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MONOTONICITY AND SHARP INEQUALITIES RELATED TO GAMMA FUNCTION

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Abstract. In this paper, we investigate the monotonicity pattern of the function

$$x \mapsto \frac{\ln \Gamma(x+1)}{\ln(x^2+a) - \ln(x+a)}$$

on $(0,1)$ for $a \geqslant 1$ and resolve an open problem. From which we prove that the double inequality

$$\left(\frac{x^2+a}{x+a}\right)^{(1-\gamma)(a+1)} < \Gamma(x+1) < \left(\frac{x^2+b}{x+b}\right)^{(1-\gamma)(b+1)}$$

holds for $x \in (0,1)$ if and only if $0 < a \leqslant (1-\gamma)/(2\gamma-1)$ and $b \geqslant (\pi^2-6\gamma)/(18-12\gamma-\pi^2)$, while the double inequality

$$\left(\frac{x^2+a}{x+a}\right)^{\gamma a} < \Gamma(x+1) < \left(\frac{x^2+b}{x+b}\right)^{\gamma b}$$

holds for $x \in (0,1)$ if and only if $a \geqslant (1-\gamma)/(2\gamma-1)$ and $0 < b \leqslant 6\gamma/(\pi^2-12\gamma)$, where $\gamma = 0.577\ldots$ denotes Euler-Mascheroni's constant. These greatly improve some existing results.

1. Introduction

For $x > 0$ the classical Euler's gamma function Γ and psi (digamma) function ψ are defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \qquad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \tag{1.1}$$

respectively. The derivatives ψ', ψ'', ψ''', ... are known as polygamma functions.

There has an extensive literature on approximates for gamma function $\Gamma(x)$, similar to Stirling formula, more of which are related to x is enough large. Due to $\Gamma(x+1) = x\Gamma(x)$, in this paper, we are interested in those approximates for gamma function $\Gamma(x)$ on the interval $(0,1)$. In [7], Ivády present a very simple bound of rational functions for the gamma function on $(0,1)$. He proved that the double inequality

$$\frac{x^2+1}{x+1} < \Gamma(x+1) < \frac{x^2+2}{x+2} \tag{1.2}$$

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holds for $x\in(0,1)$, which improves some gamma function inequalities of Alzer [3], Baricz [5] and Elezović et al. [6].

Zhao et al. [16, Theorem 2] proved the function

$$Q(x)=\frac{\ln\Gamma(x+1)}{\ln(x^2+1)-\ln(x+1)}$$

is strictly increasing on $(0,1)$, with the limits

$$\lim_{x\to 0^+}Q(x)=\gamma \text{ and } \lim_{x\to 1^-}Q(x)=2(1-\gamma),$$

where $\gamma=0.577\ldots$ denotes Euler-Mascheroni's constant. It follows that the double inequality

$$\left(\frac{x^2+1}{x+1}\right)^{\alpha}<\Gamma(x+1)<\left(\frac{x^2+1}{x+1}\right)^{\beta} \tag{1.3}$$

holds on $(0,1)$ if and only if $\alpha\geqslant 2(1-\gamma)$ and $\beta\leqslant\gamma$, which clearly refines the first inequality in (1.2).

At the end of the same paper, they posted an open problem as follows.

PROBLEM 1. What is the largest number $a>1$ (or the smallest number $a<6$ respectively) for the function

$$Q_a(x)=\frac{\ln\Gamma(x+1)}{\ln(x^2+a)-\ln(x+a)} \tag{1.4}$$

to be strictly increasing (or decreasing respectively) on $(0,1)$?

Recently, the Problem 1 was solved by Kupán and Szász in [8]. They proved that Q_a is strictly increasing if and only if $a\in(0,a_{10}]$ and decreasing if and only if $a\in[a_{50},\infty)$, where

$$a_{10}=\frac{6\gamma}{\pi^2-12\gamma}\approx 1.1768 \text{ and } a_{50}=\frac{\pi^2-6\gamma}{18-\pi^2-12\gamma}\approx 5.3217.$$

As consequences, they obtained the following sharp inequalities:

$$\left(\frac{x^2+a_{10}}{x+a_{10}}\right)^{(1-\gamma)(1+a_{10})}\leqslant\Gamma(x+1)\leqslant\left(\frac{x^2+a_{10}}{x+a_{10}}\right)^{\gamma a_{10}},\ x\in[0,1], \tag{1.5}$$

$$\left(\frac{x^2+a_{50}}{x+a_{50}}\right)^{\gamma a_{50}}\leqslant\Gamma(x+1)\leqslant\left(\frac{x^2+a_{50}}{x+a_{50}}\right)^{(1-\gamma)(1+a_{50})},\ x\in[0,1]. \tag{1.6}$$

The first aim of this paper is to characterize the monotonicity pattern of the function Q_a on $(0,1)$ and give another approach to solve Problem 1. The second aim is to determine the best constants $a,b>0$ such that the double inequalities

$$\left(\frac{x^2+a}{x+a}\right)^{(1-\gamma)(1+a)}\leqslant\Gamma(x+1)\leqslant\left(\frac{x^2+b}{x+b}\right)^{(1-\gamma)(1+b)},$$

$$\left(\frac{x^2+a}{x+a}\right)^{\gamma a}\leqslant\Gamma(x+1)\leqslant\left(\frac{x^2+b}{x+b}\right)^{\gamma b}$$

hold for $x\in(0,1)$.

The paper is organized as follows. In Section 2, some lemmas as our tools are introduced. In Section 3, the monotonicity pattern of function Q_a on $(0,1)$ is described for $a\geqslant 1$, and Problem 1 is resolved. Two best double inequalities for gamma function are presented in Section 4. In the last section, we give some remarks.

2. Tools

To prove our results, we need some lemmas as tools. The first lemma is called "L'Hospital Monotone Rule" (or, for short, LMR).

LEMMA 1. ([4, Theorem 2], [10]) *For $-\infty<a<b<\infty$, let $f,g:[a,b]\to\mathbb{R}$ be continuous functions that are differentiable on (a,b), with $f(a)=g(a)=0$ or $f(b)=g(b)=0$. Assume that $g'(x)\neq 0$ for each x in (a,b). If f'/g' is increasing (decreasing) on (a,b), then so is f/g.*

The second and third lemmas are based on the auxiliary function

$$H_{f,g}:=\frac{f'}{g'}g-f, \tag{2.1}$$

and called "L'Hospital Piecewise Monotone Rules", for short, LPMR (see [13, Remark 1]).

LEMMA 2. ([11, Proposition 4.4], [13, Theorem 8]) *For $-\infty\leqslant a<b\leqslant\infty$, let f and g be differentiable functions on (a,b). Suppose that (i) $g'(x)\neq 0$ on (a,b); (ii) $f(a^+)=g(a^+)=0$; (iii) there is a $c\in(a,b)$ such that f'/g' is increasing (decreasing) on (a,c) and decreasing (increasing) on (c,b). Then*

(i) when $\operatorname{sgn}g'(x)\operatorname{sgn}H_{f,g}(b^-)\geqslant(\leqslant)0$, f/g is increasing (decreasing) on (a,b).

(ii) when $\operatorname{sgn}g'(x)\operatorname{sgn}H_{f,g}(b^-)<(>)0$, there is a unique number $x_0\in(a,b)$ such that f/g is increasing (decreasing) on (a,x_0) and decreasing (increasing) on (x_0,b).

LEMMA 3. ([11, Proposition 4.4], [13, Theorem 9]) *For $-\infty\leqslant a<b\leqslant\infty$, let f and g be differentiable functions on (a,b) with $gg'\neq 0$ on (a,b). Suppose that (i) $g'(x)\neq 0$ on (a,b); (ii) $f(b^-)=g(b^-)=0$; (iii) there is a $c\in(a,b)$ such that f'/g' is increasing (decreasing) on (a,c) and decreasing (increasing) on (c,b). Then*

(i) when $\operatorname{sgn}g'(x)\operatorname{sgn}H_{f,g}(a^+)\leqslant(\geqslant)0$, f/g is decreasing (increasing) on (a,b);

(ii) when $\operatorname{sgn}g'(x)\operatorname{sgn}H_{f,g}(a^+)>(<)0$, there is a unique number $x_0\in(a,b)$ such that f/g is increasing (decreasing) on (a,x_0) and decreasing (increasing) on (x_0,b).

