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GENERALIZED WEIGHTED OSTROWSKI AND OSTROWSKI–GRÜSS TYPE INEQUALITIES ON TIME SCALES VIA A PARAMETER FUNCTION

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Abstract. We prove generalized weighted Ostrowski and Ostrowski–Grüss type inequalities on time scales via a parameter function. In particular, our result extends a result of Dragomir and Barnett. Furthermore, we apply our results to the continuous, discrete, and quantum cases, to obtain some interesting new inequalities.

1. Introduction

In order to estimate the absolute deviation of a differentiable function from its integral mean, Dragomir and Barnett [10] obtained in 1999 the following Ostrowski type inequality.

THEOREM 1. (See [10]) *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and twice differentiable on (a,b) with second derivative $f'':(a,b)\to\mathbb{R}$. Then,*

$$\left|f(x)-\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^a f(t)dt\right| \leqslant \frac{M}{2}\left\{\left[\frac{(x-\frac{a+b}{2})^2}{(b-a)^2}+\frac{1}{4}\right]^2+\frac{1}{12}\right\}(b-a)^2$$

for all $x\in[a,b]$, where $M=\sup\limits_{a<t<b}|f''(t)|<\infty$.

By introducing a parameter, Liu [13] established in 2010 the following perturbed weighted generalized three-point integral inequality with bounded derivative.

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THEOREM 2. (See [13]) *Let* $0 \leqslant k \leqslant 1$ *and* $f : [a,b] \to \mathbb{R}$ *be a differentiable mapping. Assume there exists a constant* $\gamma \in \mathbb{R}$ *such that* $\gamma \leqslant f'(x)$ *for* $x \in [a,b]$, $g : [a,b] \to [0,\infty)$ *is continuous and positive on* (a,b), *and let* $h : [a,b] \to \mathbb{R}$ *be differentiable such that* $h'(t) = g(t)$ *on* $[a,b]$. *Then,*

$$\begin{aligned}
&\Bigg| \left\{ (1-k)f(x) - k\left[\frac{\int_a^x g(t)dt}{\int_a^b g(t)dt} f(a) + \frac{\int_x^b g(t)dt}{\int_a^b g(t)dt} f(b) \right] \right\} \int_a^b g(t)dt \\
&\quad - \gamma \Bigg\{ (1-k)\left[(h(b)-h(a))\left(x - \frac{a+b}{2} \right) \right. \\
&\quad \left. + (b-a)\left(h(x) - \frac{h(a)+h(b)}{2} \right) \right] \int_a^b (h(t)-h(x))dt \Bigg\} - \int_a^b f(t)g(t)dt \Bigg| \\
&\leqslant \begin{cases} (1-k)\left[\frac{1}{2}\int_a^b g(t)dt + \left| h(x) - \frac{h(a)+h(b)}{2} \right| \right](S-\gamma)(b-a), & k \in [0,\frac{1}{2}] \\ k\left[\frac{1}{2}\int_a^b g(t)dt + \left| h(x) - \frac{h(a)+h(b)}{2} \right| \right](S-\gamma)(b-a), & k \in (\frac{1}{2},1] \end{cases}
\end{aligned}$$

for all $x \in [a,b]$.

In order to unify the continuous and discrete calculus in a consistent manner, Hilger introduced in 1988 the theory of time scales [8, 11]. Since the advent of this calculus, many researchers have been able to extend known classical integral inequalities to time scales. We refer the interested reader to papers [6, 7, 14, 15, 16, 18, 19, 20, 21, 22], books [1, 3, 5], and references therein. In 2014, Liu et al. [17] obtained the following inequality on time scales.

THEOREM 3. (See [17]) *Let* $0 \leqslant k \leqslant 1$, $g : [a,b] \to [0,\infty)$ *be rd-continuous and positive, and* $h : [a,b] \to \mathbb{R}$ *be differentiable such that* $h^{\Delta}(t) = g(t)$ *on* $[a,b]$. *Let* $a,b,t,x \in \mathbb{T}$, $a < b$, *and* $f : [a,b] \to \mathbb{R}$ *be twice differentiable. Then,*

$$\begin{aligned}
&\Bigg| (1-k)^2 f(x) - \frac{1}{(\int_a^b g(t)\Delta t)^2}\left(\int_a^b S(x,t)\Delta t \right)\left(\int_a^b g(t) f^{\Delta}(\sigma(t))\Delta t \right) \\
&\quad + \frac{k}{(\int_a^b g(t)\Delta t)^2} \int_a^b S(x,t)\left(f^{\Delta}(a)\int_a^t g(s)\Delta s + f^{\Delta}(b)\int_t^b g(s)\Delta s \right)\Delta t \\
&\quad + \frac{k(1-k)}{(\int_a^b g(t)\Delta t)}\left(f(a)\int_a^x g(t)\Delta t + f(b)\int_x^b g(t)\Delta t \right) - \frac{1-k}{\int_a^b g(t)\Delta t}\int_a^b g(t) f(\sigma(t))\Delta t \Bigg| \\
&\leqslant \frac{M}{(\int_a^b g(t)\Delta t)^2}\int_a^b\int_a^b |S(x,t)||S(t,s)|\Delta s\Delta t
\end{aligned}$$

for all $x \in [a,b]$, *where* $M = \sup_{a<t<b} |f^{\Delta\Delta}(t)| < \infty$ *and*

$$S(x,t) = \begin{cases} h(t) - ((1-k)h(a) + kh(x)), & a \leqslant t < x, \\ h(x) - (kh(x) + (1-k)h(b)), & x \leqslant t \leqslant b. \end{cases}$$

Recently, in 2017, by using a different weighted Peano kernel, Nwaeze obtained in [19] the following weighted Ostrowski type inequality.

