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BOUNDEDNESS OF SOME SUBLINEAR OPERATORS ON WEIGHTED VARIABLE HERZ–MORREY SPACES

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Abstract. A new class of generalized Herz-Morrey spaces with weight and variable exponent is defined. In addition, the boundedness of some sublinear operators on such spaces is also considered. The approach is based on the Muckenhoupt theory with variable exponent and function decomposition.

1. Introduction

Following the fundamental work of Kováčik and Rákosník [15] in the early 1990s, function spaces with variable exponents were investigated by a significant number of authors, see [1, 2, 6, 7, 8, 16, 19, 24, 25, 26, 27, 28] and the references therein. The theory of these spaces had a remarkable development in part due to its useful applications. For instance, they appear in the modeling of electrorheological fluids, in the study of image processing and in PDE with non-standard growth conditions, for an overview we refer to [5, 9, 20].

On the other hand, there do exist many different properties between the variable function spaces and the classical cases. For instance, the variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ are not translation invariant. More precisely, if $p(\cdot)$ is non-constant in $\mathbb{R}^n$, then there always exists $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $h \in \mathbb{R}^n$, such that $f(x+h)$ is not in $L^{p(\cdot)}(\mathbb{R}^n)$. In addition, if $p(\cdot)$ is unbounded, then $L^{p(\cdot)}(\mathbb{R}^n)$ is no longer separable and it can happen that $L^\infty(E) \subset L^{p(\cdot)}(E)$ when the set $E \subset \mathbb{R}^n$ has infinite measure, see [2] for further details. Therefore, apart from useful application considerations, the motivation to study such spaces have an intrinsic interest.

Herz spaces $\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$ and Herz-Morrey spaces $M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$ with variable exponent $p(\cdot)$ were introduced first by Izuki [12, 13]. Very recently, the boundedness of some important operators in harmanic analysis, such as the Hardy-Littlewood maximal operator, Marcinkiewicz integrals and some sublinear operators and so on, were obtained on these spaces, see [14, 22, 23]. In 2016, Izuki and Noi [10, 11] defined new generalized Herz spaces having weight and variable exponent, namely, weighted Herz spaces with variable exponent. Under proper assumptions on each exponent and weight,

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they proved the boundedness of fractional integral operators and intrinsic square functions on those spaces. Inspired of their work [10, 13, 23], the aim of this paper is to define weighted Herz-Morrey spaces with variable exponent and study the boundedness of some sublinear operators on these spaces under certain weak size conditions, which are similar to those introduced by Soria and Weiss in [21].

We usually denote the ball with radius r and center x by B. f_B denotes the integral average of f on B, i.e. $f_B = \frac{1}{|B|}\int_B f(x)dx$. $p'(\cdot)$ means the conjugate exponent defined by $1/p(\cdot)+1/p'(\cdot)=1$. The letter C stands for a positive constant, which may vary from line to line.

2. Preliminaries and lemmas

2.1. Lebesgue spaces with variable exponent

Let $p(\cdot)$ be a measurable function with values in $[1,\infty)$. The set $\mathscr{P}(\mathbb{R}^n)$ consists of all variable exponents $p(\cdot)$ satisfying

$$1 < p_- \leqslant p(x) \leqslant p_+ < \infty,$$

where $p_- := \text{ess inf}\{p(x) : x\in\mathbb{R}^n\}$, $p_+ := \text{ess sup}\{p(x) : x\in\mathbb{R}^n\}$. The variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ is the class of all measurable functions f on $\mathbb{R}^n$ such that

$$I_{p(\cdot)}(f) := \int_{\mathbb{R}^n} |f(x)|^{p(x)}dx < \infty.$$

This set becomes a Banach space when equipped with the norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \inf\{\lambda > 0 : I_{p(\cdot)}(f/\lambda) \leqslant 1\}.$$

It is easy to see that the variable Lebesgue exponent norm has the following property

$$\|f^\sigma\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \|f\|^\sigma_{L^{\sigma p(\cdot)}(\mathbb{R}^n)}, \quad \sigma \geqslant 1/p_-.$$