The following lemma offers a simple criterion to determine the sign of a class of special polynomial on given interval contained in $(0,\infty)$ without using Descartes' Rule of Signs, which is very crucial to prove our results.

LEMMA 4. ([14, Lemma 7]) *For $n \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{0\}$ with $n > m$, let $P_n(t)$ be an n degrees polynomial defined by*

$$P_n(t) = \sum_{i=m+1}^{n} a_i t^i - \sum_{i=0}^{m} a_i t^i, \tag{2.2}$$

where $a_n, a_m > 0$, $a_i \geqslant 0$ for $0 \leqslant i \leqslant n-1$ with $i \neq m$. Then there is a unique number $t_{m+1} \in (0,\infty)$ to satisfy $P_n(t) = 0$ such that $P_n(t) < 0$ for $t \in (0, t_{m+1})$ and $P_n(t) > 0$ for $t \in (t_{m+1}, \infty)$.

Consequently, for given $t_0 > 0$, if $P_n(t_0) > 0$ then $P_n(t) > 0$ for $t \in (t_0,\infty)$ and if $P_n(t_0) < 0$ then $P_n(t) < 0$ for $t \in (0,t_0)$.

LEMMA 5. ([1, p. 260.]) *Let $x > 0$ and $n \in \mathbb{N}$. Then*

$$\psi^{(n)}(x+1) - \psi^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}}. \tag{2.3}$$

LEMMA 6. ([12, Lemma 3]) *For $n \in \mathbb{N}$, the double inequality*

$$\frac{(n-1)!}{x^n} + \frac{n!}{2x^{n+1}} < (-1)^{n+1} \psi^{(n)}(x) < \frac{(n-1)!}{x^n} + \frac{n!}{x^{n+1}}$$

holds on $(0,\infty)$. In particular, for $x > 0$, we have

$$\psi'(x) > \frac{1}{x} + \frac{1}{2x^2}, \tag{2.4}$$

$$\psi''(x) > -\frac{1}{x^2} - \frac{2}{x^3}, \tag{2.5}$$

$$\psi'''(x) > \frac{2}{x^3} + \frac{3}{x^4}. \tag{2.6}$$

3. The monotonicity pattern of $x \mapsto Q_a(x)$

Let

$$Q_a(x) = \frac{\ln\Gamma(x+1)}{\ln(x^2+a) - \ln(x+a)} := \frac{f_1(x)}{f_2(x)}. \tag{3.1}$$

Then $f_1(0) = f_2(0) = f_1(1) = f_2(1) = 0$ and

$$\lim_{x\to 0} Q_a(x) = \lim_{x\to 0} \frac{f_1(x)}{f_2(x)} = \gamma a, \quad \lim_{x\to 1} Q_a(x) = \lim_{x\to 1} \frac{f_1(x)}{f_2(x)} = (1-\gamma)(a+1).$$

Differentiation yields

$$f_1'(x) = \psi(x+1) \text{ and } f_2'(x) = \frac{x^2+2ax-a}{(x^2+a)(x+a)}. \tag{3.2}$$

Since

$$p_2(x,a) := x^2+2ax-a = \left[x+a+\sqrt{a(a+1)}\right]\left[x-\left(\sqrt{a(a+1)}-a\right)\right], \tag{3.3}$$

it is seen that $f_2'(x) < 0$ for $x \in (0, x_a)$ and $f_2'(x) > 0$ for $x \in (x_a, 1)$, where

$$x_a \equiv x(a) = \sqrt{a(a+1)} - a \in \left(\sqrt{2} - 1, 1/2\right) \text{ for } a \in [1, \infty), \tag{3.4}$$

which follows from

$$\frac{dx_a}{da} = \frac{1}{2} \frac{2a+1}{\sqrt{a(a+1)}} - 1 > 0$$

and therefore,

$$\sqrt{2} - 1 = x(1) < x_a < x(\infty) = \frac{1}{2}.$$

We now discuss the monotonicity of f_1/f_2 on $(0, x_a)$ and $(x_a, 1)$ under the condition $a \geqslant 1$. We have that for $x \neq x_a$,

$$\frac{f_1'(x)}{f_2'(x)} = \frac{\left(x^2 + a\right)(x+a)}{x^2 + 2ax - a} \psi(x+1),$$

$$\left(\frac{f_1'(x)}{f_2'(x)}\right)' = \frac{p_4(x,a)}{(x^2 + 2ax - a)^2} \psi(x+1) + \frac{\left(x^2 + a\right)(x+a)}{(x^2 + 2ax - a)} \psi'(x+1),$$

where

$$p_4(x,a) = \left(x^4 + 4ax^3 - 2a(2-a)x^2 - 4a^2 x - a^2(2a+1)\right). \tag{3.5}$$

Since the polynomial $p_4(x,a)$ satisfies the conditions for coefficients in Lemma 4 whether the coefficient of x^2 is positive or non-positive, and $p_4(1,a) = -(2a-1)(a+1)^2 < 0$, it is easily seen that $p_4(x,a) < 0$ for all $x \in (0,1)$. Then $(f_1'/f_2')'$ can be written as

$$\left(\frac{f_1'(x)}{f_2'(x)}\right)' = \frac{p_4(x,a)}{(x^2 + 2ax - a)^2} f_3(x), \quad (x \neq x_a), \tag{3.6}$$

where

$$f_3(x) = \psi(x+1) + \frac{\left(x^2 + 2ax - a\right)\left(x^2 + a\right)(x+a)}{p_4(x,a)} \psi'(x+1). \tag{3.7}$$

This implies that for $x \in (0,1)$ with $x \neq x_a$,

$$\operatorname{sgn}\left(\frac{f_1'(x)}{f_2'(x)}\right)' = -\operatorname{sgn} f_3(x). \tag{3.8}$$

A simple computation gives

$$f_3(0) = \frac{\pi^2 - 12\gamma}{6(2a+1)} \left(a - \frac{6\gamma}{\pi^2 - 12\gamma}\right) \begin{cases} > 0 \text{ if } a \in (a_{10}, \infty), \\ < 0 \text{ if } a \in (1, a_{10}), \end{cases} \tag{3.9}$$

$$f_3(1) = \frac{18 - \pi^2 - 12\gamma}{6(2a-1)} \left(a - \frac{\pi^2 - 6\gamma}{18 - \pi^2 - 12\gamma}\right) \begin{cases} > 0 \text{ if } a \in (a_{50}, \infty), \\ < 0 \text{ if } a \in (1, a_{50}), \end{cases} \tag{3.10}$$

where

$$a_{10} = \frac{6\gamma}{\pi^2 - 12\gamma} \approx 1.177 \text{ and } a_{50} = \frac{\pi^2 - 6\gamma}{18 - \pi^2 - 12\gamma} \approx 5.322. \tag{3.11}$$

Also, it was listed in [1, p. 259] that there is a $x_1 \in (0.4616321, 0.4616322)$ such that $\psi(x+1) < 0$ for $x \in (-1, x_1)$ and $\psi(x+1) > 0$ for $x \in (x_1, \infty)$. So by (3.4) there is a unique

$$a_{21} := \frac{x_1^2}{1-2x_1} \in (2.777, 2.778) \tag{3.12}$$

such that

$$f_3(x_a) = \psi(x_a+1) \begin{cases} < 0 \text{ if } a \in (1, a_{21}), \\ > 0 \text{ if } a \in (a_{21}, \infty). \end{cases} \tag{3.13}$$

