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SCHUR CONVEX FUNCTIONS AND THE BONNESEN STYLE ISOPERIMETRIC INEQUALITIES FOR PLANAR CONVEX POLYGONS

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Abstract. In this note, we continue to investigate Bonnesen-type isoperimetric inequalities for planar convex polygons. We shall first establish some analytic isoperimetric inequalities for a special class of Schur convex functions. Subsequently, by using these analytic isoperimetric inequalities, Bonnesen-type isoperimetric inequalities and related inverse inequalities for the planar convex polygons are obtained.

1. Introduction

Schur convex functions [4] play an important role in the study of analytic inequalities and geometric inequalities. Let us recall some notions and lemmas.

Let $I \subset \boldsymbol{R}$ and $I^n = I \times I \times \cdots \times I$ (n copies).

LEMMA 1.1. ([13]) *An $n \times n$ matrix $S = [s_{ij}]$ is said to be a doubly stochastic matrix if $s_{ij} \geqslant 0$ for $1 \leqslant i < j \leqslant n$, and*

$$\sum_{j=1}^{n} s_{ij} = 1, \quad i = 1,2\cdots,n; \quad \sum_{i=1}^{n} s_{ij} = 1, \quad j = 1,2,\cdots,n.$$

LEMMA 1.2. ([13])
(1). A permutation matrix is a doubly stochastic matrix.
(2). $S = [s_{ij}]$ with $s_{ij} = \frac{1}{n}$, $1 \leqslant i, j \leqslant n$, is a doubly stochastic matrix.

LEMMA 1.3. ([13]) *A real function $f : I^n \to \boldsymbol{R}$ $(n > 1)$ is called to be Schur convex function if for any doubly stochastic matrix S and all $\boldsymbol{x} \in I^n$, $f(S\boldsymbol{x}) \leqslant f(\boldsymbol{x})$. It is called to be strictly Schur convex if inequality is strict. f is said to be Schur concave (resp. strictly Schur concave) if $-f$ is Schur convex.*

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LEMMA 1.4. ([4]) *Let $\Omega \in \boldsymbol{R}^n$ be symmetric and convex set with nonempty interior, and let $f : \Omega \to \boldsymbol{R}$ be differentiable in the interior of Ω. Then f is Schur convex (Schur concave) on Ω if and only if f is symmetric on Ω and*

$$(x_1 - x_2)\left(\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2}\right) \geqslant 0 (\leqslant 0) \qquad \text{for all} \quad x \in \Omega^0,$$

where Ω^0 is the interior of Ω.

The above definitions and example can be found in many references such as [4] and [14].

The classical isoperimetric inequality states that for a domain K with the boundary composing of the simple curve $\mathscr{C}$ of length L and area A

$$L^2 - 4\pi A \geqslant 0, \tag{1.1}$$

where equality holds if K is a circle. The isoperimetric deficit of K is defined as $\Delta(K) = L^2 - 4\pi A$. Bonnesen in [8] gave a low bound for the isoperimetric deficit $\Delta(K)$, as follows

$$\Delta(K) = L^2 - 4\pi A \geqslant \pi^2 (R - r)^2,$$

where R is the circumradius and r is the inradius of the curve $\mathscr{C}$.

Later Bonnesen proved a series of inequalities of the form

$$\Delta(K) = L^2 - 4\pi A \geqslant B,$$

where the equality B is an invariant of geometric significance having the following basic properties:

1. B is non-negative;
2. B is vanish only when K is a disc.

Many Bs are discovered in the last century and mathematicians are still working on those unknown invariants of geometric significance. See references [1, 2, 3, 6, 7, 8, 9, 10] for more details.

Here are some of the different forms of Bonnesen-style isoperimetric inequality.

$$L^2 - 4\pi A \geqslant 4\pi d^2; \qquad L^2 - 4\pi A \geqslant \pi^2 (r_e - r_i)^2;$$

$$L^2 - 4\pi A \geqslant (L - 2\pi r_i)^2; \qquad L^2 - 4\pi A \geqslant (L - 2\pi r_e)^2;$$

$$L^2 - 4\pi A \geqslant (\frac{A}{r} - \pi r)^2; \qquad L^2 - 4\pi A \geqslant L^2 \left(\frac{r_e - r_i}{r_e + r_i}\right)^2;$$

$$L^2 - 4\pi A \geqslant A^2 \left(\frac{1}{r_i} - \frac{1}{r_e}\right)^2; \qquad L^2 - 4\pi A \geqslant A^2 \left(\frac{1}{r} - \frac{1}{r_e}\right)^2.$$

It is difficult to compare those isoperimetric deficit lower bounds and to determine which lower bound is the best.