THEOREM 4. (See [19]) *Let* $\nu : [a,b] \to [0,\infty)$ *be rd-continuous and positive and* $w : [a,b] \to \mathbb{R}$ *be differentiable such that* $w^{\Delta}(t) = \nu(t)$ *on* $[a,b]$*. Suppose also that* $a,b,s,t \in \mathbb{T}$*,* $a < b$*,* $f : [a,b] \to \mathbb{R}$ *is differentiable, and* ψ *is a function of* $[0,1]$ *into* $[0,1]$*. Then,*

$$\left| \left[\frac{1+\psi(1-\lambda)-\psi(\lambda)}{2} f(t) + \frac{\psi(\lambda)f(a)+(1-\psi(1-\lambda))f(b)}{2} \right] \int_a^b \nu(t)\Delta t \right.$$
$$\left. - \int_a^b \nu(s) f(\sigma(s))\Delta s \right| \leqslant M \int_a^b |K(s,t)|\Delta s,$$

where

$$K(s,t) = \begin{cases} w(s) - \left(w(a) + \psi(\lambda)\frac{w(b)-w(a)}{2} \right), & s \in [a,t), \\ w(s) - \left(w(a) + (1+\psi(1-\lambda))\frac{w(b)-w(a)}{2} \right), & s \in [t,b], \end{cases} \tag{1}$$

and $M = \sup_{a<t<b} |f^{\Delta}(t)| < \infty$.

Inspired by the ideas employed in [17, 19], here we obtain generalized Ostrowski and Ostrowski–Grüss inequalities on time scales via a parameter function. Our results are different from the ones given in [17] since we are using the generalized weighted Peano kernel (1). Furthermore, we apply our results to the continuous, discrete, and quantum cases, to obtain some interesting new inequalities. More corollaries are also obtained by considering different parameter and weight functions. In particular, we generalize and extend Theorem 1 to time scales (see Remark 20).

The paper is organized as follows. In Section 2, we provide the reader with essentials on the calculus on time scales. Our results are then stated and proved in Section 3.

2. Preliminaries on time scales

In this section, we briefly recall the theory of time scales. For further details and proofs we refer the reader to Hilger's original work [11] and to the books [8, 9].

DEFINITION 5. A time scale is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$.

Throughout this work, we assume $\mathbb{T}$ to be a time scale with the topology that is inherited from the standard topology on $\mathbb{R}$. It is also assumed throughout that in $\mathbb{T}$ the interval $[a,b]$ means the set $\{t \in \mathbb{T}\colon a \leqslant t \leqslant b\}$ for points $a < b$ in $\mathbb{T}$. Since a time scale may not be connected, we need the following concept of jump operators.

DEFINITION 6. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$, respectively.

The jump operators σ and ρ allow the classification of points in $\mathbb{T}$ as follows.

DEFINITION 7. If $\sigma(t) > t$, then t is right-scattered, while if $\rho(t) < t$, then we say that t is left-scattered. Points that are simultaneously right-scattered and left-scattered are called isolated. If $\sigma(t) = t$, then t is called right-dense, and if $\rho(t) = t$, then t is left-dense. Points that are both right-dense and left-dense are said to be dense.

DEFINITION 8. The (forward) graininess function $\mu : \mathbb{T} \to [0, \infty)$ is given by $\mu(t) = \sigma(t) - t$, $t \in \mathbb{T}$. The set $\mathbb{T}^\kappa$ is defined as follows: if $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$; otherwise, $\mathbb{T}^\kappa = \mathbb{T}$.

If $\mathbb{T} = \mathbb{R}$, then $\mu(t) \equiv 0$; if $\mathbb{T} = \mathbb{Z}$, then we have $\mu(t) \equiv 1$.

DEFINITION 9. Assume $f : \mathbb{T} \to \mathbb{R}$ and fix $t \in \mathbb{T}^\kappa$. Then the (delta) derivative $f^\Delta(t) \in \mathbb{R}$ at $t \in \mathbb{T}^\kappa$ is defined to be the number (provided it exists) with the property that given any $\varepsilon > 0$ there exists a neighborhood U of t such that

$$\left| f(\sigma(t)) - f(s) - f^\Delta(t)\,[\sigma(t) - s] \right| \leqslant \varepsilon\,|\sigma(t) - s| \quad \forall s \in U.$$

If $\mathbb{T} = \mathbb{R}$, then $f^\Delta(t) = \frac{df(t)}{dt}$; if $\mathbb{T} = \mathbb{Z}$, then $f^\Delta(t) = \Delta f(t) = f(t+1) - f(t)$.

THEOREM 10. *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^\kappa$. Then the product $fg : \mathbb{T} \to \mathbb{R}$ is differentiable at t with $(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t)$.*

DEFINITION 11. Function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it is continuous at all right-dense points $t \in \mathbb{T}$ and its left-sided limits exist at all left-dense points $t \in \mathbb{T}$. Then, we write: $f \in C_{rd}(\mathbb{T}, \mathbb{R})$.

It turns out that every rd-continuous function has an anti-derivative (see, e.g., Theorem 1.74 of [9]).

DEFINITION 12. Function $F : \mathbb{T} \to \mathbb{R}$ is a delta anti-derivative of $f : \mathbb{T} \to \mathbb{R}$ provided $F^\Delta(t) = f(t)$ for any $t \in \mathbb{T}^\kappa$. In this case, one defines the Δ-integral of f by

$$\int_a^b f(s)\Delta s := F(b) - F(a) \quad \text{for all} \quad a, b \in \mathbb{T}.$$

THEOREM 13. *Let f, g be rd-continuous, $a, b, c \in \mathbb{T}$ and $\alpha, \beta \in \mathbb{R}$. Then,*

1. $\int_a^b [\alpha f(t) + \beta g(t)]\,\Delta t = \alpha \int_a^b f(t)\Delta t + \beta \int_a^b g(t)\Delta t$,
2. $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t$,
3. $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t$,

4. $\int_a^b f(t)g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(\sigma(t))\Delta t$.