Differentiation again yields

$$\begin{aligned} f_3'(x) &= \psi'(x+1) + \left(\frac{(x^2+2ax-a)(x^2+a)(x+a)}{p_4(x,a)} \right)' \psi'(x+1) \\ &\quad + \frac{(x^2+2ax-a)(x^2+a)(x+a)}{p_4(x,a)} \psi''(x+1) \\ &= 2\frac{(x^2+2ax-a)\, p_6(x,a)}{p_4(x,a)^2} \psi'(x+1) \\ &\quad + \frac{(x^2+2ax-a)(x^2+a)(x+a)}{p_4(x,a)} \psi''(x+1), \end{aligned}$$

where

$$p_6(x,a) = x^6 + 6ax^5 + 3a(2a-3)x^4 + 2a^2(a-9)x^3 - 3a^2(6a+1)x^2 - 6a^4x - a^3. \tag{3.14}$$

Similarly, whether $a \in [1, 3/2]$ or $a \in (3/2, 9)$ or $a \in [9, \infty)$, the polynomial $p_6(x,a)$ always satisfies the conditions for coefficients in Lemma 4 and $p_6(1,a) = -(6a-1)(a+1)^3 < 0$, so we get that $p_6(x,a) < 0$ for all $x \in (0,1)$. From the expression of $f_3'(x)$ we have that for $x \neq x_a$,

$$\frac{p_4(x,a)^2}{p_2(x,a)} f_3'(x) = 2p_6(x,a)\psi'(x+1) + (x^2+a)(x+a)\, p_4(x,a)\, \psi''(x+1) := f_4(x), \tag{3.15}$$

and then,

$$\operatorname{sgn} f_3'(x) = \operatorname{sgn}(x-x_a) \operatorname{sgn} f_4(x) \tag{3.16}$$

Now we deal with the monotonicity of f_4 on $(0,1)$ for $a \geqslant 1$.

LEMMA 7. *For $a \in [1, \infty)$, the function f_4 defined by (3.15) is strictly decreasing on $(0,1)$.*

Proof. (i) Differentiating for $f_4(x)$ leads to

$$f_4'(x) = l_1(x,a)\psi'(x+1) + l_2(x,a)\psi''(x+1) + l_3(x,a)\psi'''(x+1),$$

where

$$l_1(x,a) = 2\frac{\partial p_6}{\partial x} = 12\left[x^5 + 5ax^4 + 2a(2a-3)x^3 + a^2(a-9)x^2 - a^2(6a+1)x - a^4\right],$$

$$l_2(x,a) = 2p_6(x,a) + \frac{\partial}{\partial x}\left[\left(x^2+a\right)(x+a)p_4(x,a)\right] = 3\left(3x^2+2ax+a\right)p_4(x,a),$$

$$l_3(x,a) = \left(x^2+a\right)(x+a)p_4(x,a).$$

Clearly, whether $a \in [1,3/2]$ or $a \in (3/2,9)$ or $a \in [9,\infty)$, the polynomial $l_1(x,a)$ satisfies the conditions for coefficients in Lemma 4 and

$$l_1(1,a) = -12(a+1)^2\left(a^2+3a-1\right) < 0,$$

so $l_1(x,a) < 0$ for all $x \in (0,1)$. While $l_2(x,a)$, $l_3(x,a) < 0$ is due to $p_4(x,a) < 0$ for $x \in (0,1)$. Using the inequalities (2.4), (2.5) and (2.6) we have

$$f_4'(x) < l_1(x,a)\left(\frac{1}{x+1} + \frac{1}{2(x+1)^2}\right) - l_2(x,a)\left(\frac{1}{(x+1)^2} + \frac{2}{(x+1)^3}\right)$$
$$+ l_3(x,a)\left(\frac{2}{(x+1)^3} + \frac{3}{(x+1)^4}\right) := \frac{p_8(x,a)}{(x+1)^4},$$

where

$$p_8(x,a) = \sum_{k=6}^{8} b_k x^k + b_5 x^7 - \sum_{k=0}^{4} b_k x^k,$$

here

$$b_8 = 5, \quad b_7 = (28a+11), \quad b_6 = \left(18a^2+22a+21\right),$$

$$\begin{aligned} b_5 &= 4a^3 - 36a^2 - 21a + 18,\ b_4 = a\left(32a^2+136a+99\right), \\ b_3 &= a\left(96a^2+199a+108\right),\ b_2 = a^2(134a+147), \\ b_1 &= a^2\left(4a^3+20a^2+47a+18\right),\ b_0 = a^3\left(10a^2+5a-9\right). \end{aligned}$$

It is evident that all $b_k > 0$ for $0 \leqslant k \leqslant 8$ except b_5. However, whether $b_5 > 0$ or $b_5 \leqslant 0$, all coefficients of the polynomial $p_8(x,a)$ meet the conditions in Lemma 4, and

$$p_8(1,a) = -(a+1)^2\left(14a^3 - 3a^2 + 288a - 55\right) < 0.$$

By Lemma 4 we get that $p_8(x,a) < 0$ for $x \in (0,1)$. Therefore, $f_4'(x) < 0$ for $x \in (0,1)$. □

LEMMA 8. *For $a \in [1,\infty)$, let the function f_4 be defined on $(0,1)$ by (3.15).*
(i) We have $f_4(0) > 0$ for all $a \in [1,\infty)$.
(ii) We have $f_4(1) < 0$ for $a \in [1,a_8)$ and $f_4(x_a) > 0$ for $a \in (a_8,\infty)$, where

$$a_8 = \frac{3\pi^2 - 3\zeta(3) - 15 + \sqrt{3}\sqrt{3\pi^4 - 26\pi^2 - 10\pi^2\zeta(3) + 27(\zeta(3))^2 + 6\zeta(3) + 75}}{12(\zeta(3)-1)} \approx 8.953,$$

here $\zeta(s)=\sum_{n=1}^{\infty}n^{-s}$ $(s>1)$.

(iii) There is a unique $a_{22}\in(2.817,2.818)$ *such that* $f_4(x_a)<0$ *for* $a\in[1,a_{22})$ *and* $f_4(x_a)>0$ *for* $a\in(a_{22},\infty)$, *where* $x_a=\sqrt{a(a+1)}-a\in[\sqrt{2}-1,1/2)$.

Proof. (i) It is easily derived that for $a\geqslant 1$,

$$\begin{aligned} f_4(0) &= 2a^4(2a+1)\zeta(3)-\frac{1}{3}\pi^2a^3 \\ &> 2a^4(2a+1)-\frac{1}{3}\pi^2a^3=\frac{a^3}{3}\left(12a^2+6a-\pi^2\right)>0, \end{aligned}$$

here we have used the known inequality $\zeta(3)>1$.

(ii) We have

$$\begin{aligned} f_4(1) &= -2\left(\frac{\pi^2}{6}-1\right)(6a-1)(a+1)^3+2(\zeta(3)-1)(2a-1)(a+1)^4 \\ &= \frac{1}{3}(a+1)^3\left[12(\zeta(3)-1)a^2-6a\left(\pi^2-\zeta(3)-5\right)+\left(\pi^2-6\zeta(3)\right)\right]. \end{aligned}$$

It is easily known that the quadratic polynomial in the square brackets has two roots, that are $a_8\approx 8.953$ and

$$-\frac{3\zeta(3)-3\pi^2+15+\sqrt{3}\sqrt{3\pi^4-26\pi^2-10\pi^2\zeta(3)+27(\zeta(3))^2+6\zeta(3)+75}}{12\zeta(3)-12}\approx 0.122,$$

which imply the second assertion.