Given an open set $\Omega\subset\mathbb{R}^n$. For all compact subsets $F\subset\Omega$, the space $L^{p(\cdot)}_{loc}(\Omega)$ is defined by

$$L^{p(\cdot)}_{loc}(\Omega) = \{f : f\in L^{p(\cdot)}(F)\}.$$

A measurable function $g(\cdot):\mathbb{R}^n\to(0,\infty)$ is called globally log-Hölder continuous if it satisfies

$$|g(x)-g(y)| \leqslant \frac{-C}{\log(|x-y|)}, \quad |x-y|\leqslant 1/2,\ x,y\in\mathbb{R}^n, \tag{2.1}$$

$$|g(x)-g_\infty| \leqslant \frac{C}{\log(e+|x|)}, \quad x\in\mathbb{R}^n, \tag{2.2}$$

for some real constant g_∞. The set of $p(\cdot)$ satisfying (2.1) and (2.2) is denoted by $LH(\mathbb{R}^n)$. As is well-known, if $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)\bigcap LH(\mathbb{R}^n)$, then the Hardy-Littlewood maximal operator M, which is defined by

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy,$$

is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, see [3].

2.2. Weighted function spaces with variable exponent

Let ω be a weight function on $\mathbb{R}^n$, that is, ω is real-valued, non-negative and locally integrable. The weighted variable exponent Lebesgue space $L^{p(\cdot)}(\omega)$ is defined by

$$L^{p(\cdot)}(\omega) := \{f : f\omega^{\frac{1}{p(\cdot)}} \in L^{p(\cdot)}(\mathbb{R}^n)\}.$$

The space $L^{p(\cdot)}(\omega)$ is a Banach space equipped with the norm

$$\|f\|_{L^{p(\cdot)}(\omega)} = \|f\omega^{\frac{1}{p(\cdot)}}\|_{L^{p(\cdot)}}.$$

A weight is said to be a Muckenhoupt A_1 weight if

$$M\omega(x) \leqslant C\omega(x) \quad a.e.\ x \in \mathbb{R}^n.$$

For $1 < p < \infty$, we say that ω is an A_p weight if

$$\sup_B \left(\frac{1}{|B|} \int_B \omega(x) dx \right) \left(\frac{1}{|B|} \int_B \omega(x)^{1-p'} dx \right)^{p-1} < \infty.$$

The Muckenhoupt A_p class with constant exponent $p \in (1, \infty)$ was recently generalized by Izuki and Noi [10, 11] as follows.

DEFINITION 2.1. Let $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$. A weight is said to be an $A_{p(\cdot)}$ weight if

$$\sup_B \frac{1}{|B|} \|\omega^{1/p(\cdot)}\chi_B\|_{L^{p(\cdot)}} \|\omega^{-1/p(\cdot)}\chi_B\|_{L^{p'(\cdot)}} < \infty.$$

REMARK 2.1. If $p(\cdot) \equiv p \in (1, \infty)$, then we see immediately that the definition reduces to the classical Muckenhoupt A_p class. Cruz-Uribe et al. [4] showed that $\omega \in A_{p(\cdot)}$ if and only if the Hardy-Littlewood maximal operator M is bounded on the space $L^{p(\cdot)}(\omega)$.

REMARK 2.2. Suppose that $p(\cdot), q(\cdot) \in \mathscr{P}(\mathbb{R}^n)\bigcap LH(\mathbb{R}^n)$ and $p(\cdot) \leqslant q(\cdot)$, then we have $A_1 \subset A_{p(\cdot)} \subset A_{q(\cdot)}$, see [11].

DEFINITION 2.2. Let $0 < \beta < n$, and $p_1(\cdot), p_2(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ such that $1/p_2(x) \equiv 1/p_1(x) - \beta/n$. A weight ω is said to be an $A(p_1(\cdot), p_2(\cdot))$ weight if

$$\|\omega\chi_B\|_{L^{p_2(\cdot)}} \|\omega^{-1}\chi_B\|_{L^{p_1'(\cdot)}} \leqslant C|B|^{1-\frac{\beta}{n}}.$$

REMARK 2.3. Let $p_1(\cdot), p_2(\cdot)$ and ω be as in Definition 2.2, we then have

$$\omega \in A(p_1(\cdot), p_2(\cdot)) \Leftrightarrow \omega^{p_2(\cdot)} \in A_{1+p_2(\cdot)/p_1'(\cdot)}.$$

Let $B_k = \{x \in \mathbb{R}^n : |x| \leqslant 2^k\}$, $R_k = B_k \setminus B_{k-1}$ and $\chi_k = \chi_{R_k}$ be the characteristic function of the set R_k for $k \in \mathbb{Z}$.