We use the following result to prove our generalized weighted Ostrowski inequality on time scales.

THEOREM 14. *If f is Δ-integrable on $[a,b]$, then so is $|f|$, and*

$$\left|\int_a^b f(t)\Delta t\right| \leqslant \int_a^b |f(t)|\Delta t.$$

We also make use of the h_k polynomials, $k \in \mathbb{N}_0$. They are defined as follows: $h_0(t,s) = 1$ for all $s,t \in \mathbb{T}$ and then, recursively, by $h_{k+1}(t,s) = \int_s^t h_k(\tau,s)\Delta\tau$, $s,t \in \mathbb{T}$.

3. Main results

For the proof of our main results (Theorems 16 and 24), we use the following useful lemma.

LEMMA 15. (See [19]) *Let $\nu : [a,b] \to [0,\infty)$ be rd-continuous and positive and $w : [a,b] \to \mathbb{R}$ be delta-differentiable such that $w^{\Delta}(t) = \nu(t)$ on $[a,b]$. Moreover, suppose also that $a,b,s,t \in \mathbb{T}$, $a < b$, $f : [a,b] \to \mathbb{R}$ is delta-differentiable, and ψ is a function of $[0,1]$ into $[0,1]$. Then the following equality holds:*

$$\left[\frac{1+\psi(1-\lambda)-\psi(\lambda)}{2}f(t) + \frac{\psi(\lambda)f(a)+(1-\psi(1-\lambda))f(b)}{2}\right]\int_a^b \nu(t)\Delta t$$
$$= \int_a^b K(s,t)f^{\Delta}(s)\Delta s + \int_a^b \nu(s)f(\sigma(s))\Delta s,$$

where $K(\cdot,\cdot)$ is given by (1).

3.1. Generalized weighted Ostrowski inequality on time scales

We now state and prove our first main result.

THEOREM 16. *Let $\nu : [a,b] \to [0,\infty)$ be rd-continuous and positive, and function $w : [a,b] \to \mathbb{R}$ be delta-differentiable such that $w^{\Delta}(t) = \nu(t)$ on $[a,b]$. Suppose also that $a,b,t,x \in \mathbb{T}$, $a < b$, $f : [a,b] \to \mathbb{R}$ is twice delta-differentiable, and ψ is a function of $[0,1]$ into $[0,1]$. Then, the inequality*

$$\begin{aligned}
&\Bigg|\Phi^2(\lambda)f(x) - \frac{1}{\left(\int_a^b \nu(t)\Delta t\right)^2}\left(\int_a^b K(t,x)\Delta t\right)\left(\int_a^b \nu(s)f^\Delta(\sigma(s))\Delta s\right) \\
&\quad + \frac{\psi(\lambda)f^\Delta(a)+(1-\psi(1-\lambda))f^\Delta(b)}{2\int_a^b \nu(t)\Delta t}\int_a^b K(t,x)\Delta t \\
&\quad - \frac{\Phi(\lambda)}{\int_a^b \nu(t)\Delta t}\int_a^b \nu(t)f(\sigma(t))\Delta t + \frac{\psi(\lambda)f(a)+(1-\psi(1-\lambda))f(b)}{2}\Phi(\lambda)\Bigg| \\
&\leqslant \frac{M}{\left(\int_a^b \nu(t)\Delta t\right)^2}\int_a^b\int_a^b |K(t,x)||K(s,t)|\Delta s\Delta t
\end{aligned} \tag{2}$$

holds for all $x \in [a,b]$ *and* $\lambda \in [0,1]$, *where*

$$\Phi(\lambda) = \frac{1+\psi(1-\lambda)-\psi(\lambda)}{2}, \tag{3}$$

$M = \sup_{a<t<b}\left|f^{\Delta\Delta}(t)\right| < \infty$, *and* $K(\cdot,\cdot)$ *is given by* (1).

Proof. With $\Phi(\lambda)$ given by (3), it follows from Lemma 15 that

$$\begin{aligned}
\Phi(\lambda)f(x) = \frac{1}{\int_a^b \nu(t)\Delta t}\int_a^b K(t,x)f^\Delta(t)\Delta t + \frac{1}{\int_a^b \nu(t)\Delta t}\int_a^b \nu(t)f(\sigma(t))\Delta t \\
- \frac{\psi(\lambda)f(a)+(1-\psi(1-\lambda))f(b)}{2}.
\end{aligned} \tag{4}$$

From (4) we get

$$\begin{aligned}
\Phi^2(\lambda)f(x) = \frac{\Phi(\lambda)}{\int_a^b \nu(t)\Delta t}\int_a^b K(t,x)f^\Delta(t)\Delta t + \frac{\Phi(\lambda)}{\int_a^b \nu(t)\Delta t}\int_a^b \nu(t)f(\sigma(t))\Delta t \\
- \frac{\psi(\lambda)f(a)+(1-\psi(1-\lambda))f(b)}{2}\Phi(\lambda)
\end{aligned} \tag{5}$$

and

$$\begin{aligned}
\Phi(\lambda)f^\Delta(t) = \frac{1}{\int_a^b \nu(t)\Delta t}\int_a^b K(s,t)f^{\Delta\Delta}(s)\Delta s + \frac{1}{\int_a^b \nu(t)\Delta t}\int_a^b \nu(s)f^\Delta(\sigma(s))\Delta s \\
- \frac{\psi(\lambda)f^\Delta(a)+(1-\psi(1-\lambda))f^\Delta(b)}{2}.
\end{aligned} \tag{6}$$