However, the literature on the study of Bonnesen-type isoperimetric inequalities for planar convex polygon is relatively less (see [5, 11, 12, 13]).

In 1998, Zhang [13] proved a form of Bonnesen-style isoperimetric inequality for planar convex polygon, as follows.

Let $\mathscr{C}_n$ be an n-sided plane convex polygon *inscribed in a circle* of radius R with side-length a_i $(i=1,2,\cdots,n)$ and perimeter L_n, enclosing a domain of area A_n.

$$(L_n)^2 - 4n\tan\frac{\pi}{n}A_n \geqslant \left[L_n - L_n^*\right]^2. \tag{1.2}$$

where L_n^* is the perimeter of the regular convex n-sides polygon *inscribed in the same circle* with $\mathscr{C}_n$.

In 2015, L. Ma [5] obtained a new Bonnesen-style inequality for planar convex polygon

$$(L_n)^2 - 4n\tan\frac{\pi}{n}A_n \geqslant \frac{1}{R^2}\left[A_n - A_n^*\right]^2, \tag{1.3}$$

where A_n^* is the area of the regular convex n-sides polygon *inscribed in the same circle* with $\mathscr{C}_n$.

But Zhang's result and Ma's result are for the planar convex polygon *inscribed in a circle* of radius R.

In the note, we continue to investigate the Bonnesen-type isoperimetric inequalities for the planar convex polygon, but our results are for the planar convex polygon *circumscribed in a circle* of radius r.

2. Some analytic inequalities

In order to simplify the statements. We set

$$I=(0,l); \quad H_n=\{\Theta=(\theta_1,\cdots,\theta_n)\in\mathbb{R}^n | \sum_{i=1}^n \theta_i = ml\} \quad (0<m<n);$$

$$D_n = I^n \cap H_n; \quad \Omega=(\sigma,\sigma,\cdots,\sigma) \quad \text{where} \quad \sigma=\frac{1}{n}\sum_{i=1}^n\theta_i=\frac{ml}{n}.$$

THEOREM 2.1. *Suppose that a real function $f(\theta)$ is positive and strictly convex. Then we have for $\alpha>0$*

$$\left(\sum_{i=1}^n f(\theta_i)\right)^{2\alpha} - (nf(\sigma))^\alpha\left(\sum_{i=1}^n f(\theta_i)\right)^\alpha \geqslant \left[(nf(\sigma))^\alpha - \left(\sum_{i=1}^n f(\theta_i)\right)^\alpha\right]^2. \tag{2.1}$$

In order to prove above result, we need a lemma below.

LEMMA 2.1. ([13]) *If real function $f: I^n \to \mathbb{R}$ is Schur convex, then $f(\Omega)$ is a global minimum in D_n. If f is a strictly Schur convex function, then $f(\Omega)$ is the unique global minimum in D_n.*

Proof of Theorem 2.1. Consider the function

$$F(\Theta)=\left(\sum_{i=1}^{n} f(\theta_i)\right)^{2\alpha}-(nf(\sigma))^{\alpha}\left(\sum_{i=1}^{n} f(\theta_i)\right)^{\alpha}-\left[(nf(\sigma))^{\alpha}-\left(\sum_{i=1}^{n} f(\theta_i)\right)^{\alpha}\right]^{2},$$

we observe that $F(\Omega)=0$. We shall prove that $F(\Theta)$ is strictly Schur convex function on I^n where $I=(0,l)$. Obviously, $F(\Theta)$ is a symmetric function on I^n. Hence, by Lemma 1.4, to guarantee $F(\Theta)$ is strictly Schur convex, it suffices to verify that

$$\Delta=(\theta_1-\theta_2)\left(\frac{\partial F}{\partial\theta_1}-\frac{\partial F}{\partial\theta_2}\right),\ if\ \theta_1\neq\theta_2.$$

Furthermore, we set $T_n=\sum_{i=1}^{n} f(\theta_i)$. Then

$$\frac{\partial F}{\partial\theta_i}=2\alpha(T_n)^{2\alpha-1}f'(\theta_i)-(nf(\sigma))^{\alpha}\alpha(T_n)^{\alpha-1}f'(\theta_i)+2\left[nf(\sigma))^{\alpha}-(T_n)^{\alpha}\right]\alpha(T_n)^{\alpha-1}f'(\theta_i)$$

$$=\alpha(nf(\sigma))^{\alpha}(T_n)^{\alpha-1}f'(\theta_i),\quad i=1,2. \tag{2.2}$$

$$\Delta=(\theta_1-\theta_2)\left(\frac{\partial F}{\partial\theta_1}-\frac{\partial F}{\partial\theta_2}\right)=(\theta_1-\theta_2)\alpha(nf(\sigma))^{\alpha}(T_n)^{\alpha-1}\left[f'(\theta_1)-f'(\theta_2)\right]. \tag{2.3}$$

