SCHUR CONVEX FUNCTIONS AND THE BONNESEN STYLE ISOPERIMETRIC INEQUALITIES FOR PLANAR CONVEX POLYGONS

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Abstract. In this note, we continue to investigate Bonnesen-type isoperimetric inequalities for planar convex polygons. We shall first establish some analytic isoperimetric inequalities for a special class of Schur convex functions. Subsequently, by using these analytic isoperimetric inequalities, Bonnesen-type isoperimetric inequalities and related inverse inequalities for the planar convex polygons are obtained.

1. Introduction

Schur convex functions [4] play an important role in the study of analytic inequalities and geometric inequalities. Let us recall some notions and lemmas.

Let $I \subset \mathbb{R}$ and $I^n = I \times I \times \cdots \times I$ ($n$ copies).

**Lemma 1.1.** ([13]) An $n \times n$ matrix $S = [s_{ij}]$ is said to be a doubly stochastic matrix if $s_{ij} \geq 0$ for $1 \leq i < j \leq n$, and

$$
\sum_{j=1}^{n} s_{ij} = 1, \quad i = 1, 2, \cdots, n; \quad \sum_{i=1}^{n} s_{ij} = 1, \quad j = 1, 2, \cdots, n.
$$

**Lemma 1.2.** ([13])

(1). A permutation matrix is a doubly stochastic matrix.

(2). $S = [s_{ij}]$ with $s_{ij} = \frac{1}{n}$, $1 \leq i, j \leq n$, is a doubly stochastic matrix.

**Lemma 1.3.** ([13]) A real function $f : I^n \to \mathbb{R}$ ($n > 1$) is called to be Schur convex function if for any doubly stochastic matrix $S$ and all $x \in I^n$, $f(Sx) \leq f(x)$. It is called to be strictly Schur convex if inequality is strict. $f$ is said to be Schur concave (resp. strictly Schur concave) if $-f$ is Schur convex.


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Lemma 1.4. ([4]) Let $\Omega \in \mathbb{R}^n$ be symmetric and convex set with nonempty interior, and let $f : \Omega \rightarrow \mathbb{R}$ be differentiable in the interior of $\Omega$. Then $f$ is Schur convex (Schur concave) on $\Omega$ if and only if $f$ is symmetric on $\Omega$ and

$$(x_1 - x_2) \left( \frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2} \right) \geq 0 (\leq 0) \quad \text{for all} \quad x \in \Omega^0,$$

where $\Omega^0$ is the interior of $\Omega$.

The above definitions and example can be found in many references such as [4] and [14].

The classical isoperimetric inequality states that for a domain $K$ with the boundary composing of the simple curve $\mathcal{C}$ of length $L$ and area $A$

$$L^2 - 4\pi A \geq 0,$$

where equality holds if $K$ is a circle. The isoperimetric deficit of $K$ is defined as $\Delta(K) = L^2 - 4\pi A$. Bonnesen in [8] gave a low bound for the isoperimetric deficit $\Delta(K)$, as follows

$$\Delta(K) = L^2 - 4\pi A \geq \pi^2 (R - r)^2,$$

where $R$ is the circumradius and $r$ is the inradius of the curve $\mathcal{C}$.

Later Bonnesen proved a series of inequalities of the form

$$\Delta(K) = L^2 - 4\pi A \geq B,$$

where the equality $B$ is an invariant of geometric significance having the following basic properties:

1. $B$ is non-negative;
2. $B$ is vanish only when $K$ is a disc.

Many $B$s are discovered in the last century and mathematicians are still working on those unknown invariants of geometric significance. See references [1, 2, 3, 6, 7, 8, 9, 10] for more details.

Here are some of the different forms of Bonnesen-style isoperimetric inequality.

$$L^2 - 4\pi A \geq 4\pi d^2; \quad L^2 - 4\pi A \geq \pi^2 (r_e - r_i)^2;$$

$$L^2 - 4\pi A \geq (L - 2\pi r_i)^2; \quad L^2 - 4\pi A \geq (L - 2\pi r_e)^2;$$

$$L^2 - 4\pi A \geq \frac{A}{r} - \pi r)^2; \quad L^2 - 4\pi A \geq L^2 \left( \frac{r_e - r_i}{r_e + r_i} \right)^2;$$

$$L^2 - 4\pi A \geq A^2 \left( \frac{1}{r_i} - \frac{1}{r_e} \right)^2; \quad L^2 - 4\pi A \geq A^2 \left( \frac{1}{r} - \frac{1}{r_e} \right)^2.$$

It is difficult to compare those isoperimetric deficit lower bounds and to determine which lower bound is the best.

