f(t)}{g(t)} = \alpha.$$

This together with the facts that

$$\alpha = \lim_{t\to 0^+} \frac{f(t)}{g(t)} = \frac{1}{2},$$

$$\beta_p = \lim_{t\to\infty} \frac{f(t)}{g(t)} = \begin{cases} 2^{-p} & \text{if } p > 0, \\ 1 & \text{if } p = 0, \\ 1 & \text{if } p < 0 \end{cases} \tag{2.6}$$

implies the double inequality (1.19).

(ii) When $p \leqslant 2/3$, similarly, f/g is increasing on $(0,\infty)$. This leads to the reverse of (1.19).

(iii) When $2/3 < p < 2$, by Lemma 4 we see that $t \mapsto p - h(t) := j(t)$ is decreasing on $(0,\infty)$ but $j(0^+) = p - 2/3 > 0$ and $j(\infty) = p - 2 < 0$, which implies that there is a $t_1 \in (0,\infty)$ such that $j(t) > 0$ for $t \in (0,t_1)$ and $j(t) < 0$ for $t \in (t_1,\infty)$. This in conjunction with (2.5) indicates that (f'/g') is decreasing on $(0,t_1)$ and increasing on (t_1,∞).

It is easy to check that $f(0^+) = g(0^+) = 0$, $g'(t) = p\cosh^{p-1} t \sinh t > 0$ and

$$
\begin{aligned}
H_{f,g}(\infty) &= \lim_{t\to\infty}\left(\frac{f'(t)}{g'(t)}g(t) - f(t)\right)\\
&= \lim_{t\to\infty}\left(\frac{\cosh^{p+1} t}{e^{pt\tanh t}}\frac{\sinh t}{t+\sinh t\cosh t}\frac{e^{pt\tanh t}-1}{p} - \frac{(\cosh t)^p - 1}{p}\right)\\
&= \frac{1}{p}\lim_{t\to\infty}\left(1 - \frac{\left(\frac{\cosh t}{e^{t\tanh t}}\right)^p + \frac{t}{\sinh t\cosh^{1-p} t}}{\frac{t}{\cosh t\sinh t}+1}\right) = \frac{1-2^{-p}}{p} > 0,
\end{aligned}
$$

where the last equality holds due to

$$
\begin{aligned}
\frac{\cosh t}{e^{t\tanh t}} &= \frac{1+e^{-2t}}{2}\exp\left(-\frac{t}{e^t\cosh t}\right) \to \frac{1}{2} \text{ as } t\to\infty,\\
\frac{t}{\sinh t\cosh^{1-p} t} &= \frac{\cosh t}{\sinh t}\frac{t}{\cosh^{2-p} t} \to 0 \text{ as } t\to\infty.
\end{aligned}
$$

Then by Lemma 2 we see that there is a $t_0 \in (0,\infty)$ such that f/g is decreasing on $(0,t_0)$ and increasing on (t_0,∞). Thus we conclude that

$$
\begin{aligned}
\gamma_p &= \frac{f(t_0)}{g(t_0)} < \frac{f(t)}{g(t)} < \lim_{t\to 0^+}\frac{f(t)}{g(t)} = \frac{1}{2} \text{ for } t\in(0,t_0),\\
\gamma_p &= \frac{f(t_0)}{g(t_0)} < \frac{f(t)}{g(t)} < \lim_{t\to\infty}\frac{f(t)}{g(t)} = \beta_p = 2^{-p} \text{ for } t\in(t_0,\infty).
\end{aligned}
$$

So for $t \in (0,\infty)$, it holds that

$$
\gamma_p < \frac{(\cosh t)^p - 1}{e^{pt\tanh t}-1} < \max_{p\in(2/3,2)}\left(\frac{1}{2}, 2^{-p}\right) = \delta_p,
$$

which proves (1.20).

In particular, if $p = 1$, then $\delta_p = \max_{p\in(2/3,2)}\left(2^{-1}, 2^{-p}\right) = 1/2$. Numeric computation yields $t_0 \approx 1.6223026$, and so

$$
\gamma_1 = \frac{\cosh t_0 - 1}{e^{t_0\tanh t_0} - 1} \approx 0.46814.
$$

Thus we complete the proof. □

Proof of Theorem 4. (i) Necessity. The first inequality of (1.22) is equivalent to $f(t)/g(t) > 1/2$ or $g(t) - 2f(t) < 0$, where $f(t)$ and $g(t)$ are defined by (2.2) and (2.3), respectively. If $g(t) - 2f(t) < 0$ holds for $t > 0$, then there must be

$$
\lim_{t\to 0^+}\frac{g(t)-2f(t)}{t^4} \leqslant 0. \tag{2.7}
$$

Expanding in power series yields

$$g(t)-2f(t)=\frac{1}{p}\left(e^{pt\tanh t}+1-2(\cosh t)^{p}\right)=\frac{1}{4}\left(p-\frac{2}{3}\right)t^{4}+O\left(t^{6}\right),$$

which together with (2.7) gives $p\leqslant 2/3$.