Since f is strictly convex, then $f''>0$ and

$$(\theta_1-\theta_2)\left[f'(\theta_1)-f'(\theta_2)\right]>0. \tag{2.4}$$

Combine (2.4) and (2.3), inequality (2.1) can be derived. □

By using the strictly convex properties of $f(\theta)=\tan\theta$ and $f(\theta)=\frac{1}{\sin\theta}$ for $\theta\in(0,\pi/2)$ and Theorem 2.1, we get the following results.

COROLLARY 2.1. *Let* $\theta_i\in(0,\pi/2)$, $i=1,2,\cdots,n$; *and* $\sum_{i=1}^{n}\theta_i=\pi$. *Then for* $\alpha>0$

$$\left(\sum_{i=1}^{n}\tan\theta_i\right)^{2\alpha}-(n\tan\frac{\pi}{n})^{\alpha}\left(\sum_{i=1}^{n}\tan\theta_i\right)^{\alpha}\geqslant\left[(n\tan\frac{\pi}{n})^{\alpha}-\left(\sum_{i=1}^{n}\tan\theta_i\right)^{\alpha}\right]^{2}. \tag{2.5}$$

In particular, take $\alpha=1$, *we have*

$$\left(\sum_{i=1}^{n}\tan\theta_i\right)^{2}-n\tan\frac{\pi}{n}\left(\sum_{i=1}^{n}\tan\theta_i\right)\geqslant\left[n\tan\frac{\pi}{n}-\sum_{i=1}^{n}\tan\theta_i\right]^{2}. \tag{2.6}$$

COROLLARY 2.2. *Let* $\theta_i\in(0,\pi/2)$, $i=1,2,\cdots,n$; *and* $\sum_{i=1}^{n}\theta_i=\pi$. *Then for* $\alpha>0$

$$\left(\sum_{i=1}^{n}\frac{1}{\sin\theta_i}\right)^{2\alpha}-\left(\frac{n}{\sin\frac{\pi}{n}}\right)^{\alpha}\left(\sum_{i=1}^{n}\frac{1}{\sin\theta_i}\right)^{\alpha}\geqslant\left[\left(\frac{n}{\sin\frac{\pi}{n}}\right)^{\alpha}-\left(\sum_{i=1}^{n}\frac{1}{\sin\theta_i}\right)^{\alpha}\right]^{2}. \tag{2.7}$$

COROLLARY 2.3. *Let $x_i \in (0,1)$, $i = 1,2,\cdots,n$; and $\sum_{i=1}^{n} x_i = m$. Then for $\alpha > 0$*

$$\left(\sum_{i=1}^{n} x_i^2\right)^{2\alpha} - \left(\frac{m^2}{n}\right)^{\alpha}\left(\sum_{i=1}^{n} x_i^2\right)^{\alpha} \geqslant \left[\left(\frac{m^2}{n}\right)^{\alpha} - \left(\sum_{i=1}^{n} x_i^2\right)^{\alpha}\right]^2. \tag{2.8}$$

Where we use the fact that $f(x) = x^2$ in $(0,1)$ is strictly convex function.

3. Bonnensen style isoperimetric inequalities of plane convex polygon

In this section, by using above analytic isoperimetric inequalities, we establish some Bonnesen-type isoperimetric inequalities and related inverse inequalities for the planar convex polygon. Our first main result is stated as follows.