(iii) Direct computation yields

$$p_6(x_a,a)=-12a^3(a+1)^2\left(\sqrt{a+1}-\sqrt{a}\right)^2,$$

$$\left(x_a^2+a\right)(x_a+a)\,p_4(x_a,a)=-8a^3\sqrt{a(a+1)}(a+1)^2\left(\sqrt{a+1}-\sqrt{a}\right)^2.$$

Then we have

$$\begin{aligned} f_4(x_a) &= -24a^3(a+1)^2\left(\sqrt{a+1}-\sqrt{a}\right)^2\psi'(x_a+1) \\ &\quad -8a^3\sqrt{a(a+1)}(a+1)^2\left(\sqrt{a+1}-\sqrt{a}\right)^2\psi''(x_a+1), \end{aligned}$$

which shows that

$$\frac{f_4(x_a)}{8a^3(a+1)^2\left(\sqrt{a+1}-\sqrt{a}\right)^2}=-3\psi'(x_a+1)-\sqrt{a(a+1)}\psi''(x_a+1).$$

Due to the relation $x_a=\sqrt{a(a+1)}-a$, we obtain

$$a=\frac{x_a^2}{1-2x_a}\quad\text{and}\quad\sqrt{a(a+1)}=\frac{x_a(1-x_a)}{1-2x_a}. \tag{3.17}$$

Then we have

$$\frac{f_4(x_a)}{8a^3(a+1)^2\left(\sqrt{a+1}-\sqrt{a}\right)^2} = -3\psi'(x_a+1)-\sqrt{a(a+1)}\psi''(x_a+1) \tag{3.18}$$

$$= -3\psi'(x_a+1)-\frac{x_a(1-x_a)}{1-2x_a}\psi''(x_a+1) := \frac{f_5(x_a)}{1-2x_a},$$

where $x_a \in [\sqrt{2}-1, 1/2)$ and

$$f_5(t) = 3(2t-1)\psi'(t+1)+t(t-1)\psi''(t+1).$$

We now show that f_5 is strictly increasing on $(0,\infty)$. Differentiation leads us to

$$f_5'(t) = 6\psi'(t+1)+4(2t-1)\psi''(t+1)+t(t-1)\psi'''(t+1).$$

Utilizing the recurrence formulas (2.3) we get

$$f_5'(t+1)-f_5'(t) = 8\psi''(t)+2t\psi'''(t)+\frac{2\left(2t^4+3t^3+6t^2+6t+2\right)}{t^3(t+1)^3} := f_6(t),$$

$$f_6(t+1)-f_6(t) = 2\psi'''(t)-2\frac{2t^7+9t^6+47t^5+187t^4+378t^3+396t^2+216t+48}{t^4(t+1)^3(t+2)^3} := f_7(t),$$

$$f_7(t+1)-f_7(t) = -\frac{8\left(12t^5+123t^4+498t^3+998t^2+994t+395\right)}{(t+1)^4(t+2)^3(t+3)^3} < 0,$$

for all $t>0$. Then we have

$$f_7(t) > f_7(t+1) > \ldots > \lim_{n\to\infty} f_7(t+n) = 0,$$

which implies that

$$f_6(t) < f_6(t+1) < \ldots < \lim_{n\to\infty} f_6(t+n) = 0.$$

This in turn indicates that

$$f_5'(t) > f_5'(t+1) > \ldots > \lim_{n\to\infty} f_5'(t+n) = 0$$

for all $t>0$, that is, f_5 is strictly increasing on $(0,\infty)$.

Consequently, for $t\in(0,\infty)$, there is a unique $t_0 \in (0.462104, 0.462105)$ such that $f_5(t)<0$ for $t\in(0,t_0)$ and $f_5(t)>0$ for $t\in(t_0,\infty)$.

Thus by the relation (3.18) it is clearly seen that $f_4(x_a)<0$ for $x_a \in \left(\sqrt{2}-1, t_0\right)$ and $f_4(x_a)<0$ for $x_a\in(t_0,1/2)$, where $t_0\in(0.462104,0.462105)$ implies by (3.17) that

$$a = \frac{x_a^2}{1-2x_a} \in (2.817, 2.818).$$

This completes the proof. $\square$

Based on the monotonicity of f_4 and the signs of $f_4(0)$, $f_4(1)$ and $f_4(x_a)$, together with the relation (3.16), namely, $\operatorname{sgn} f_3'(x) = \operatorname{sgn}(x - x_a) \operatorname{sgn} f_4(x)$, we can list the monotonic pattern of f_3 on $(0,1)$ for $a \in [1,\infty)$ as follows:

Table 1: monotonicity of f_3

a	$f_4(0)$	$(0,x_a)$	$f_4(x_a)$	$(x_a,1)$	$f_4(1)$
$(1,a_{22})$	$+$	$f_3 \searrow\nearrow$	$-$	$f_3 \searrow$	$-$
a_{22}	$+$	$f_3 \searrow$	0	$f_3 \searrow$	$-$
(a_{22},a_8)	$+$	$f_3 \searrow$	$+$	$f_3 \nearrow\searrow$	$-$
a_8	$+$	$f_3 \searrow$	$+$	$f_3 \nearrow$	0
(a_8,∞)	$+$	$f_3 \searrow$	$+$	$f_3 \nearrow$	$+$

Making use of the monotonicity of f_3 given in Table 1 and $f_3(0)$, $f_3(1)$ and $f_3(x_a)$ presented in (3.9), (3.10) and (3.13), respectively, we have the following

Table 2: the signs of f_3

0	a	$f_3(0)$	$\operatorname{sgn} f_3(x)$	on $(0,x_a)$	$f_3(x_a)$	$\operatorname{sgn} f_3(x)$	on $(x_a,1)$	$f_3(1)$
1	$(1,a_{10}]$	$\leqslant 0$	$f_3 \searrow\nearrow$	f_3-	$-$	$f_3 \searrow$	f_3-	$-$
2	$(a_{10},a_{21}]$	$+$	$f_3 \searrow\nearrow$	f_3+-	$\leqslant 0$	$f_3 \searrow$	f_3-	$-$
3	(a_{21},a_{22})	$+$	$f_3 \searrow\nearrow$	$f_3+(+)+$	$+$	$f_3 \searrow$	f_3+-	$-$
4	a_{22}	$+$	$f_3 \searrow$	f_3+	$+$	$f_3 \searrow$	f_3+-	$-$
5	(a_{22},a_{50})	$+$	$f_3 \searrow$	f_3+	$+$	$f_3 \nearrow\searrow$	f_3+-	$-$
6	$[a_{50},a_8)$	$+$	$f_3 \searrow$	f_3+	$+$	$f_3 \nearrow\searrow$	f_3+	$\geqslant 0$
7	$[a_8,\infty)$	$+$	$f_3 \searrow$	f_3+	$+$	$f_3 \nearrow$	f_3+-	$+$

REMARK 1. From Table 2, when $a \in (a_{21},a_{22})$ we see that f_3 is decreasing then increasing on $(0,x_a)$ and $f_3(0)$, $f_3(x_a) > 0$. This contains two cases of sign of f_3, one of which is "$+$" on $(0,x_a)$, another one is "$+-+$". We guess that $f_3(x) > 0$ on $(0,x_a)$.

Now we are in a position to state and prove the monotonicity pattern of Q_a.

THEOREM 1. *Let $a_{10} \approx 1.177$, $a_{21} \approx 2.777$, $a_{22} \approx 2.817$, $a_{50} \approx 5.322$ be defined in (3.11), (3.12) and Lemma 8, respectively. For $a \in [1,\infty)$, let Q_a be defined on $(0,1)$ by (1.4).*

(i) The function Q_a is strictly increasing on $(0,1)$ if and only if $a \in [1,a_{10}]$, and therefore, we have

$$\left(\frac{x^2+a}{x+a}\right)^{(1-\gamma)(a+1)} < \Gamma(x+1) < \left(\frac{x^2+a}{x+a}\right)^{\gamma a} \tag{3.19}$$

holds for all $x \in (0,1)$ with the best constants $(1-\gamma)(a+1)$ and γa.