Substituting (6) into (5) results to

$$\begin{aligned}
\Phi^2(\lambda)f(x) = & \frac{1}{\int_a^b \nu(t)\Delta t}\int_a^b K(t,x)\Bigg[\frac{1}{\int_a^b \nu(t)\Delta t}\int_a^b K(s,t)f^{\Delta\Delta}(s)\Delta s \\
& + \frac{1}{\int_a^b \nu(t)\Delta t}\int_a^b \nu(s)f^{\Delta}(\sigma(s))\Delta s - \frac{\psi(\lambda)f^{\Delta}(a)+(1-\psi(1-\lambda))f^{\Delta}(b)}{2}\Bigg]\Delta t \\
& + \frac{\Phi(\lambda)}{\int_a^b \nu(t)\Delta t}\int_a^b \nu(t)f(\sigma(t))\Delta t - \frac{\psi(\lambda)f(a)+(1-\psi(1-\lambda))f(b)}{2}\Phi(\lambda) \\
= & \frac{1}{\left(\int_a^b \nu(t)\Delta t\right)^2}\left(\int_a^b K(t,x)\Delta t\right)\left(\int_a^b \nu(s)f^{\Delta}(\sigma(s))\Delta s\right) \\
& + \frac{1}{\left(\int_a^b \nu(t)\Delta t\right)^2}\int_a^b\int_a^b K(t,x)K(s,t)f^{\Delta\Delta}(s)\Delta s\Delta t \\
& - \frac{\psi(\lambda)f^{\Delta}(a)+(1-\psi(1-\lambda))f^{\Delta}(b)}{2\int_a^b \nu(t)\Delta t}\int_a^b K(t,x)\Delta t \\
& + \frac{\Phi(\lambda)}{\int_a^b \nu(t)\Delta t}\int_a^b \nu(t)f(\sigma(t))\Delta t - \frac{\psi(\lambda)f(a)+(1-\psi(1-\lambda))f(b)}{2}\Phi(\lambda).
\end{aligned} \tag{7}$$

The desired inequality (2) is obtained by rearranging (7) and applying Theorem 14. □

REMARK 17. Let $w(t)=t$. Then

$$\begin{aligned}
\int_a^b |K(s,t)|\Delta s = & h_2\left(a, a+\psi(\lambda)\frac{b-a}{2}\right) + h_2\left(t, a+\psi(\lambda)\frac{b-a}{2}\right) \\
& + h_2\left(t, a+(1+\psi(1-\lambda))\frac{b-a}{2}\right) + h_2\left(b, a+(1+\psi(1-\lambda))\frac{b-a}{2}\right)
\end{aligned}$$

and

$$\begin{aligned}
\int_a^b K(s,t)\Delta s = & h_2\left(t, a+\psi(\lambda)\frac{b-a}{2}\right) - h_2\left(a, a+\psi(\lambda)\frac{b-a}{2}\right) \\
& + h_2\left(b, a+(1+\psi(1-\lambda))\frac{b-a}{2}\right) - h_2\left(t, a+(1+\psi(1-\lambda))\frac{b-a}{2}\right)
\end{aligned}$$

hold for all $\lambda \in [0,1]$ such that $a+\psi(\lambda)\frac{b-a}{2}$ and $a+(1+\psi(1-\lambda))\frac{b-a}{2}$ are in $\mathbb{T}$ and $t \in \left[a+\psi(\lambda)\frac{b-a}{2}, a+(1+\psi(1-\lambda))\frac{b-a}{2}\right]$.

COROLLARY 18. *Let $a,b,t,x \in \mathbb{T}$, $a<b$, and $f:[a,b]\to\mathbb{R}$ be twice delta-differentiable. Then, for all $x \in [a,b]$, the following inequality holds for all $\lambda \in [0,1]$*

such that $a+\lambda\frac{b-a}{2}$ *and* $a+(2-\lambda)\frac{b-a}{2}$ *are in* $\mathbb{T}$ *and* $t\in\left[a+\lambda\frac{b-a}{2},a+(2-\lambda)\frac{b-a}{2}\right]$*:*

$$\begin{aligned}
&\Bigg|(\lambda^2-2\lambda+1)f(x)-\frac{\int_a^b f^{\Delta}(\sigma(s))\Delta s}{(b-a)^2}\left[h_2\left(x,a+\lambda\frac{b-a}{2}\right)-h_2\left(a,a+\lambda\frac{b-a}{2}\right)\right.\\
&\quad\left.+h_2\left(b,a+(2-\lambda)\frac{b-a}{2}\right)-h_2\left(x,a+(2-\lambda)\frac{b-a}{2}\right)\right]\\
&\quad+\lambda\frac{f^{\Delta}(a)+f^{\Delta}(b)}{2(b-a)}\left[h_2\left(x,a+\lambda\frac{b-a}{2}\right)-h_2\left(a,a+\lambda\frac{b-a}{2}\right)\right.\\
&\quad\left.+h_2\left(b,a+(2-\lambda)\frac{b-a}{2}\right)-h_2\left(x,a+(2-\lambda)\frac{b-a}{2}\right)\right]\\
&\quad-\frac{1-\lambda}{b-a}\int_a^b f(\sigma(t))\Delta t+\lambda(1-\lambda)\frac{f(a)+f(b)}{2}\Bigg|\\
&\leqslant\frac{M}{(b-a)^2}\int_a^b|K(t,x)|\left[h_2\left(a,a+\lambda\frac{b-a}{2}\right)+h_2\left(t,a+\lambda\frac{b-a}{2}\right)\right.\\
&\quad\left.+h_2\left(t,a+(2-\lambda)\frac{b-a}{2}\right)+h_2\left(b,a+(2-\lambda)\frac{b-a}{2}\right)\right]\Delta t,
\end{aligned}$$

where $M=\sup_{a<t<b}\left|f^{\Delta\Delta}(t)\right|<\infty$*, and*

$$K(t,x)=\begin{cases}t-\left(a+\lambda\frac{b-a}{2}\right), & t\in[a,x),\\ t-\left(a+(2-\lambda)\frac{b-a}{2}\right), & t\in[x,b].\end{cases}$$

Proof. Let $w(t)=t$ and $\psi(\lambda)=\lambda$. The result follows from Theorem 16 by using Remark 17. □

As a particular case of Corollary 18, we obtain an extension of Theorem 1 to an arbitrary time scale $\mathbb{T}$.