Sufficiency. It has been proved in part (ii) of Theorem 3.

(ii) Necessity. It follows from $\lim_{t\to\infty}[f(t)/g(t)]\leqslant 1/2$, which by the limit relation (2.6) gives $2^{-q}\leqslant 1/2$, that is, $q\geqslant 1$.

Sufficiency. The increasing property of power mean in its parameter in combination with the second inequality of (1.21) gives the sufficiency.

This completes the proof. □

3. Further results and remarks

REMARK 1. Lemma 4 reveals a double inequality

$$\frac{2}{3}<\frac{t\cosh^{2}t+t\sinh^{2}t-\cosh t\sinh t}{t+\cosh t\sinh t}\frac{\cosh t}{t\sinh t}<2 \tag{3.1}$$

for $t>0$. Note that

$$\cosh^{2}t+\sinh^{2}t=\cosh 2t,\ \ \cosh t\sinh t=\frac{1}{2}\sinh 2t,\ \frac{\cosh t}{\sinh t}=\frac{\sinh 2t}{\cosh 2t-1},$$

and with $2t\to t$, inequalities (3.1) can be rewritten as

$$\frac{1}{3}<\frac{\cosh t-\frac{\sinh t}{t}}{1+\frac{\sinh t}{t}}\frac{\sinh t}{t}\frac{1}{\cosh t-1}<1.$$

By the relations given in (1.18) and

$$L(a,b)=\frac{a-b}{\ln a-\ln b}=\sqrt{ab}\frac{\sinh t}{t},$$

the above double inequality is equivalent to

$$\frac{1}{3}<\frac{A-L}{G+L}\frac{L}{A-G}<1,$$

or

$$\frac{A-L}{G+L}<\frac{A-G}{L}<3\frac{A-L}{G+L}.$$

This together with (1.10) gives

$$\frac{A-L}{G+L}<\ln\frac{Z_{1/2}}{G}=\frac{A-G}{L}<3\frac{A-L}{G+L},$$

that is,

$$G\exp\left(\frac{A-L}{G+L}\right)<Z_{1/2}<G\exp^{3}\left(\frac{A-L}{G+L}\right),$$

where $Z_{1/2}=Z^{2}\left(\sqrt{a},\sqrt{b}\right)$.

REMARK 2. In the double inequality (1.19), the right hand side is obviously increasing in p on $\mathbb{R}$. Also, we claim that the left one is decreasing in p on $(0,\infty)$. For this end, it suffices to show that the function U defined on $(0,\infty)^2$ by

$$U(p,x) = \ln\left(\beta_p e^{pt\tanh t} + 1 - \beta_p\right)^{1/p} = \frac{1}{p}\ln\left(2^{-p}e^{px} + 1 - 2^{-p}\right) \tag{3.2}$$
$$= -\ln 2 + \frac{1}{p}\ln\left(e^{px} + 2^p - 1\right)$$

is decreasing in p on $(0,\infty)$, where $x = t\tanh t$.

Differentiations give

$$\frac{\partial U}{\partial p} = \frac{1}{p}\frac{xe^{px} + 2^p\ln 2}{e^{px} + 2^p - 1} - \frac{1}{p^2}\ln\left(e^{px} + 2^p - 1\right)$$
$$\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial p}\right) = \frac{(2^p - 1)e^{px}}{(e^{px} + 2^p - 1)^2}\left(x - \frac{2^p\ln 2}{2^p - 1}\right).$$

It is easy to see that $x \mapsto \partial U/\partial p$ is increasing for $x > (2^p\ln 2)/(2^p - 1)$ and decreasing for $0 < x < (2^p\ln 2)/(2^p - 1)$. So, to show that $\partial U/\partial p < 0$ for $p > 0$, it suffices to check that

$$\lim_{x\to 0^+}\frac{\partial U}{\partial p} = 0 \text{ and } \lim_{x\to\infty}\frac{\partial U}{\partial p} = 0.$$