(ii) When $a \in (a_{10},a_{21}] \cup [a_{22},a_{50})$, there is a unique $x_{01} \in (0,1)$ such that Q_a is decreasing on $(0,x_{01})$ and increasing on $(x_{01},1)$, and therefore, the double inequality

$$\left(\frac{x^2+a}{x+a}\right)^{\alpha} < \Gamma(x+1) \leqslant \left(\frac{x^2+a}{x+a}\right)^{\beta} \tag{3.20}$$

holds for all $x \in (0,1)$*, where*

$$\alpha = \max\left(\gamma a, (1-\gamma)(a+1)\right) \quad \text{and} \quad \beta = \frac{\ln \Gamma(x_{01}+1)}{\ln\left(x_{01}^2+a\right) - \ln(x_{01}+a)}$$

are the best constants, here x_{01} *is the sole solution of the equation*

$$\frac{d}{dx} \frac{\ln \Gamma(x+1)}{\ln(x^2+a) - \ln(x+a)} = 0$$

on $(0,1)$*. In particular, we have*

$$\Gamma(x+1) > \left(\frac{x^2+a_{20}}{x+a_{20}}\right)^{(1-\gamma)(a_{20}+1)} = \left(\frac{x^2+a_{20}}{x+a_{20}}\right)^{\gamma a_{20}} \tag{3.21}$$

for $x \in (0,1)$*, where*

$$a_{20} = \frac{1-\gamma}{2\gamma-1} \approx 2.738. \tag{3.22}$$

(iii) The function Q_a *is strictly decreasing on* $(0,1)$ *if and only if* $a \in [a_{50}, \infty)$*, and consequently, the double inequality*

$$\left(\frac{x^2+a}{x+a}\right)^{\gamma a} < \Gamma(x+1) < \left(\frac{x^2+a}{x+a}\right)^{(1-\gamma)(a+1)} \tag{3.23}$$

holds for all $x \in (0,1)$ *with the best constants* γa *and* $(1-\gamma)(a+1)$*.*

Proof. (i) The necessary condition for the function $Q_a = f_1/f_2$ to be strictly increasing on $(0,1)$ follows from the following limit relation

$$\lim_{x \to 0^+} Q_a'(x) = -\frac{\pi^2 - 12\gamma}{12}\left(a - \frac{6\gamma}{\pi^2 - 12\gamma}\right) \geqslant 0.$$

When $a \in [1, a_{10}]$, by Line 1 in Table 2 with the sign relation (3.8), that is, $\operatorname{sgn}(f_1'/f_2')' = -\operatorname{sgn} f_3$, we have $(f_1'/f_2')' > 0$ for $x \in (0, x_a)$ and $x \in (x_a, 1)$. Note that $f_1(0^+) = f_2(0^+) = 0$, it follows from Lemma 1 that f_1/f_2 is strictly increasing on $(0, x_a)$. Similarly, since $f_1(1^-) = f_2(1^-) = 0$, by Lemma 1 it follows that f_1/f_2 is also strictly increasing on $(x_a, 1)$.

In view of the continuity of f_1/f_2 on $(0,1)$, it is obtained that f_1/f_2 is strictly increasing on $(0,1)$, which proves the sufficiency.

Therefore, we obtain

$$\gamma a = \lim_{x \to 0^+} \frac{f_1(x)}{f_2(x)} < \frac{f_1(x)}{f_2(x)} < \lim_{x \to 1^-} \frac{f_1(x)}{f_2(x)} = (1-\gamma)(a+1), \tag{3.24}$$

which is equivalent to the double inutility (3.19).

(ii) When $a \in (a_{10}, a_{21}] \cup [a_{22}, a_{50})$, we distinguish two cases:

Case 1: $a\in(a_{10},a_{21}]$. By Line 2 in Table 2 and $\operatorname{sgn}(f_1'/f_2')'=-\operatorname{sgn}f_3$, we see that there is a $x_{00a}\in(0,x_a)$ such that (f_1'/f_2') is decreasing on $(0,x_{00a})$ and increasing on (x_{00a},x_a), and increasing on $(x_a,1)$.

On the other hand, we have that

$$\begin{aligned}
H_{f_1,f_2}(x) &= \frac{f_1'(x)}{f_2'(x)}f_2(x)-f_1(x)\\
&= \frac{(x^2+a)(x+a)}{x^2+2ax-a}\psi(x+1)\ln\frac{x^2+a}{x+a}-\ln\Gamma(x+1)\\
&= \frac{(x^2+a)(x+a)}{(x-x_a)\left(x+a+\sqrt{a(a+1)}\right)}\psi(x+1)\ln\frac{x^2+a}{x+a}-\ln\Gamma(x+1)\\
&\to -[\operatorname{sgn}(x-x_a)\operatorname{sgn}\psi(x_a+1)]\infty \text{ as } x\to x_a
\end{aligned}\tag{3.25}$$

which in conjunction with the facts that $\psi(x_a+1)<0$ by (3.13) and $f_2'(x)<0$ for $x\in(0,x_a)$ lead to $\operatorname{sgn}f_2'(x)\operatorname{sgn}H_{f_1,f_2}(x_a^-)>0$. Note that $f_1(0)=f_2(0)=0$, by part (ii) of Lemma 2 it follows that there is a unique number $x_{0a}'\in(0,x_a)$ such that f_1/f_2 is decreasing on $(0,x_{0a}')$ and increasing on (x_{0a}',x_a).

Also, the increasing property of (f_1'/f_2') on $(x_a,1)$ and $f_1(1^-)=f_2(1^-)=0$ imply, by Lemma 1, that f_1/f_2 is increasing on $(x_a,1)$.

Using the continuity of f_1/f_2 at $x=x_a$, we easily see that f_1/f_2 is decreasing on $(0,x_{01})$ and increasing on $(x_{01},1)$, where $x_{01}=x_{0a}'\in(0,x_a)$. Therefore, we obtain

$$\begin{aligned}
\frac{f_1(x_{01})}{f_2(x_{01})} &< \frac{f_1(x)}{f_2(x)} < \lim_{x\to0^+}\frac{f_1(x)}{f_2(x)} = \gamma a, \text{ for } x\in(0,x_{01}),\\
\frac{f_1(x_{01})}{f_2(x_{01})} &< \frac{f_1(x)}{f_2(x)} < \lim_{x\to1^-}\frac{f_1(x)}{f_2(x)} = (1-\gamma)(a+1) \text{ for } x\in(x_{01},1),
\end{aligned}$$

that is,

$$\beta=\frac{f_1(x_{01})}{f_2(x_{01})}\leqslant\frac{f_1(x)}{f_2(x)}<\max(\gamma a,(1-\gamma)(a+1))=\alpha \text{ for } x\in(0,1),\tag{3.26}$$

which proves (3.20).

Letting $\gamma a=(1-\gamma)(a+1)$ yields $a=a_{20}=(1-\gamma)/(2\gamma-1)\approx2.738\in(a_{10},a_{21}]$, and the inequality (3.21) follows.

Case 2: $a\in[a_{22},a_{50})$. By Line 5 in Table 2 and $\operatorname{sgn}(f_1'/f_2')'=-\operatorname{sgn}f_3$, it follows that $(f_1'/f_2')'<0$ for $x\in(0,x_a)$ and there is a $x_{a11}\in(x_a,1)$ such that $(f_1'/f_2')'<0$ for $x\in(x_a,x_{a11})$ and $(f_1'/f_2')'>0$ for $x\in(x_{a11},1)$.

Analogously, applying Lemma 1 to f_1/f_2 on $(0,x_a)$ gives that f_1/f_2 is decreasing on the interval $(0,x_a)$.

On the interval $(x_a,1)$, we note that $f_1(1^-)=f_2(1^-)=0$ and (f_1'/f_2') is decreasing on (x_a,x_{a11}) and increasing on $(x_{a11},1)$. And, from (3.13), (3.2) and (3.25), we see that $\psi(x_a+1)>0$, $f_2'(x)>0$, $H_{f_1,f_2}(x_a^+)=-\infty$, and then $\operatorname{sgn}f_2'(x)\operatorname{sgn}H_{f_1,f_2}(x_a^+)<0$. By part (ii) of Lemma 3 it is derived that there is a unique number $x_{a1}'\in(x_a,1)$ such that f_1/f_2 is decreasing on (x_a,x_{a1}') and increasing on $(x_{a1}',1)$.

Thus we get that f_1/f_2 is decreasing on $(0,x_{01})$ and increasing on $(x_{01},1)$, where $x_{01}=x'_{a1}\in(x_a,1)$, and the double inequality (3.20) follows similarly.

(iii) The necessary condition for the function Q_a to be strictly decreasing on $(0,1)$ can be deduced by the inequality

$$\lim_{x\to 1^-} Q'_a(x) = -\frac{18-\pi^2-12\gamma}{12}\left(a-\frac{\pi^2-6\gamma}{(18-\pi^2-12\gamma)}\right)\leqslant 0,$$

which implies $a\geqslant a_{50}$.