COROLLARY 19. *Let* $a,b,t,x\in\mathbb{T}$*,* $a<b$*, and* $f:[a,b]\to\mathbb{R}$ *be twice delta-differentiable. Then, the inequality*

$$\begin{aligned}
&\left|f(x)-\frac{\int_a^b f^{\Delta}(\sigma(s))\Delta s}{(b-a)^2}\Big[h_2(x,a)-h_2(x,b)\Big]-\frac{1}{b-a}\int_a^b f(\sigma(t))\Delta t\right|\\
&\qquad\leqslant\frac{M}{(b-a)^2}\int_a^b|K(t,x)|\Big[h_2(t,a)+h_2(t,b)\Big]\Delta t
\end{aligned}$$

holds for all $x\in[a,b]$*, where* $M=\sup_{a<t<b}\left|f^{\Delta\Delta}(t)\right|<\infty$ *and* $K(t,x)=\begin{cases}t-a, & t\in[a,x),\\ t-b, & t\in[x,b].\end{cases}$

Proof. Choose $\lambda = 0$ in Corollary 18. □

REMARK 20. Theorem 1 is obtained from Corollary 19 by choosing $\mathbb{T} = \mathbb{R}$.

To the best of our knowledge, Theorem 16 is new even when we consider particular time scales as $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$ or $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$.

COROLLARY 21. *Let $\nu : [a,b] \to [0,\infty)$ be continuous and positive and function $w : [a,b] \to \mathbb{R}$ be differentiable such that $w'(t) = \nu(t)$ on $[a,b]$. Suppose also that $a < b$, $f : [a,b] \to \mathbb{R}$ is twice differentiable, and ψ is a function of $[0,1]$ into $[0,1]$. Then, the inequality*

$$\begin{aligned}
&\Bigg|\Phi^2(\lambda)f(x) - \frac{1}{\left(\int_a^b \nu(t)dt\right)^2}\left(\int_a^b K(t,x)dt\right)\left(\int_a^b \nu(s)f'(s)ds\right) \\
&\quad + \frac{\psi(\lambda)f'(a) + (1-\psi(1-\lambda))f'(b)}{2\int_a^b \nu(t)dt}\int_a^b K(t,x)dt \\
&\quad - \frac{\Phi(\lambda)}{\int_a^b \nu(t)dt}\int_a^b \nu(t)f(t)dt + \frac{\psi(\lambda)f(a) + (1-\psi(1-\lambda))f(b)}{2}\Phi(\lambda)\Bigg| \\
&\leqslant \frac{M}{\left(\int_a^b \nu(t)dt\right)^2}\int_a^b\int_a^b |K(t,x)||K(s,t)|dsdt
\end{aligned}$$

holds for all $x \in [a,b]$ and $\lambda \in [0,1]$, where $K(\cdot,\cdot)$ is given by (1), $\Phi(\lambda)$ by (3), and $M = \sup_{a<t<b}\left|f''(t)\right| < \infty$.

Proof. Choose $\mathbb{T} = \mathbb{R}$ in Theorem 16. □

COROLLARY 22. *Let $a,b \in \mathbb{Z}$, $a < b$, $\nu : \{a,\ldots,b\} \to [0,\infty)$ be positive and function $w : \{a,\ldots,b\} \to \mathbb{R}$ be such that $\Delta w(t) = \nu(t)$, $t = a,\ldots,b-1$. Consider also the given functions $f : \{a,\ldots,b\} \to \mathbb{R}$ and $\psi : [0,1] \to [0,1]$. Then, the inequality*

$$\begin{aligned}
&\Bigg|\Phi^2(\lambda)f(x) - \frac{1}{\left(\sum_{t=a}^{b-1} \nu(t)\right)^2}\left(\sum_{t=a}^{b-1} K(t,x)\right)\left(\sum_{s=a}^{b-1} \nu(s)(f(s+2) - f(s+1))\right) \\
&\quad + \frac{\psi(\lambda)(f(a+1) - f(a)) + (1-\psi(1-\lambda))(f(b+1) - f(b))}{2\sum_{t=a}^{b-1} \nu(t)}\sum_{t=a}^{b-1} K(t,x) \\
&\quad - \frac{\Phi(\lambda)}{\sum_{t=a}^{b-1} \nu(t)}\sum_{t=a}^{b-1} \nu(t)f(t+1) + \frac{\psi(\lambda)f(a) + (1-\psi(1-\lambda))f(b)}{2}\Phi(\lambda)\Bigg| \\
&\leqslant \frac{M}{\left(\sum_{t=a}^{b-1} \nu(t)\right)^2}\sum_{t=a}^{b-1}\sum_{s=a}^{b-1} |K(t,x)||K(s,t)|
\end{aligned}$$

holds for all $x \in \{a, \dots, b\}$ *and* $\lambda \in [0,1]$, *where* $K(\cdot,\cdot)$ *is given by* (1), $\Phi(\lambda)$ *by* (3), *and* $M = \sup_{a<t<b} \left| f(t+1) - 2f(t) + f(t-1) \right| < \infty$.