However, the literature on the study of Bonnesen-type isoperimetric inequalities for planar convex polygon is relatively less (see [5, 11, 12, 13]).

Let $C_n$ be an $n$-sided plane convex polygon inscribed in a circle of radius $R$ with side-length $a_i$ ($i = 1, 2, \ldots, n$) and perimeter $L_n$, enclosing a domain of area $A_n$.

$$(L_n)^2 - 4n \tan \frac{\pi}{n} A_n \geq \left[ L_n - L^*_n \right]^2. \tag{1.2}$$

where $L^*_n$ is the perimeter of the regular convex $n$-sides polygon inscribed in the same circle with $C_n$.

In 2015, L. Ma [5] obtained a new Bonnesen-style inequality for planar convex polygon

$$(L_n)^2 - 4n \tan \frac{\pi}{n} A_n \geq \frac{1}{R^2} \left[ A_n - A^*_n \right]^2, \tag{1.3}$$

where $A^*_n$ is the area of the regular convex $n$-sides polygon inscribed in the same circle with $C_n$.

But Zhang’s result and Ma’s result are for the planar convex polygon inscribed in a circle.

In the note, we continue to investigate the Bonnesen-type isoperimetric inequalities for the planar convex polygon, but our results are for the planar convex polygon circumscribed in a circle of radius $r$.

2. Some analytic inequalities

In order to simplify the statements. We set

$I = (0, l); \quad H_n = \{ \Theta = (\theta_1, \cdots, \theta_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} \theta_i = ml \} \quad (0 < m < n);$

$D_n = I^n \cap H_n; \quad \Omega = (\sigma, \sigma, \cdots, \sigma) \quad \text{where} \quad \sigma = \frac{1}{n} \sum_{i=1}^{n} \theta_i = \frac{ml}{n}.$

THEOREM 2.1. Suppose that a real function $f(\theta)$ is positive and strictly convex. Then we have for $\alpha > 0$

$$\left( \sum_{i=1}^{n} f(\theta_i) \right)^{2\alpha} - (nf(\sigma))^{\alpha} \left( \sum_{i=1}^{n} f(\theta_i) \right)^{\alpha} \geq \left[ (nf(\sigma))^{\alpha} - \left( \sum_{i=1}^{n} f(\theta_i) \right)^{\alpha} \right]^2. \tag{2.1}$$

In order to prove above result, we need a lemma below.

LEMMA 2.1. ([13]) If real function $f : I^n \to \mathbb{R}$ is Schur convex, then $f(\Omega)$ is a global minimum in $D_n$. If $f$ is a strictly Schur convex function, then $f(\Omega)$ is the unique global minimum in $D_n$. 
Proof of Theorem 2.1. Consider the function

\[ F(\Theta) = \left( \sum_{i=1}^{n} f(\theta_i) \right)^{2\alpha} - (nf(\sigma))^{\alpha} \left( \sum_{i=1}^{n} f(\theta_i) \right)^{\alpha} - \left[ (nf(\sigma))^{\alpha} - \left( \sum_{i=1}^{n} f(\theta_i) \right)^{\alpha} \right]^2, \]

we observe that \( F(\Omega) = 0 \). We shall prove that \( F(\Theta) \) is strictly Schur convex function on \( I^n \) where \( I = (0, 1) \). Obviously, \( F(\Theta) \) is a symmetric function on \( I^n \). Hence, by Lemma 1.4, to guarantee \( F(\Theta) \) is strictly Schur convex, it suffices to verify that

\[ \Delta = (\theta_1 - \theta_2) \left( \frac{\partial F}{\partial \theta_1} - \frac{\partial F}{\partial \theta_2} \right), \]

if \( \theta_1 \neq \theta_2 \).