If $a\in[a_{50},\infty)$, then it follows from Lines 6 and 7 in Table 2 that (f'_1/f'_2) is decreasing on $(0,x_a)$ and $(x_a,1)$. Applying Lemma 1 to the ratio f_1/f_2 on the two intervals and noting that f_1/f_2 is continuous at $x=x_a$, we conclude that f_1/f_2 is decreasing on $(0,1)$, and the sufficiency follows. And, the double inequality (3.24) is reversed, that is, inequality (3.23) holds true for $x\in(0,1)$.

This completes the proof. □

4. Sharp approximations for gamma function

For latter use, the following lemma is needed.

LEMMA 9. *For $a\in(0,\infty)$ and $x\in(0,1)$, the functions*

$$A(x,a)=\left(\frac{x^2+a}{x+a}\right)^{a+1} \quad \text{and} \quad B(x,a)=\left(\frac{x^2+a}{x+a}\right)^{a}$$

are increasing and decreasing with respect to a on $(0,\infty)$, respectively, and we have

$$\lim_{a\to\infty} A(x,a)=\lim_{a\to\infty} B(x,a)=e^{-x(1-x)}.$$

Proof. Logarithm differentiation yields

$$\frac{\partial \ln A}{\partial a} = \ln\frac{x^2+a}{x+a}+(a+1)\left(\frac{1}{x^2+a}-\frac{1}{x+a}\right),$$

$$\frac{\partial^2 \ln A}{\partial a^2} = -x(x-1)^2\frac{2x^2+(a+1)x+2a}{(x^2+a)^2(x+a)^2}<0,$$

which shows that $a\mapsto \partial A/\partial a$ is decreasing on $(0,\infty)$. Hence, it is derived that

$$\frac{\partial \ln A}{\partial a} > \lim_{a\to\infty}\frac{\partial \ln A}{\partial a}=0,$$

namely, A is increasing in a on $(0,\infty)$.

Similarly, we have

$$\frac{\partial \ln B}{\partial a} = \ln\frac{x^2+a}{x+a}+a\left(\frac{1}{x^2+a}-\frac{1}{x+a}\right),$$

$$\frac{\partial^2 \ln B}{\partial a^2} = x^2 (1-x) \frac{2x^2 + ax + a}{(x^2+a)^2 (x+a)^2} > 0.$$

It follows that

$$\frac{\partial \ln B}{\partial a} < \lim_{a\to\infty} \frac{\partial \ln B}{\partial a} = 0,$$

which proves the monotonicity of B with respect to a on $(0,\infty)$.

Straightforward computation leads to the desired limits. □

THEOREM 2. *For $a,b \in (0,\infty)$, the double inequality*

$$\left(\frac{x^2+a}{x+a}\right)^{(1-\gamma)(a+1)} < \Gamma(x+1) < \left(\frac{x^2+b}{x+b}\right)^{(1-\gamma)(b+1)} \tag{4.1}$$

holds for all $x \in (0,1)$ if and only if

$$0 < a \leqslant a_{20} = \frac{1-\gamma}{2\gamma-1} \approx 2.738 \text{ and } b \geqslant a_{50} = \frac{\pi^2 - 6\gamma}{18 - 12\gamma - \pi^2} \approx 5.322.$$

Proof. (i) The necessity for the first inequality in (4.1) can be obtained from the limit relation

$$\lim_{x\to 0+} \frac{\ln\Gamma(x+1) - (1-\gamma)(a+1)\ln\frac{x^2+a}{x+a}}{x} = -(2\gamma-1)\left(a - \frac{1-\gamma}{2\gamma-1}\right) \geqslant 0.$$

If $a \leqslant a_{20}$, then the inequality (3.21) and the increasing property of $a \mapsto A(x,a)^{1-\gamma}$ by Lemma 9 reveal the sufficiency.

(ii) If the second inequality in (4.1) holds for all $x \in (0,1)$, then we have

$$\lim_{x\to 1-} \frac{\ln\Gamma(x+1) - (1-\gamma)(b+1)\ln\frac{x^2+b}{x+b}}{(1-x)^2} = -\left(18-12\gamma-\pi^2\right)\left(b - \frac{\pi^2-6\gamma}{18-12\gamma-\pi^2}\right) \leqslant 0.$$

Solving the inequality for b yields $b \geqslant a_{50} = \left(\pi^2 - 6\gamma\right)/\left(18-12\gamma-\pi^2\right)$, which proves the necessity.

The sufficiency follows from part (iii) of Theorem 1.

The proof ends. □

Taking $a = 0,\ 1,\ \left(1-\gamma^2\right)/\gamma \approx 1.155,\ \gamma/(1-\gamma) \approx 1.365,\ 2,\ (\gamma+2)/(1-\gamma) \approx 2.548$ and $b = a_{51} = 2(1-\gamma)/(2\gamma-1) \approx 5.475,\ 6, \infty$ in Theorem 2 and using Lemma 9 we have the following corollary.

COROLLARY 1. *For $x \in (0,1)$, we have*

$$x^{1-\gamma} < \left(\frac{x^2+1}{x+1}\right)^{2(1-\gamma)} < \left(\frac{x^2+1/\gamma}{x+1/\gamma}\right)^{\left(1-\gamma^2\right)/\gamma} < \frac{x^2+\gamma/(1-\gamma)}{x+\gamma/(1-\gamma)}$$

$$
\begin{aligned}
&< \left(\frac{x^2+2}{x+2}\right)^{3(1-\gamma)} < \left(\frac{x^2+(\gamma+2)/(1-\gamma)}{x+(\gamma+2)/(1-\gamma)}\right)^{3/2} < \Gamma(x+1) \qquad (4.2)\\
&< \left(\frac{x^2+2(1-\gamma)/(2\gamma-1)}{x+2(1-\gamma)/(2\gamma-1)}\right)^{(1-\gamma)/(2\gamma-1)} < \left(\frac{x^2+6}{x+6}\right)^{7(1-\gamma)} < e^{-(1-\gamma)\left(x-x^2\right)}.
\end{aligned}
$$

For ease of use, sometimes we prefer certain simpler bounds for the gamma function, such as $\left(x^2+a\right)/(x+a)$.

THEOREM 3. *For $a \in (0,\infty)$, the inequality*

$$\Gamma(x+1) > \frac{x^2+a}{x+a} \tag{4.3}$$

holds for all $x \in (0,1)$ if and only if $0 < a \leqslant a_{11} = \gamma/(1-\gamma) \approx 1.365$.

Proof. The necessity follows from the limit relation

$$\lim_{x\to 1^-} \frac{\ln\Gamma(x+1) - \ln\frac{x^2+a}{x+a}}{1-x} = \gamma - 1 + \frac{1}{a+1} \geqslant 0.$$

Since for $x \in (0,1)$,

$$\frac{\partial}{\partial a}\frac{x^2+a}{x+a} = \frac{x(1-x)}{(x+a)^2} > 0,$$

to prove the sufficiency, it suffices to prove the inequality (4.4) holds when $a = a_{11}$, which follows from (4.3).

This completes the proof. □

REMARK 2. Due to

$$\lim_{x\to 0^+} \frac{\ln\Gamma(x+1) - \ln\frac{x^2+a}{x+a}}{x} = \frac{1}{a} - \gamma,$$

a possible best constant such that the reverse inequality (4.3) holds is $a = 1/\gamma \approx 1.732$. However, this guess is not valid.

THEOREM 4. *For $a,b \in (0,\infty)$, the double inequality*

$$\left(\frac{x^2+a}{x+a}\right)^{\gamma a} < \Gamma(x+1) < \left(\frac{x^2+b}{x+b}\right)^{\gamma b} \tag{4.4}$$

holds for all $x \in (0,1)$ if and only if

$$a \geqslant a_{20} = \frac{1-\gamma}{2\gamma-1} \approx 2.738 \ \text{ and } \ 0 < b \leqslant a_{10} = \frac{6\gamma}{\pi^2 - 12\gamma} \approx 1.177.$$

Proof. (i) The necessary condition for the first inequality in (4.4) follows from the limit relation

$$\lim_{x\to 1}\frac{\ln\Gamma(x+1)-\gamma a\ln\frac{x^2+a}{x+a}}{1-x}=\frac{2\gamma-1}{a+1}\left(a-\frac{1-\gamma}{2\gamma-1}\right)\geqslant 0.$$

If $a\geqslant a_{20}$, then the inequality (3.21) and the decreasing property of $a\mapsto B(x,a)^{\gamma}$ given in Lemma 9 imply the sufficiency.