Proof. Choose $\mathbb{T} = \mathbb{Z}$ in Theorem 16. $\square$

An interesting (quantum) calculus is obtained by choosing $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$ [12]. In this case, $\sigma(t) = qt$ and $f^{\Delta}(t) = D_q f(t) := \frac{f(qt)-f(t)}{(q-1)t}$. The corresponding integral is known in the literature as the $d_q t$ integral [12].

COROLLARY 23. *Let* $m,n \in \mathbb{N}$ *with* $m < n$, $\nu : [q^m, q^n] \to [0,\infty)$ *be positive, and function* $w : [q^m, q^n] \to \mathbb{R}$ *be such that* $D_q w(t) = \nu(t)$ *on* $[q^m, q^n]$. *Consider also functions* $f : [q^m, q^n] \to \mathbb{R}$ *and* $\psi : [0,1] \to [0,1]$. *Then, the inequality*

$$
\begin{aligned}
&\Bigg| \Phi^2(\lambda) f(x) - \frac{1}{\left(\int_{q^m}^{q^n} \nu(t) d_q t\right)^2} \left(\sum_{j=m}^{n-1} K(q^j, x) \right) \left(\int_{q^m}^{q^n} \nu(s) \frac{f(q^2 s) - f(qs)}{(q-1)qs} d_q s \right) \\
&\quad + \frac{q^n \psi(\lambda) \left[f(q^{m+1}) - f(q^m) \right] + q^m (1 - \psi(1-\lambda)) \left[f(q^{n+1}) - f(q^n) \right]}{2 q^{m+n} (q-1) \int_{q^m}^{q^n} \nu(t) d_q t} \sum_{j=m}^{n-1} K(q^j, x) \\
&\quad - \frac{\Phi(\lambda)}{\int_{q^m}^{q^n} \nu(t) d_q t} \int_{q^m}^{q^n} \nu(t) f(qt) d_q t + \frac{\psi(\lambda) f(q^m) + (1 - \psi(1-\lambda)) f(q^n)}{2} \Phi(\lambda) \Bigg| \\
&\leqslant \frac{M}{\left(\int_{q^m}^{q^n} \nu(t) d_q t\right)^2} \sum_{j=m}^{n-1} \sum_{i=m}^{n-1} |K(q^j, x)| |K(q^i, q^j)|
\end{aligned}
$$

holds for all $x \in [q^m, q^n]$ *and* $\lambda \in [0,1]$, *where*

$$
M = \sup_{q^m < t < q^n} \left| \frac{f(q^2 t) - (q+1) f(qt) + q f(t)}{q(q-1)^2 t^2} \right| < \infty,
$$

$$
K(q^j, x) = \begin{cases} w(q^j) - \left(w(q^m) + \psi(\lambda) \frac{w(q^n) - w(q^m)}{2} \right), & q^j \in [q^m, x), \\ w(q^j) - \left(w(q^m) + (1 + \psi(1-\lambda)) \frac{w(q^n) - w(q^m)}{2} \right), & q^j \in [x, q^n], \end{cases} \tag{8}
$$

and $\Phi(\lambda)$ *is given by* (3).

Proof. Choose $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, and $a = q^m$ and $b = q^n$, $m < n$, in Theorem 16. $\square$

3.2. Generalized weighted Ostrowski–Grüss inequality on time scales

Follows the second main result of our paper.

THEOREM 24. *Let* $\nu : [a,b] \to [0,\infty)$ *be rd-continuous and positive, and function* $w : [a,b] \to \mathbb{R}$ *be delta-differentiable such that* $w^{\Delta}(t) = \nu(t)$ *on* $[a,b]$. *Suppose also* *that $a,b,t,x \in \mathbb{T}$, $a<b$, $f:[a,b]\to\mathbb{R}$ is delta-differentiable and ψ is a function of $[0,1]$ into $[0,1]$. Then, the inequality*

$$\begin{aligned}
&\Bigg|\left[\frac{1+\psi(1-\lambda)-\psi(\lambda)}{2(b-a)}f(x)+\frac{\psi(\lambda)f(a)+(1-\psi(1-\lambda))f(b)}{2(b-a)}\right]\int_a^b \nu(t)\Delta t \\
&\quad -\frac{1}{b-a}\int_a^b \nu(t)f(\sigma(t))\Delta t-\left(\frac{f(b)-f(a)}{(b-a)^2}\int_a^b K(t,x)\Delta t\right)\Bigg| \\
&\leqslant \left[\frac{1}{b-a}\int_a^b K^2(t,x)\Delta t-\left(\frac{1}{b-a}\int_a^b K(t,x)\Delta t\right)^2\right]^{\frac{1}{2}} \\
&\quad \times\left[\frac{1}{b-a}\int_a^b \left(f^{\Delta}(t)\right)^2\Delta t-\left(\frac{1}{b-a}\int_a^b f^{\Delta}(t)\Delta t\right)^2\right]^{\frac{1}{2}}
\end{aligned} \tag{9}$$

holds for all $x\in[a,b]$ and $\lambda\in[0,1]$, where $K(\cdot,\cdot)$ is defined as in (1).