In fact, a simple calculation yields $\lim_{x\to 0^+}\partial U/\partial p = 0$. And

$$p\frac{\partial U}{\partial p} = \frac{2^p\ln 2}{e^{px} + 2^p - 1} + \left(\frac{xe^{px}}{e^{px} + 2^p - 1} - x\right) - \frac{1}{p}\ln\left(1 + (2^p - 1)e^{-px}\right)$$
$$= \frac{2^p\ln 2}{e^{px} + 2^p - 1} - \frac{x(2^p - 1)}{e^{px} + 2^p - 1} - \frac{1}{p}\ln\left(1 + (2^p - 1)e^{-px}\right) \to 0$$

as $x \to \infty$. Then we obtain $\lim_{x\to\infty}\partial U/\partial p = 0$

From (3.2), employing L'Hospital rule yields

$$\lim_{p\to 0^+} U(p,x) = -\ln 2 + \lim_{p\to 0^+}\frac{\ln\left(e^{px} + 2^p - 1\right)}{p} = \lim_{p\to 0^+}\frac{xe^{px} + 2^p\ln 2}{e^{px} + 2^p - 1} - \ln 2 = x.$$

Similarly, we have

$$\lim_{p\to\infty} U(p,x) = -\ln 2 + \lim_{p\to\infty}\frac{\ln\left(e^{px} + 2^p - 1\right)}{p} = \lim_{p\to\infty}\frac{xe^{px} + 2^p\ln 2}{e^{px} + 2^p - 1} - \ln 2$$
$$= \lim_{p\to\infty}\frac{e^{-px}\ln 2 + (x - \ln 2)}{1 + (2e^{-x})^p - e^{-px}} = \max\left(x - \ln 2, 0\right).$$

Thus, taking $p = 2,\infty$ in part (i) and $p = 2/3,\ 1/2,\ 0^+$ in part (ii) of Theorem 3 we get the following corollary.

COROLLARY 1. *Let $t>0$. Then the following inequalities hold:*

$$\max\left(\frac{1}{2}e^{t\tanh t},1\right)<\left(\frac{1}{4}e^{2t\tanh t}+\frac{3}{4}\right)^{1/2}<\cosh t<\left(\frac{1}{2}e^{2t\tanh t}+\frac{1}{2}\right)^{1/2}<e^{t\tanh t},$$

$$\sqrt{e^{t\tanh t}}<\left(\frac{1}{2}\sqrt{e^{t\tanh t}}+\frac{1}{2}\right)^{2}<\left(\frac{1}{2}\left(e^{t\tanh t}\right)^{2/3}+\frac{1}{2}\right)^{3/2}<\cosh t$$
$$<\frac{1}{2}\left(\left(e^{t\tanh t}\right)^{2/3}+2^{2/3}-1\right)^{3/2}<\frac{1}{2}\left(\sqrt{e^{t\tanh t}}+\sqrt{2}-1\right)^{2}<e^{t\tanh t}.$$

REMARK 3. By the relations (1.18), two inequalities in the above corollary are equivalent to two ones for means:

$$\max\left(\frac{Z}{2},G\right)<\left(\frac{Z^2+3G^2}{4}\right)^{1/2}<A<\left(\frac{Z^2+G^2}{2}\right)^{1/2}<Z,$$

$$\sqrt{ZG}<\left(\frac{\sqrt{Z}+\sqrt{G}}{2}\right)^{2}<\left(\frac{Z^{2/3}+G^{2/3}}{2}\right)^{3/2}<A$$
$$<\frac{1}{2}\left(Z^{2/3}+\left(2^{2/3}-1\right)G^{2/3}\right)^{3/2}<\frac{1}{2}\left(\sqrt{Z}+\left(\sqrt{2}-1\right)\sqrt{G}\right)^{2}<Z.$$

We now give the sharp lower and upper bounds for the ratio f_1/f_2, where

$$f_1(t)=w(\cosh t)^p+1-w,\quad (w>0)$$
$$f_2(t)=\exp(pt\tanh t).$$

Differentiation yields

$$f_1'(t)=pw(\cosh t)^{p-1}\sinh t,$$
$$f_2'(t)=p\exp(pt\tanh t)\frac{\cosh t\sinh t+t}{\cosh^2 t},$$

then we have

$$\frac{f_1'(t)}{f_2'(t)}=w\frac{\cosh^{p+1}t}{e^{pt\tanh t}}\frac{\sinh t}{t+\sinh t\cosh t}=w\frac{f'(t)}{g'(t)},$$
$$\left(\frac{f_1'(t)}{f_2'(t)}\right)'=w\left(\frac{f'(t)}{g'(t)}\right)'=-\frac{\cosh^{p-1}t}{e^{pt\tanh t}}\frac{t\sinh t}{t+\sinh t\cosh t}(p-h(t)),$$