(ii) Solving the inequality

$$\lim_{x\to 0}\frac{\ln\Gamma(x+1)-\gamma b\ln\frac{x^2+b}{x+b}}{x^2}=\frac{\pi^2-12\gamma}{12b}\left(b-\frac{6\gamma}{\pi^2-12\gamma}\right)\leqslant 0$$

for b gives $b\leqslant a_{10}=6\gamma/\left(\pi^2-12\gamma\right)$, which yields the necessity.

The sufficiency follows from part (i) of Theorem 1 and the decreasing property of $a\mapsto B(x,a)^{\gamma}$ on $(0,\infty)$.

Thus we finish the proof. □

Taking $a=\infty,2/\gamma\approx 3.465,3$ and $b=1,1/(2\gamma)\approx 0.866$ in Theorem 4 and using Lemma 9 give the following corollary.

COROLLARY 2. *For $x\in(0,1)$, we have*

$$e^{-\gamma\left(x-x^2\right)}<\left(\frac{x^2+2/\gamma}{x+2/\gamma}\right)^2<\left(\frac{x^2+3}{x+3}\right)^{3\gamma}<\Gamma(x+1)<\left(\frac{x^2+1}{x+1}\right)^{\gamma}<\sqrt{\frac{2\gamma x^2+1}{2\gamma x+1}}. \tag{4.5}$$

COROLLARY 3. *For $a\geqslant a_{20}=(1-\gamma)/(2\gamma-1)\approx 2.738$, we have*

$$\Gamma(x+1)>2^{\gamma a}\left(\sqrt{a(a+1)}-a\right)^{\gamma a}$$

for $x\in(0,1)$. Moreover, the lower bound is decreasing with respect to a, and

$$\lim_{a\to\infty}2^{\gamma a}\left(\sqrt{a(a+1)}-a\right)^{\gamma a}=e^{-\gamma/4}.$$

In particular, we have that for $x\in(0,1)$,

$$\Gamma(x+1)>\left(2\frac{\sqrt{\gamma(1-\gamma)}-(1-\gamma)}{2\gamma-1}\right)^{\gamma(1-\gamma)/(2\gamma-1)}\approx 0.880>e^{-\gamma/4}\approx 0.866.$$

Proof. As shown previously, for $f_2(x)=\ln\left(x^2+a\right)-\ln(x+a)$ we have $f_2'(x)<0$ for $x\in(0,x_a)$ and $f_2'(x)>0$ for $x\in(x_a,1)$, where $x_a=\sqrt{a(a+1)}-a$. Hence, we get that

$$\frac{x^2+a}{x+a}\geqslant\frac{x_a^2+a}{x_a+a}=2\left(\sqrt{a(a+1)}-a\right).$$

It follows from Theorem 2 that for $a \geqslant a_{20}$,

$$\Gamma(x+1) > \left(\frac{x^2+a}{x+a}\right)^{\gamma a} \geqslant 2^{\gamma a}\left(\sqrt{a(a+1)}-a\right)^{\gamma a} := h(a).$$

Logarithm differentiation leads us to

$$\begin{aligned}(\ln h(a))' &= \gamma \ln 2 + \gamma \ln\left(\sqrt{a(a+1)}-a\right) + \frac{\gamma}{2}\frac{\sqrt{a+1}-\sqrt{a}}{\sqrt{a+1}},\\ (\ln h(a))'' &= \frac{\gamma}{4a(a+1)\sqrt{a^2+a}}\left(2(a+1)\sqrt{a^2+a}-a(2a+3)\right) > 0,\end{aligned}$$

where the inequality holds due to

$$\left[2(a+1)\sqrt{a^2+a}\right]^2 - [a(2a+3)]^2 = a(3a+4) > 0.$$

Therefore, it is derived that

$$(\ln h(a))' < \lim_{a\to\infty}(\ln h(a))' = 0,$$

which proves the corollary. □

REMARK 3. The above corollary gives a constant lower bound for gamma function $\Gamma(x+1)$ on $(0,1)$.

5. Comparisons and remarks

Lastly, we compare our results with some known inequalities.

PROPOSITION 1. *For $x \in (0,1)$, the inequalities*

$$\frac{x^2+1}{x+1} < \left(\frac{x^2+a}{x+a}\right)^{(1-\gamma)/(a+1)} < \Gamma(x+1) < \left(\frac{x^2+b}{x+b}\right)^{(1-\gamma)/(b+1)} < \frac{x^2+2}{x+2}$$

hold if and only if $a \in [a_{01}, a_{20}]$ and $b \in [a_{50}, a_{51}]$, where $a_{01} = (1-\gamma)/\gamma \approx 0.732$, $a_{50} \approx 5.322$ is given in (3.11) and $a_{51} = 2(1-\gamma)/(2\gamma-1) \approx 5.475$.

Proof. By Theorem 2, it is enough to prove the first and last inequalities hold for $x \in (0,1)$ if and only if $a \geqslant a_{01}$ and $b \leqslant a_{51}$, respectively. Denote by

$$\begin{aligned}g_1(x) &:= (1-\gamma)(a+1)\ln\frac{x^2+a}{x+a} - \ln\frac{x^2+1}{x+1},\\ g_2(x) &:= (1-\gamma)(b+1)\ln\frac{x^2+b}{x+b} - \ln\frac{x^2+2}{x+2}.\end{aligned}$$

(i) If the first inequality holds for $x \in (0,1)$, then we have

$$\lim_{x \to 0^+} \frac{g_1(x)}{x} = \frac{a\gamma - (1-\gamma)}{a} \geqslant 0,$$

which implies that $a \geqslant (1-\gamma)/\gamma = a_{01}$.

Since $a \mapsto A(x,a)^{1-\gamma}$ is increasing on $(0,\infty)$, to prove sufficiency, it suffices to prove $g_1(x) > 0$ for $x \in (0,1)$ when $a = a_{01}$. Differentiation yields

$$g_1(x) = -\frac{(2\gamma-1)x}{\gamma^2} \frac{q_1(x)}{(x^2+1)(x+1)(x^2+a_{01})(x+a_{01})},$$

where

$$q_1(x) = \gamma x^4 + 2x^3 - (2\gamma-1)x^2 - (1-\gamma).$$

Since the polynomial $q_1(x)$ satisfies the conditions for coefficients in Lemma 4 and $q_1(1) = 2 > 0$, so there is a $x_0 \in (0,1)$ such that $q_1(x) < 0$ for $x \in (0,x_0)$ and $q_1(x) > 0$ for $x \in (x_0,1)$. This reveals that g_1 is increasing on $(0,x_0)$ and decreasing on $(x_0,1)$, which leads to $g_1(x) > \min(g_1(0), g_1(1)) = 0$.

(ii) If the last inequality holds for $x \in (0,1)$, then we have

$$\lim_{x \to 0^+} \frac{g_2(x)}{x} = \frac{(2\gamma-1)b - 2(1-\gamma)}{2b} \leqslant 0,$$

which reveals that $b \leqslant 2(1-\gamma)/(2\gamma-1) = a_{51}$. Similarly, it suffices to prove $g_2(x) < 0$ for $x \in (0,1)$ when $b = a_{51}$. Differentiation yields

$$g_2(x) = \frac{(2-3\gamma)x}{(2\gamma-1)^2} \frac{q_2(x)}{(x^2+2)(x+2)(x^2+a_{51})(x+a_{51})},$$

where

$$q_2(x) = (2\gamma-1)x^4 + 4\gamma x^3 + 2(4-5\gamma)x^2 - 4(1-\gamma).$$

Due to the polynomial $q_2(x)$ satisfies the conditions for coefficients in Lemma 4 and $q_2(1) = 3 > 0$, so there is a $x_0 \in (0,1)$ such that $q_2(x) < 0$ for $x \in (0,x_0)$ and $q_2(x) > 0$ for $x \in (x_0,1)$. This reveals that g_2 is decreasing on $(0,x_0)$ and increasing on $(x_0,1)$, which leads to $g_2(x) < \max(g_2(0), g_2(1)) = 0$.