Proof. We start by making the following computations:

$$\begin{aligned}
&\int_a^b\int_a^b \Big(K(t,x)-K(s,x)\Big)\Big(f^{\Delta}(t)-f^{\Delta}(s)\Big)\Delta t\Delta s \\
&=(b-a)\int_a^b K(t,x)f^{\Delta}(t)\Delta t-\left(\int_a^b K(t,x)\Delta t\right)\left(\int_a^b f^{\Delta}(s)\Delta s\right) \\
&\quad -\left(\int_a^b K(s,x)\Delta s\right)\left(\int_a^b f^{\Delta}(t)\Delta t\right)+(b-a)\int_a^b K(s,x)f^{\Delta}(s)\Delta s \\
&=2(b-a)\int_a^b K(t,x)f^{\Delta}(t)\Delta t-2\left(\int_a^b K(t,x)\Delta t\right)\left(\int_a^b f^{\Delta}(s)\Delta s\right).
\end{aligned}$$

This implies that

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b K(t,x)f^{\Delta}(t)\Delta t-\left(\frac{1}{b-a}\int_a^b K(t,x)\Delta t\right)\left(\frac{1}{b-a}\int_a^b f^{\Delta}(s)\Delta s\right) \\
&\qquad =\frac{1}{2(b-a)^2}\int_a^b\int_a^b \Big(K(t,x)-K(s,x)\Big)\Big(f^{\Delta}(t)-f^{\Delta}(s)\Big)\Delta t\Delta s.
\end{aligned} \tag{10}$$

Following the same process, one gets

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b K^2(t,x)\Delta t-\left(\frac{1}{b-a}\int_a^b K(t,x)\Delta t\right)^2 \\
&\qquad =\frac{1}{2(b-a)^2}\int_a^b\int_a^b \Big(K(t,x)-K(s,x)\Big)^2\Delta t\Delta s
\end{aligned} \tag{11}$$

and

$$\frac{1}{b-a}\int_a^b \left(f^{\Delta}(t)\right)^2\Delta t - \left(\frac{1}{b-a}\int_a^b f^{\Delta}(t)\Delta t\right)^2 = \frac{1}{2(b-a)^2}\int_a^b\int_a^b \left(f^{\Delta}(t)-f^{\Delta}(s)\right)^2\Delta t\Delta s. \tag{12}$$

From Lemma 15, we also get that

$$\left[\frac{1+\psi(1-\lambda)-\psi(\lambda)}{2}f(x)+\frac{\psi(\lambda)f(a)+(1-\psi(1-\lambda))f(b)}{2}\right]\int_a^b \nu(t)\Delta t - \int_a^b \nu(t)f(\sigma(t))\Delta t = \int_a^b K(t,x)f^{\Delta}(t)\Delta t. \tag{13}$$

Using the Cauchy–Schwarz inequality on time scales (see [2]), we get

$$\begin{aligned}
&\left|\frac{1}{2(b-a)^2}\int_a^b\int_a^b \left(K(t,x)-K(s,x)\right)\left(f^{\Delta}(t)-f^{\Delta}(s)\right)\Delta t\Delta s\right| \\
&\leqslant \left[\frac{1}{2(b-a)^2}\int_a^b\int_a^b \left(K(t,x)-K(s,x)\right)^2\Delta t\Delta s\right]^{\frac{1}{2}} \\
&\quad\times\left[\frac{1}{2(b-a)^2}\int_a^b\int_a^b \left(f^{\Delta}(t)-f^{\Delta}(s)\right)^2\Delta t\Delta s\right]^{\frac{1}{2}}.
\end{aligned} \tag{14}$$

Inequality (9) is achieved by applying (10)–(13) and Definition 12 to (14). □

COROLLARY 25. *Let $\mathbb{T}$ be a time scale with $a,b\in\mathbb{T}$, $a<b$, and $f:[a,b]\to\mathbb{R}$ be delta-differentiable. Then the inequality*

$$\begin{aligned}
&\Bigg|\left(\frac{1-\lambda}{b-a}f(x)+\lambda\frac{f(a)+f(b)}{2(b-a)}\right)\int_a^b (\sigma(t)+t)\Delta t \\
&\qquad -\frac{1}{b-a}\int_a^b (\sigma(t)+t)f(\sigma(t))\Delta t - \left(\frac{f(b)-f(a)}{(b-a)^2}\int_a^b K(t,x)\Delta t\right)\Bigg| \\
&\leqslant \left[\frac{1}{b-a}\int_a^b K^2(t,x)\Delta t - \left(\frac{1}{b-a}\int_a^b K(t,x)\Delta t\right)^2\right]^{\frac{1}{2}} \\
&\quad\times\left[\frac{1}{b-a}\int_a^b \left(f^{\Delta}(t)\right)^2\Delta t - \left(\frac{1}{b-a}\int_a^b f^{\Delta}(t)\Delta t\right)^2\right]^{\frac{1}{2}}
\end{aligned}$$

holds for all $x\in[a,b]$ and $\lambda\in[0,1]$, where $K(\cdot,\cdot)$ is defined by

$$K(t,x)=\begin{cases} t^2-a^2-\frac{\lambda}{2}(b^2-a^2), & t\in[a,x),\\ t^2-a^2-\frac{2-\lambda}{2}(b^2-a^2), & t\in[x,b]. \end{cases} \tag{15}$$

Proof. Let $\psi(\lambda)=\lambda$ and $w(t)=t^2+c$, $c\in\mathbb{R}$, in Theorem 24. The result follows because $\nu(t)=\sigma(t)+t$ for $t\in[a,b]$. $\square$

COROLLARY 26. *Let $\mathbb{T}$ be a time scale with $a,b\in\mathbb{T}$, $a<b$, and $f:[a,b]\to\mathbb{R}$ be delta-differentiable. Then the inequality*

$$\left|\frac{1}{b-a}f(x)\int_a^b(\sigma(t)+t)\Delta t-\frac{1}{b-a}\int_a^b(\sigma(t)+t)f(\sigma(t))\Delta t-\left(\frac{f(b)-f(a)}{(b-a)^2}\int_a^b K(t,x)\Delta t\right)\right|$$
$$\leqslant\left[\frac{1}{b-a}\int_a^b K^2(t,x)\Delta t-\left(\frac{1}{b-a}\int_a^b K(t,x)\Delta t\right)^2\right]^{\frac{1}{2}}$$
$$\times\left[\frac{1}{b-a}\int_a^b\left(f^\Delta(t)\right)^2\Delta t-\left(\frac{1}{b-a}\int_a^b f^\Delta(t)\Delta t\right)^2\right]^{\frac{1}{2}}$$

holds for all $x\in[a,b]$, where $K(\cdot,\cdot)$ is defined by (15).