where $f(t)$, $g(t)$ and $h(t)$ are defined by (2.2), (2.3) and (2.1), respectively. We have shown in the proof of Theorem 3 that

$$\left(\frac{f_1'(t)}{f_2'(t)}\right)'\begin{cases}\leqslant 0 \text{ for } t\in(0,\infty) & \text{if } p\geqslant 2,\\ \begin{cases}<0 \text{ for } t\in(0,t_1)\\ >0 \text{ for } t\in(t_1,\infty)\end{cases} & \text{if } \frac{2}{3}<p<2,\\ \geqslant 0 \text{ for } t\in(0,\infty) & \text{if } 0<p\leqslant\frac{2}{3}.\end{cases}$$

It is easy to verify that

$$\left(\frac{f_1}{f_2}\right)' = \frac{f_2'}{f_2}\left(\frac{f_1'}{f_2'}f_2 - f_1\right) = \frac{f_2'}{f_2}H_{f_1,f_2},$$
$$H_{f_1,f_2}' = \left(\frac{f_1'}{f_2'}\right)' f_2,$$

which in combination with the facts that $f_2(t), f_2'(t) > 0$ lead us to

$$\operatorname{sgn}\left(\frac{f_1}{f_2}\right)' = \operatorname{sgn} H_{f_1,f_2} \quad \text{and} \quad \operatorname{sgn} H_{f_1,f_2}' = \operatorname{sgn}\left(\frac{f_1'}{f_2'}\right)'.$$

Also, we have

$$\begin{aligned} H_{f_1,f_2}(t) &= \frac{f_1'(t)}{f_2'(t)}f_2(t) - f_1(t) \\ &= w\frac{\cosh^{p+1}t}{e^{pt\tanh t}}\frac{\sinh t}{t+\sinh t\cosh t}e^{pt\tanh t} - (w(\cosh t)^p + 1 - w) \\ &= -w\frac{t\cosh^p t}{t+\cosh t\sinh t} - 1 + w \to \begin{cases} \frac{w-2}{2} & \text{as } t\to 0, \\ \begin{matrix} -\infty \text{ if } p\geqslant 2 \\ w-1 \text{ if } p<2 \end{matrix} & \text{as } t\to\infty. \end{cases} \end{aligned}$$

Then the ratio f_1/f_2 has the following monotonicity pattern:

p	w	$\operatorname{sgn} H_{f_1,f_2}'$	H_{f_1,f_2}	$H_{f_1,f_2}(0)$	$H_{f_1,f_2}(\infty)$	$\operatorname{sgn} H_{f_1,f_2}$	f_1/f_2
$[2,\infty)$	$(2,\infty)$	$-$	$\searrow$	$+$	$-\infty$	$+-$	$\nearrow\searrow$
$[2,\infty)$	$(0,2]$	$-$	$\searrow$	$\leqslant 0$	$-\infty$	$-$	$\searrow$
$(\frac{2}{3},2)$	$(2,\infty)$	$-+$	$\searrow\nearrow$	$+$	$+$	$+?+$	$\nearrow?\nearrow$
$(\frac{2}{3},2)$	$(1,2]$	$-+$	$\searrow\nearrow$	$\leqslant 0$	$+$	$-+$	$\searrow\nearrow$
$(\frac{2}{3},2)$	$(0,1]$	$-+$	$\searrow\nearrow$	$-$	$\leqslant 0$	$-$	$\searrow$
$(0,\frac{2}{3}]$	$[2,\infty)$	$+$	$\nearrow$	$\geqslant 0$	$+$	$+$	$\nearrow$
$(0,\frac{2}{3}]$	$(1,2)$	$+$	$\nearrow$	$-$	$+$	$-+$	$\searrow\nearrow$
$(0,\frac{2}{3}]$	$(0,1]$	$+$	$\nearrow$	$-$	$\leqslant 0$	$-$	$\searrow$,

where "?" denotes $\operatorname{sgn} H_{f_1,f_2}$ and the monotonicity of f_1/f_2 are indeterminate, respectively.

From the above table and the limits

$$\lim_{t\to 0}\frac{f_1(t)}{f_2(t)} = \lim_{t\to 0}\frac{w(\cosh t)^p + 1 - w}{\exp(pt\tanh t)} = 1,$$
$$\lim_{t\to 0}\frac{f_1(t)}{f_2(t)} = \lim_{t\to\infty}\frac{w(\cosh t)^p + 1 - w}{\exp(pt\tanh t)} = w2^{-p} \text{ for } p > 0,$$

we immediately obtain the results as follows.