This completes the proof. □

REMARK 4. From Corollaries 1 and 2 and Proposition 1, it is easily seen that our Theorems 1 and 2 are refinements of Ivády's and Zhao et al.'s results.

Alzer [2, Theorem 2] proved that

$$x^{\theta(x-1)-\gamma} < \Gamma(x) < x^{\vartheta(x-1)-\gamma} \tag{5.1}$$

with the best constants

$$\theta = 1-\gamma \approx 0.423 \text{ and } \vartheta = \frac{1}{2}\left(\frac{\pi^2}{6} - \gamma\right) \approx 0.534, \tag{5.2}$$

and that if $x \in (1,\infty)$, then (5.1) holds with the best possible constants

$$\theta = \frac{1}{2}\left(\frac{\pi^2}{6} - \gamma\right) \text{ and } \vartheta = 1.$$

PROPOSITION 2. *For $x \in (0,1)$, the inequalities*

$$x^{(1-\gamma)x} < \left(\frac{x^2+a}{x+a}\right)^{(1-\gamma)(a+1)} < \Gamma(x+1) < \left(\frac{x^2+b}{x+b}\right)^{(1-\gamma)(b+1)} < x^{\vartheta(x-1)+1-\gamma}$$

hold if and only if $a \in [2, a_{20}]$ and $b = a_{50}$, where $a_{20} \approx 2.738$ and $a_{50} \approx 5.322$ are as in Theorem 2, ϑ is given in (5.2).

Proof. By Theorem 2, it suffices to prove the first inequality and the last one hold if and only if $a \geqslant 2$ and $b \leqslant a_{50}$. For $x \in (0,1)$, let us define

$$g_3(x) = (a+1)\ln\frac{x^2+a}{x+a} - x\ln x,$$
$$g_4(x) = (1-\gamma)(b+1)\ln\frac{x^2+b}{x+b} - (\vartheta(x-1)+1-\gamma)\ln x.$$

(i) We first prove that the first inequality holds for $x \in (0,1)$ if and only if $a \geqslant 2$. The necessity follows from the limit relation

$$\lim_{x\to 1^-}\frac{g_3(x)}{(1-x)^2} = \frac{1}{2}\frac{a-2}{a+1} \geqslant 0.$$

Since $a \mapsto A(x,a)$ is increasing on $(0,\infty)$ by Lemma 9, to prove the sufficiency, it is enough to prove $g_3(x) > 0$ for $x \in (0,1)$ when $a = 2$. Differentiation yields

$$g_3'(x) = \frac{3\left(x^2+4x-2\right)}{(x^2+2)(x+2)} - \ln x - 1,$$
$$g_3''(x) = \frac{1-x}{x(x^2+2)^2(x+2)^2} q_3(x),$$

where

$$q_3(x) = x^5 + 8x^4 + 40x^3 + 56x^2 + 28x - 16.$$

Since $q_3(1) = 117 > 0$, by Lemma 4 it is deduced that there is a $x_0 \in (0,1)$ such that $q_3(x) < 0$ for $x \in (0,x_0)$ and $q_3(x) > 0$ for $x \in (x_0,1)$, which implies that g_3' is decreasing on $(0,x_0)$ and increasing on $(x_0,1)$. It follows that $g_3'(x) < g_3'(1) = 0$ for $x \in (x_0,1)$, which together with $\lim_{x\to 0^+} g_3'(x) = \infty$ yields that there is a $x_1 \in (0,x_0)$ such that $g_3'(x) > 0$ for $x \in (0,x_1)$ and $g_3'(x) < 0$ for $x \in (x_1,1)$. Therefore, we conclude that $g_3(x) > \min(g_3(0^+), g_3(1^-)) = 0$, which proves the sufficiency.

(ii) We now prove that the last inequality holds for $x \in (0,1)$ if and only if $b \leqslant a_{50}$. The necessity can be obtained by solving the inequality

$$\lim_{x\to 1^-}\frac{g_4(x)}{(1-x)^2} = -\vartheta + \frac{3b(1-\gamma)}{2(b+1)} \leqslant 0$$

for b, which gives $b \leqslant 2\vartheta/(3-2\vartheta-3\gamma) = a_{50}$.

Next we prove $g_4(x) < 0$ for $x \in (0,1)$ when $b = a_{50}$ or $\vartheta = 3b(1-\gamma)/(2b+2)$. Differentiation yields

$$\begin{aligned}\frac{g_4'(x)}{1-\gamma} &= (b+1)\frac{\left(x^2+2bx-b\right)}{(x^2+b)(x+b)} + \frac{\gamma+\vartheta-1}{x(1-\gamma)} - \frac{\vartheta}{1-\gamma} - \frac{\vartheta}{1-\gamma}\ln x \\ &= (b+1)\frac{\left(x^2+2bx-b\right)}{(x^2+b)(x+b)} + \frac{b-2}{2(b+1)}\frac{1}{x} - \frac{3b}{2(b+1)} - \frac{3b}{2(b+1)}\ln x, \\ \frac{g_4''(x)}{1-\gamma} &= \frac{b}{2(b+1)}\frac{1-x}{x^2(x^2+b)^2(x+b)^2}q_4(x),\end{aligned}$$

where

$$\begin{aligned}q_4(x) &= 3x^6 + 8(b+1)x^5 + \left(11b^2+32b+12\right)x^4 + 4b\left(b^2+6b+5\right)x^3 \\ &\quad + b\left(2b^2+15b+4\right)x^2 - 4b^2\left(b^2-1\right)x - b^3(b-2).\end{aligned}$$

Clearly, for $b = a_{50} > 5$, the polynomial $q_4(x)$ satisfies the conditions for coefficients in Lemma 4 and $q_4(1) = -(5b-23)(b+1)^3 < 0$, hence we have $q_4(x) < 0$ for $x \in (0,1)$. This implies that $g_4''(x) < 0$, which in turn indicates that $g_4'(x) > g_4'(1) = 0$ for $x \in (0,1)$, that is, g_4 is strictly increasing on $(0,1)$. Then we obtain $g_4(x) < g_4(1) = 0$ for $x \in (0,1)$.

The proposition is proved. □

Recently, Laforgia and Natalini [9, Theorem 2.1] presented a new lower bound for gamma function, which states that for $0 \leqslant x \leqslant 1$,

$$\Gamma(x+1) \geqslant e^{(1-\gamma)(x-1)}.$$

Now we prove

PROPOSITION 3. *The inequalities*

$$\Gamma(x+1) > \left(\frac{x^2+a}{x+a}\right)^{(1-\gamma)(a+1)} > e^{(1-\gamma)(x-1)}$$

hold for $x \in (0,1)$ if and only if $a \in [1/2, a_{20}]$, where $a_{20} \approx 2.738$ is as in Theorem 2.

Proof. By Theorem 2, it suffices to prove the second inequality holds for $x \in (0,1)$ if and only if $a \geqslant 1/2$. For $x \in (0,1)$, let g_5 be defined by

$$g_5(x) = (a+1)\ln\frac{x^2+a}{x+a} - (x-1).$$

The necessity follows from

$$\lim_{x\to 1-}\frac{g_5(x)}{(1-x)^2} = \frac{1}{2}\frac{2a-1}{a+1} \geqslant 0.$$

Similarly, to prove sufficiency, it is enough to prove $g_5(x) < 0$ for $x \in (0,1)$ when $a = 1/2$. Differentiation leads to

$$g_5'(x) = -\frac{4(x-1)^2(x+1)}{(2x+1)(2x^2+1)} < 0,$$

which yields $g_5(x) > g_5(1) = 0$. $\square$

REMARK 5. Clearly, our Theorem 2 is stronger than Alzer's and Laforgia and Natalini's results.

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