Proof. Choose $\lambda=0$ in the inequality of Corollary 25. $\square$

Our Theorem 24 is new even for very standard time scales.

COROLLARY 27. *If $w,f:[a,b]\to\mathbb{R}$ are differentiable, $\psi:[0,1]\to[0,1]$, then*

$$\left|\left[\frac{1+\psi(1-\lambda)-\psi(\lambda)}{2(b-a)}f(x)+\frac{\psi(\lambda)f(a)+(1-\psi(1-\lambda))f(b)}{2(b-a)}\right]\int_a^b w'(t)dt\right.$$
$$\left.-\frac{1}{b-a}\int_a^b w'(t)f(t)dt-\left(\frac{f(b)-f(a)}{(b-a)^2}\int_a^b K(t,x)dt\right)\right|$$
$$\leqslant\left[\frac{1}{b-a}\int_a^b K^2(t,x)dt-\left(\frac{1}{b-a}\int_a^b K(t,x)dt\right)^2\right]^{\frac{1}{2}}$$
$$\times\left[\frac{1}{b-a}\int_a^b\left(f'(t)\right)^2dt-\left(\frac{1}{b-a}\int_a^b f'(t)dt\right)^2\right]^{\frac{1}{2}}$$

holds for all $x\in[a,b]$ and $\lambda\in[0,1]$, where $K(\cdot,\cdot)$ is given by (1).

Proof. Choose $\mathbb{T}=\mathbb{R}$ in Theorem 24. $\square$

COROLLARY 28. *If $w,f:\{a,a+1,\dots,b-1,b\}\to\mathbb{R}$, $\psi:[0,1]\to[0,1]$, then*

$$\left|\left[\frac{1+\psi(1-\lambda)-\psi(\lambda)}{2(b-a)}f(x)+\frac{\psi(\lambda)f(a)+(1-\psi(1-\lambda))f(b)}{2(b-a)}\right]\sum_{t=a}^{b-1}\Delta w(t)\right.$$
$$\left.-\frac{1}{b-a}\sum_{t=a}^{b-1}\Delta w(t)f(t+1)-\left(\frac{f(b)-f(a)}{(b-a)^2}\sum_{t=a}^{b-1}K(t,x)\right)\right|$$

$$\leqslant \left[\frac{1}{b-a}\sum_{t=a}^{b-1}K^2(t,x)-\left(\frac{1}{b-a}\sum_{t=a}^{b-1}K(t,x)\right)^2\right]^{\frac{1}{2}}\left[\frac{1}{b-a}\sum_{t=a}^{b-1}\Big(\Delta f(t)\Big)^2-\left(\frac{1}{b-a}\sum_{t=a}^{b-1}\Delta f(t)\right)^2\right]^{\frac{1}{2}}$$

holds for all $x\in\{a,a+1,\ldots,b-1,b\}$ *and* $\lambda\in[0,1]$, *where* $K(\cdot,\cdot)$ *is given by* (1) *and* Δ *denotes the forward difference operator, that is,* $\Delta\xi(t)=\xi(t+1)-\xi(t)$.

Proof. Choose $\mathbb{T}=\mathbb{Z}$ in Theorem 24. □

COROLLARY 29. *Let* $m,n\in\mathbb{N}$ *with* $m<n$, $w,f:[q^m,q^n]\to\mathbb{R}$ *and* $\psi:[0,1]\to[0,1]$. *Then, the inequality*

$$\begin{aligned}&\Bigg|\left[\frac{1+\psi(1-\lambda)-\psi(\lambda)}{2(q^n-q^m)}f(x)+\frac{\psi(\lambda)f(q^m)+(1-\psi(1-\lambda))f(q^n)}{2(q^n-q^m)}\right]\int_{q^m}^{q^n}D_qw(t)d_qt\\&\quad-\frac{1}{q^n-q^m}\int_{q^m}^{q^n}D_qw(t)f(qt)d_qt-\left(\frac{f(b)-f(a)}{(q^n-q^m)^2}\sum_{j=m}^{n-1}K(q^j,x)\right)\Bigg|\\&\leqslant\left[\frac{1}{q^n-q^m}\sum_{j=m}^{n-1}K^2(q^j,x)-\left(\frac{1}{q^n-q^m}\sum_{j=m}^{n-1}K(q^j,x)\right)^2\right]^{\frac{1}{2}}\\&\quad\times\left[\frac{1}{q^n-q^m}\sum_{j=m}^{n-1}\left(\frac{f(q^{j+1})-f(q^j)}{(q-1)q^j}\right)^2-\left(\frac{1}{q^n-q^m}\sum_{j=m}^{n-1}\frac{f(q^{j+1})-f(q^j)}{(q-1)q^j}\right)^2\right]^{\frac{1}{2}}\end{aligned}$$

holds for all $x\in[q^m,q^n]$ *and* $\lambda\in[0,1]$ *with* $K(q^j,x)$ *given by* (8).

Proof. Let $\mathbb{T}=q^{\mathbb{N}_0}$ with $q>1$, $a=q^m$ and $b=q^n$, $m<n$. Then the result is a direct consequence of Theorem 24. □

Some other special cases of our Theorem 24 can be found in [4].

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