DEFINITION 2.3. Let $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, $0 < q \leqslant \infty$ and $\alpha \in \mathbb{R}$. The homogeneous weighted Herz space $\dot{K}^{\alpha,q}_{p(\cdot)}(\omega)$ is defined as the set of all $f \in L^{p(\cdot)}_{loc}(\mathbb{R}^n \setminus \{0\}, \omega)$ such that

$$\|f\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\omega)} := \left(\sum_{k\in\mathbb{Z}} 2^{\alpha k q} \|f\chi_k\|^q_{L^{p(\cdot)}(\omega)} \right)^{1/q} < \infty,$$

with the usual modification when $q = \infty$.

DEFINITION 2.4. Let $0 \leqslant \lambda < \infty$, $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, $0 < q \leqslant \infty$ and $\alpha \in \mathbb{R}$. The homogeneous weighted Herz-Morrey space $M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\omega)$ is defined as the set of all $f \in L^{p(\cdot)}_{loc}(\mathbb{R}^n \setminus \{0\}, \omega)$ such that

$$\|f\|_{M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\omega)} := \sup_{L\in\mathbb{Z}} 2^{-L\lambda} \left(\sum_{k=-\infty}^{L} 2^{\alpha k q} \|f\chi_k\|^q_{L^{p(\cdot)}(\omega)} \right)^{1/q} < \infty,$$

with the usual modification when $q = \infty$.

REMARK 2.4. Clearly, $M\dot{K}^{\alpha,0}_{q,p(\cdot)}(\omega) = \dot{K}^{\alpha,q}_{p(\cdot)}(\omega)$. In the case $\omega \equiv 1$ and $p(\cdot) \equiv p$, then $\dot{K}^{\alpha,q}_{p(\cdot)}(\omega) = \dot{K}^{\alpha,p}_{q}(\mathbb{R}^n)$ and $M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\omega) = M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)$ are the classical Herz spaces and Herz-Morrey spaces in [18] and [17], respectively.

2.3. Key lemmas

In order to prove our main results, we need the following lemmas.

LEMMA 2.1. *Let $\mathscr{X}$ be a Banach function space on $\mathbb{R}^n$. If $f \in \mathscr{X}$ and $g \in \mathscr{X}'$, then we have*

$$\|fg\|_{L^1(\mathbb{R}^n)} \leqslant \|f\|_{\mathscr{X}} \|g\|_{\mathscr{X}'},$$

where $\mathscr{X}'$ denotes the associated space of $\mathscr{X}$.

We remark that Lemma 2.1 is the well-known generalized Hölder inequality. As a direct application of Lemma 2.1, we obtain the following result:

LEMMA 2.2. *If $\mathscr{X}$ is a Banach function space on $\mathbb{R}^n$, then we have*

$$|B| \|\chi_B\|^{-1}_{\mathscr{X}} \|\chi_B\|^{-1}_{\mathscr{X}'} \leqslant 1.$$

Lemmas 2.3 and 2.4 below have been proved by Izuki and Noi in [10, 11].

LEMMA 2.3. *Let* $p(\cdot)\in\mathscr{P}(\mathbb{R}^n)\bigcap LH(\mathbb{R}^n)$ *and* $\omega\in A_{rp(\cdot)}$, $1/p_-<r<1$. *Then there exists a constant* $0<\delta<1$ *such that for all* $k,j\in\mathbb{Z}$,

$$\frac{\|\chi_{B_k}\|_{L^{p(\cdot)}(\omega)}}{\|\chi_{B_j}\|_{L^{p(\cdot)}(\omega)}}\leqslant C2^{(k-j)n\delta},\ \ k\leqslant j,$$

and

$$\frac{\|\chi_{B_k}\|_{L^{p(\cdot)}(\omega)}}{\|\chi_{B_j}\|_{L^{p(\cdot)}(\omega)}}\leqslant C2^{(k-j)nr},\ \ k>j.$$