Furthermore, we set \( T_n = \sum_{i=1}^{n} f(\theta_i) \). Then

\[
\frac{\partial F}{\partial \theta_i} = 2\alpha (T_n)^{2\alpha - 1} f'(\theta_i) - (nf(\sigma))^{\alpha} \alpha (T_n)^{\alpha - 1} f'(\theta_i) + 2 [nf(\sigma)]^{\alpha} - (T_n)^{\alpha} \alpha (T_n)^{\alpha - 1} f'(\theta_i)
= \alpha (nf(\sigma))^{\alpha} (T_n)^{\alpha - 1} f'(\theta_i), \quad i = 1, 2.
\]

\[
\Delta = (\theta_1 - \theta_2) \left( \frac{\partial F}{\partial \theta_1} - \frac{\partial F}{\partial \theta_2} \right) = (\theta_1 - \theta_2) \alpha (nf(\sigma))^{\alpha} (T_n)^{\alpha - 1} \left[ f'(\theta_1) - f'(\theta_2) \right].
\]

(2.3)

Since \( f \) is strictly convex, then \( f'' > 0 \) and

\[
(\theta_1 - \theta_2) \left[ f'(\theta_1) - f'(\theta_2) \right] > 0.
\]

(2.4)

Combine (2.4) and (2.3), inequality (2.1) can be derived. \( \square \)

By using the strictly convex properties of \( f(\theta) = \tan \theta \) and \( f(\theta) = \frac{1}{\sin \theta} \) for \( \theta \in (0, \pi/2) \) and Theorem 2.1, we get the following results.

**Corollary 2.1.** Let \( \theta_i \in (0, \pi/2), \ i = 1, 2, \cdots, n; \) and \( \sum_{i=1}^{n} \theta_i = \pi \). Then for \( \alpha > 0 \)

\[
\left( \sum_{i=1}^{n} \tan \theta_i \right)^{2\alpha} - (\tan \frac{\pi}{n})^{\alpha} \left( \sum_{i=1}^{n} \tan \theta_i \right)^{\alpha} \geq \left[ (\tan \frac{\pi}{n})^{\alpha} - \left( \sum_{i=1}^{n} \tan \theta_i \right)^{\alpha} \right]^2.
\]

(2.5)

In particular, take \( \alpha = 1 \), we have

\[
\left( \sum_{i=1}^{n} \tan \theta_i \right)^2 - n \tan \frac{\pi}{n} \left( \sum_{i=1}^{n} \tan \theta_i \right) \geq \left[ n \tan \frac{\pi}{n} - \sum_{i=1}^{n} \tan \theta_i \right]^2.
\]

(2.6)

**Corollary 2.2.** Let \( \theta_i \in (0, \pi/2), \ i = 1, 2, \cdots, n; \) and \( \sum_{i=1}^{n} \theta_i = \pi \). Then for \( \alpha > 0 \)

\[
\left( \sum_{i=1}^{n} \frac{1}{\sin \theta_i} \right)^{2\alpha} - \left( \frac{n}{\sin \frac{\pi}{n}} \right)^{\alpha} \left( \sum_{i=1}^{n} \frac{1}{\sin \theta_i} \right)^{\alpha} \geq \left[ \left( \frac{n}{\sin \frac{\pi}{n}} \right)^{\alpha} - \left( \sum_{i=1}^{n} \frac{1}{\sin \theta_i} \right)^{\alpha} \right]^2.
\]

(2.7)
Corollary 2.3. Let \( x_i \in (0, 1), \ i = 1, 2, \ldots, n; \) and \( \sum_{i=1}^{n} x_i = m. \) Then for \( \alpha > 0 \)
\[
\left( \sum_{i=1}^{n} x_i^2 \right)^{2\alpha} - \left( \frac{m^2}{n} \right)^{\alpha} \left( \sum_{i=1}^{n} x_i^2 \right)^{\alpha} \geq \left[ \left( \frac{m^2}{n} \right)^{\alpha} - \left( \frac{\sum_{i=1}^{n} x_i^2}{n} \right)^{\alpha} \right]^2.
\] (2.8)
Where we use the fact that \( f(x) = x^2 \) in \( (0, 1) \) is strictly convex function.

3. Bonnensen style isoperimetric inequalities of plane convex polygon

In this section, by using above analytic isoperimetric inequalities, we establish some Bonnesen-type isoperimetric inequalities and related inverse inequalities for the planar convex polygon. Our first main result is stated as follows.