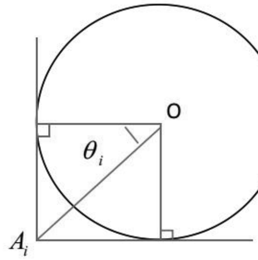
THEOREM 3.1. *Let $\mathscr{C}_n$ be an n-sided plane convex polygon circumscribed in a circle of radius r with perimeter L_n, enclosing a domain of area A_n. If $\alpha > 0$, then*

$$(L_n)^{2\alpha} - 4^{\alpha}\left(n\tan\frac{\pi}{n}\right)^{\alpha}(A_n)^{\alpha} \geqslant \frac{4^{\alpha}}{r^{2\alpha}}\left[(A_n^*)^{\alpha} - (A_n)^{\alpha}\right]^2, \tag{3.1}$$

$$\left(\frac{A_n}{r^2}\right)^{2\alpha} - \left(n\tan\frac{\pi}{n}\right)^{\alpha}\left(\frac{L_n}{2r}\right)^{\alpha} \geqslant \left[\left(\frac{A_n^*}{r^2}\right)^{\alpha} - \left(\frac{A_n}{r^2}\right)^{\alpha}\right]^2, \tag{3.2}$$

where A_n^* is the area of the regular convex n-sides polygon circumscribed in the same circle with $\mathscr{C}_n$.

Proof. We denote a_i the length of the ith side of $\mathscr{C}_n$, and θ_i the half of the central angle subtended by the ith vertex A_i of $\mathscr{C}_n$, $i = 1,2,\cdots,n$, then



$$L_n = \sum_{i=1}^{n} a_i = 2r\sum_{i=1}^{n}\tan\theta_i; \qquad A_n = \frac{1}{2}\sum_{i=1}^{n} a_i \cdot r = r^2\sum_{i=1}^{n}\tan\theta_i; \tag{3.3}$$

$$\sum_{i=1}^{n}\theta_i = \pi; \qquad A_n^* = nr^2\tan\frac{\pi}{n}. \tag{3.4}$$

Substituting (3.3) and (3.4) into (2.5), thus (3.1) and (3.2) are valid. □

REMARK 1. Inequality (3.2) can be regarded as inverse inequality of (3.1).

THEOREM 3.2. *Let $\mathscr{C}_n$ be an n-sided plane convex polygon circumscribed in a circle of radius r with perimeter L_n, enclosing a domain of area A_n. If $\alpha > 0$, Then*

$$(L_n)^{2\alpha} - 4^{\alpha}\left(n\tan\frac{\pi}{n}\right)^{\alpha}(A_n)^{\alpha} \geqslant \left[(l_n^*)^{\alpha} - (L_n)^{\alpha}\right]^2, \tag{3.5}$$

$$\left(\frac{A_n}{r^2}\right)^{2\alpha} - \left(n\tan\frac{\pi}{n}\right)^{\alpha}\left(\frac{L_n}{2r}\right)^{\alpha} \geqslant \left[\left(\frac{l_n^*}{r^2}\right)^{\alpha} - \left(\frac{L_n}{r^2}\right)^{\alpha}\right]^2, \tag{3.6}$$

where l_n^* is the perimeter of the regular convex n-sides polygon circumscribed in the same circle with $\mathscr{C}_n$.

Proof. Similar to the proof of theorem 3.1 and pay attention to the equation $l_n^* = 2nr\tan\frac{\pi}{n}$. □

REMARK 2. Inequality (3.6) can be considered as inverse inequality of (3.5). Taking $\alpha = 1$, we can derive the following inequalities.

COROLLARY 3.1. *Let $\mathscr{C}_n$ be an n-sided plane convex polygon circumscribed in a circle of radius r with perimeter L_n, enclosing a domain of area A_n. Then*

$$L_n^2 - 4\left(n\tan\frac{\pi}{n}\right)A_n \geqslant \frac{4}{r^2}\left[(A_n^*) - (A_n)\right]^2, \tag{3.7}$$

$$\left(\frac{A_n}{r^2}\right)^2 - \left(n\tan\frac{\pi}{n}\right)\frac{L_n}{2r} \geqslant \left[\left(\frac{A_n^*}{r^2}\right) - \left(\frac{A_n}{r^2}\right)\right]^2, \tag{3.8}$$

$$L_n^2 - 4\left(n\tan\frac{\pi}{n}\right)A_n \geqslant \left[(l_n^*) - (L_n)\right]^2, \tag{3.9}$$

$$\left(\frac{A_n}{r^2}\right)^2 - \left(n\tan\frac{\pi}{n}\right)\frac{L_n}{2r} \geqslant \left[\left(\frac{l_n^*}{r^2}\right) - \left(\frac{L_n}{r^2}\right)\right]^2. \tag{3.10}$$

REMARK 3. Our results (3.7) and (3.9) are different from (1.2) (Zhang's result) and (1.3) (Ma's result). Their results are mainly about an n-sided plane convex polygon *inscribed in a circle* of radius R, while our results in Theorem 3.1 and 3.2 are mainly about an n-sided plane convex polygon *circumscribed in a circle of radius* r.

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