LEMMA 2.4. *Let* $p(\cdot)\in\mathscr{P}(\mathbb{R}^n)\bigcap LH(\mathbb{R}^n)$ *and* $\omega^{p(\cdot)}\in A_1$. *Then there exists constants* $0<\delta_1,\delta_2<1$ *such that*

$$\frac{\|\chi_S\|_{L^{p(\cdot)}(\omega^{p(\cdot)})}}{\|\chi_B\|_{L^{p(\cdot)}(\omega^{p(\cdot)})}}\leqslant C\left(\frac{|S|}{|B|}\right)^{\delta_1},\quad \frac{\|\chi_S\|_{(L^{p(\cdot)}(\omega^{p(\cdot)}))'}}{\|\chi_B\|_{(L^{p(\cdot)}(\omega^{p(\cdot)}))'}}\leqslant C\left(\frac{|S|}{|B|}\right)^{\delta_2},$$

for all balls B *and all measurable sets* $S\subset B$.

REMARK 2.5. We remark that $(L^{p(\cdot)}(\omega^{p(\cdot)}))'=L^{p'(\cdot)}(\omega^{-p'(\cdot)})$ provided that $p(\cdot)\in\mathscr{P}(\mathbb{R}^n)$ and $\omega^{p(\cdot)}\in A_1$. The proof can be found in [11].

3. Main results and their proofs

In this section, we prove the boundedness of some sublinear operators under certain weak size conditions on weighted Herz-Morrey spaces with variable exponent. We consider only $0<\lambda<\infty$, the arguments are similar in the case $\lambda=0$.

Our main results can be stated as follows.

THEOREM 3.1. *Let* $0\leqslant\lambda<\infty$, $p(\cdot)\in\mathscr{P}(\mathbb{R}^n)\bigcap LH(\mathbb{R}^n)$, $0<q<\infty$, $1/p_-<r<1$, $\omega\in A_{rp(\cdot)}$ *and* $-n\delta+\lambda<\alpha<n(1-r)+\lambda$, *where* $0<\delta<1$ *is the constant appearing in Lemma 2.3. Suppose that a sublinear operator* T *satifies the size conditions*

$$|Tf(x)|\leqslant C\|f\|_{L^1(\mathbb{R}^n)}/|x|^n, \tag{3.1}$$

when $\operatorname{supp} f\subseteq R_k$ *and* $|x|\geqslant 2^{k+1}$ *with* $k\in\mathbb{Z}$, *and*

$$|Tf(x)|\leqslant C2^{-kn}\|f\|_{L^1(\mathbb{R}^n)}, \tag{3.2}$$

when $\operatorname{supp} f\subseteq R_k$ *and* $|x|\leqslant 2^{k-2}$ *with* $k\in\mathbb{Z}$. *Then, if* T *is bounded on* $L^{p(\cdot)}(\omega)$, T *is also bounded on* $M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\omega)$.

COROLLARY 3.1. *Let* $\lambda,p(\cdot),q,r,\omega$ *and* α *be as in Theorem 3.1, if a sublinear operator* T *satifies the size conditions*

$$|Tf(x)|\leqslant C\int_{\mathbb{R}^n}\frac{|f(y)|}{|x-y|^n}dy,\ \ x\notin \operatorname{supp} f, \tag{3.3}$$

for any integrable and compactly supported functions f and T is bounded on $L^{p(\cdot)}(\omega)$, then T is bounded on $M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\omega)$.

REMARK 3.1. We note that (3.3) is satisfied by many important operators in harmonic analysis, such as the Hardy-Littlewood maximal operator, Calderón-Zymund operators and Bochner-Riesz means at the critical index and so on, see [21].