Theorem 3.1. Let \( C_n \) be an \( n \)-sided plane convex polygon circumscribed in a circle of radius \( r \) with perimeter \( L_n \), enclosing a domain of area \( A_n \). If \( \alpha > 0 \), then
\[
(L_n)^{2\alpha} - 4^\alpha \left( n \tan \left( \frac{\pi}{n} \right) \right)^{\alpha} (A_n)^{\alpha} \geq 4^\alpha \frac{\alpha}{r^{2\alpha}} \left[ \left( \frac{A_n^*}{n} \right)^{\alpha} - \left( A_n \right)^{\alpha} \right]^2,
\] (3.1)
\[
\left( \frac{A_n}{r^2} \right)^{2\alpha} - \left( n \tan \left( \frac{\pi}{n} \right) \right)^{\alpha} \left( \frac{L_n}{2r} \right)^{\alpha} \geq \left[ \left( \frac{A_n^*}{r^2} \right)^{\alpha} - \left( \frac{A_n}{r^2} \right)^{\alpha} \right]^2,
\] (3.2)
where \( A_n^* \) is the area of the regular convex \( n \)-sides polygon circumscribed in the same circle with \( C_n \).

Proof. We denote \( a_i \) the length of the \( i \)-th side of \( C_n \), and \( \theta_i \) the half of the central angle subtended by the \( i \)-th vertex \( A_i \) of \( C_n \), \( i = 1, 2, \ldots, n \), then
\[
L_n = \sum_{i=1}^{n} a_i = 2r \sum_{i=1}^{n} \tan \theta_i; \quad A_n = \frac{1}{2} \sum_{i=1}^{n} a_i \cdot r = r^2 \sum_{i=1}^{n} \tan \theta_i; \quad \sum_{i=1}^{n} \theta_i = \pi; \quad A_n^* = nr^2 \tan \frac{\pi}{n}.
\] (3.3)
(3.4)
Substituting (3.3) and (3.4) into (2.5), thus (3.1) and (3.2) are valid. □

Remark 1. Inequality (3.2) can be regarded as inverse inequality of (3.1).
THEOREM 3.2. Let $C_n$ be an $n$-sided plane convex polygon circumscribed in a circle of radius $r$ with perimeter $L_n$, enclosing a domain of area $A_n$. If $\alpha > 0$, Then

\[
(L_n)^{2\alpha} - 4^{\alpha} \left( n \tan \frac{\pi}{n} \right)^{\alpha} (A_n)^{\alpha} \geq \left[ (l_n^*)^{\alpha} - (L_n)^{\alpha} \right]^2,
\]

where $l_n^*$ is the perimeter of the regular convex $n$-sides polygon circumscribed in the same circle with $C_n$.

Proof. Similar to the proof of theorem 3.1 and pay attention to the equation $l_n^* = 2nr \tan \frac{\pi}{n}$. □

REMARK 2. Inequality (3.6) can be considered as inverse inequality of (3.5). Taking $\alpha = 1$, we can derive the following inequalities.

COROLLARY 3.1. Let $C_n$ be an $n$-sided plane convex polygon circumscribed in a circle of radius $r$ with perimeter $L_n$, enclosing a domain of area $A_n$. Then

\[
L_n^2 - 4 \left( n \tan \frac{\pi}{n} \right) A_n \geq \frac{4}{r^2} \left[ (A_n^*) - (A_n) \right]^2,
\]

\[
\left( \frac{A_n}{r^2} \right)^2 - \left( n \tan \frac{\pi}{n} \right) \frac{L_n}{2r} \geq \left[ \left( \frac{A_n^*}{r^2} \right) - \left( \frac{A_n}{r^2} \right) \right]^2,
\]

\[
L_n^2 - 4 \left( n \tan \frac{\pi}{n} \right) A_n \geq \left[ (l_n^*) - (L_n) \right]^2,
\]

\[
\left( \frac{A_n}{r^2} \right)^2 - \left( n \tan \frac{\pi}{n} \right) \frac{L_n}{2r} \geq \left[ \left( \frac{l_n^*}{r^2} \right) - \left( \frac{L_n}{r^2} \right) \right]^2.
\]

REMARK 3. Our results (3.7) and (3.9) are different from (1.2) (Zhang’s result) and (1.3) (Ma’s result). Their results are mainly about an $n$-sided plane convex polygon inscribed in a circle of radius $R$, while our results in Theorem 3.1 and 3.2 are mainly about an $n$-sided plane convex polygon circumscribed in a circle of radius $r$.

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REFERENCES


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