THEOREM 5. *Let* $p,w>0$. *Then for* $(p,w)\in(0,2/3]\times[2,\infty)$, *the double inequality*

$$\frac{2}{w^{1/p}}\left(w(\cosh t)^{p}+1-w\right)^{1/p}<e^{t\tanh t}<\left(w(\cosh t)^{p}+1-w\right)^{1/p}$$

holds for $t>0$ *with the best constants* $2/w^{1/p}$ *and* 1. *It is reversed for* $(p,w)\in E_1$, *where*

$$E_1=[2,\infty)\times(0,2]\cup\left(\frac{2}{3},2\right)\times(0,1]\cup(0,\frac{2}{3}]\times(0,1].$$

In particular, taking $(p,w)=(2/3,2)$, $(2,2)$, $(p,1)$, *we have*

$$\frac{1}{\sqrt{2}}\left(2(\cosh t)^{2/3}-1\right)^{3/2}<e^{t\tanh t}<\left(2(\cosh t)^{2/3}-1\right)^{3/2}, \tag{3.3}$$

$$\sqrt{\cosh(2t)}<e^{t\tanh t}<\sqrt{2}\sqrt{\cosh(2t)}, \tag{3.4}$$

$$\cosh t<e^{t\tanh t}<2\cosh t. \tag{3.5}$$

REMARK 4. Inequalities (3.3), (3.4) and (3.5) can be equivalently rewritten in the form of means as follows:

$$\frac{1}{\sqrt{2}}\left(2A^{2/3}-G^{2/3}\right)^{3/2}<Z<\left(2A^{2/3}-G^{2/3}\right)^{3/2}, \tag{3.6}$$

$$A_2<Z<\sqrt{2}A_2, \tag{3.7}$$

$$A<Z<2A, \tag{3.8}$$

where the lower and upper bounds are sharp. The inequalities (3.6) seem to be a new comer, while (3.7) are due to [9] and [5, Theorem 4].

THEOREM 6. *(i) If* $(p,w)\in[2,\infty)\times(2,\infty)$, *then the double inequality*

$$\min\left(1,\frac{w}{2^{p}}\right)<\frac{w(\cosh t)^{p}+1-w}{\exp(pt\tanh t)}<\frac{1}{\lambda_{p,w}},$$

or equivalently,

$$\lambda_{p,w}^{1/p}\left(w(\cosh t)^{p}+1-w\right)^{1/p}<\exp(t\tanh t)<\max\left(1,\frac{2}{w^{1/p}}\right)\left(w(\cosh t)^{p}+1-w\right)^{1/p},$$

holds for $t>0$, *where*

$$\lambda_{p,w}=\frac{\exp(pt_0\tanh t_0)}{w(\cosh t_0)^{p}+1-w}, \tag{3.9}$$

and here t_0 *is the unique solution of the equation*

$$H_{f_1,f_2}(t)=w-1-w\frac{t\cosh^{p}t}{t+\cosh t\sinh t}=0$$

on $(0,\infty)$.

(ii) If $(p,w)\in(2/3,2)\times(1,2]\cup(0,2/3]\times(1,2)$, *then the double inequality*

$$\min\left(1,\frac{2}{w^{1/p}}\right)\left(w(\cosh t)^p+1-w\right)^{1/p}<\exp(t\tanh t)<\lambda_{p,w}^{1/p}\left(w(\cosh t)^p+1-w\right)^{1/p}$$

holds for $t>0$, *where* $\lambda_{p,w}$ *is defined by (3.9).*

Letting $(p,w)=(1,2)$ in Theorem 6 and solving the equation

$$H_{f_1,f_2}(t)=\left[w-1-w\frac{t\cosh^p t}{t+\cosh t\sinh t}\right]_{p=1,w=2}=0$$

for t, we find $t_0\approx 1.87897594$, and so

$$\lambda_{1,2}=\frac{\exp(t_0\tanh t_0)}{2\cosh t_0-1}\approx 1.0543.$$

Thus we have the following interesting Corollary.

COROLLARY 2. *The double inequality*

$$1<\frac{\exp(t\tanh t)}{2\cosh t-1}<\lambda_{1,2} \tag{3.10}$$

holds for $t>0$ *with the best constants* 1 *and* $\lambda_{1,2}\approx 1.0543$.

REMARK 5. The double inequality (3.10) is equivalent to the following inequalities for means:

$$1<\frac{Z}{2A-G}<\lambda_{1,2}\approx 1.0543.$$

Clearly, this double inequality is an improvement of (1.9).

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