Proof of Theorem 3.1. Let $f \in M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\omega)$. We decompose

$$f(x) = \sum_{j=-\infty}^{\infty} f(x)\chi_j(x) := \sum_{j=-\infty}^{\infty} f_j(x).$$

Then, we have

$$\begin{aligned}
\|T(f)\|^q_{M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\omega)} &= \sup_{L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{\alpha qk} \|T(f)\chi_k\|^q_{L^{p(\cdot)}(\omega)} \\
&\leqslant C \sup_{L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{\alpha qk} \left(\sum_{j=-\infty}^{k-2} \|T(f_j)\chi_k\|_{L^{p(\cdot)}(\omega)} \right)^q \\
&\quad + C \sup_{L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{\alpha qk} \left(\sum_{j=k-1}^{k+1} \|T(f_j)\chi_k\|_{L^{p(\cdot)}(\omega)} \right)^q \\
&\quad + C \sup_{L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{\alpha qk} \left(\sum_{j=k+2}^{\infty} \|T(f_j)\chi_k\|_{L^{p(\cdot)}(\omega)} \right)^q \\
&:= I_1 + I_2 + I_3.
\end{aligned}$$

We first estimate I_1. Noting that $j \leqslant k-2$ and $x \in R_k$, by (3.1), Lemmas 2.1 and 2.2, we get

$$\begin{aligned}
|T(f_j)(x)| &\leqslant C2^{-kn} \|f_j\|_{L^1(\mathbb{R}^n)} \\
&\leqslant C2^{-kn} \|f\omega^{\frac{1}{p(\cdot)}}\chi_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\omega^{-\frac{1}{p(\cdot)}}\chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \\
&\leqslant C2^{-kn} \|f\omega^{\frac{1}{p(\cdot)}}\chi_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\omega^{-\frac{1}{p(\cdot)}}\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \\
&\leqslant C2^{-kn} \|f\chi_j\|_{L^{p(\cdot)}(\omega)} \|\chi_{B_j}\|_{(L^{p(\cdot)}(\omega))'} \\
&\leqslant C2^{(j-k)n} \|f\chi_j\|_{L^{p(\cdot)}(\omega)} \|\chi_{B_j}\|^{-1}_{L^{p(\cdot)}(\omega)},
\end{aligned} \tag{3.4}$$

which combining with Lemma 2.3 yields

$$\begin{aligned}
&\|T(f_j)\chi_k\|_{L^{p(\cdot)}(\omega)} \\
&\leqslant C2^{(j-k)n} \|f\chi_j\|_{L^{p(\cdot)}(\omega)} \|\chi_{B_j}\|^{-1}_{L^{p(\cdot)}(\omega)} \|\chi_{B_k}\|_{L^{p(\cdot)}(\omega)} \\
&\leqslant C2^{(j-k)n(1-r)} \|f\chi_j\|_{L^{p(\cdot)}(\omega)}.
\end{aligned} \tag{3.5}$$

On the other hand, we see that

$$\begin{aligned}
\|f\chi_j\|_{L^{p(\cdot)}(\omega)} &= 2^{-j\alpha}\left(2^{j\alpha q}\|f\chi_j\|^q_{L^{p(\cdot)}(\omega)}\right)^{1/q} \\
&\leqslant C2^{-j\alpha}\left(\sum_{l=-\infty}^{j}2^{l\alpha q}\|f\chi_l\|^q_{L^{p(\cdot)}(\omega)}\right)^{1/q} \\
&= C2^{j(\lambda-\alpha)}\left(2^{-j\lambda}\left(\sum_{l=-\infty}^{j}2^{l\alpha q}\|f\chi_l\|^q_{L^{p(\cdot)}(\omega)}\right)^{1/q}\right) \\
&\leqslant C2^{j(\lambda-\alpha)}\|f\|_{M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\omega)}.
\end{aligned} \tag{3.6}$$

Hence, in view of $\alpha < n(1-r)+\lambda$, combining (3.5) and (3.6), we have

$$\begin{aligned}
I_1 &\leqslant C\sup_{L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{\alpha qk}\left(\sum_{j=-\infty}^{k-2}2^{(j-k)n(1-r)}\|f\chi_j\|_{L^{p(\cdot)}(\omega)}\right)^q \\
&\leqslant C\|f\|^q_{M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\omega)}\sup_{L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{\lambda qk}\left(\sum_{j=-\infty}^{k-2}2^{(j-k)(n(1-r)+\lambda-\alpha)}\right)^q \\
&\leqslant C\|f\|^q_{M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\omega)}\sup_{L\in\mathbb{Z}}2^{-L\lambda q}\left(\sum_{k=-\infty}^{L}2^{\lambda qk}\right) \\
&\leqslant C\|f\|^q_{M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\omega)}.
\end{aligned}$$

For I_2, using the boundedness of T on $L^{p(\cdot)}(\omega)$, we derive the estimate

$$I_2 \leqslant C\sup_{L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{\alpha qk}\|f\chi_k\|^q_{L^{p(\cdot)}(\omega)} \leqslant C\|f\|^q_{M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\omega)}.$$

We proceed now to estimate I_3. As argued before, we apply the size condition (3.2) and obtain

$$\begin{aligned}
|T(f_j)(x)| &\leqslant C2^{-jn}\|f_j\|_{L^1(\mathbb{R}^n)} \\
&\leqslant C2^{-jn}\|f\omega^{\frac{1}{p(\cdot)}}\chi_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\omega^{-\frac{1}{p(\cdot)}}\chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \\
&\leqslant C\|f\chi_j\|_{L^{p(\cdot)}(\omega)}\|\chi_{B_j}\|^{-1}_{L^{p(\cdot)}(\omega)}.
\end{aligned} \tag{3.7}$$

An application of Lemma 2.3 gives

$$\begin{aligned}
&\|T(f_j)\chi_k\|_{L^{p(\cdot)}(\omega)} \\
&\leqslant C\|f\chi_j\|_{L^{p(\cdot)}(\omega)}\|\chi_{B_j}\|^{-1}_{L^{p(\cdot)}(\omega)}\|\chi_{B_k}\|_{L^{p(\cdot)}(\omega)} \\
&\leqslant C2^{(k-j)n\delta}\|f\chi_j\|_{L^{p(\cdot)}(\omega)}.
\end{aligned} \tag{3.8}$$

By (3.6) and (3.8), since $\alpha > -n\delta + \lambda$, we arrive at

$$
\begin{aligned}
I_3 &\leqslant C \sup_{L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{\alpha qk} \Bigg(\sum_{j=k+2}^{\infty} 2^{(k-j)n\delta} \|f\chi_j\|_{L^{p(\cdot)}(\omega)} \Bigg)^q \\
&\leqslant C\|f\|^q_{M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\omega)} \sup_{L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{\lambda qk} \Bigg(\sum_{j=k+2}^{\infty} 2^{(k-j)(n\delta+\alpha-\lambda)} \Bigg)^q \\
&\leqslant C\|f\|^q_{M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\omega)} \sup_{L\in\mathbb{Z}} 2^{-L\lambda q} \Bigg(\sum_{k=-\infty}^{L} 2^{\lambda qk} \Bigg) \\
&\leqslant C\|f\|^q_{M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\omega)}.
\end{aligned}
$$

Consequently, the proof of Theorem 3.1 is complete. □

For the fractional singular integrals, we have a theorem similar to Theorem 3.1.

THEOREM 3.2. *Let* $0 \leqslant \lambda < \infty$, $p_2(\cdot) \in \mathscr{P}(\mathbb{R}^n)\bigcap LH(\mathbb{R}^n)$, $\omega^{p_2(\cdot)} \in A_1$, $0 < q_1 \leqslant q_2 < \infty$, $0 < \delta_1, \delta_2 < 1$ *be the constants appearing in Lemma 2.4,* $0 < \gamma < min\{n(\delta_1 + \delta_2), n\}$ *and* $-n\delta_1 + \lambda < \alpha < n\delta_2 - \gamma + \lambda$. *Define* $p_1(\cdot)$ *by* $1/p_2(\cdot) = 1/p_1(\cdot) - \gamma/n$. *Suppose that a sublinear operator* T_γ *satifies*

$$|T_\gamma f(x)| \leqslant C\|f\|_{L^1(\mathbb{R}^n)}/|x|^{n-\gamma}, \tag{3.9}$$

when $suppf \subseteq R_k$ *and* $|x| \geqslant 2^{k+1}$ *with* $k \in \mathbb{Z}$, *and*

$$|T_\gamma f(x)| \leqslant C 2^{-k(n-\gamma)} \|f\|_{L^1(\mathbb{R}^n)}, \tag{3.10}$$

when $suppf \subseteq R_k$ *and* $|x| \leqslant 2^{k-2}$ *with* $k \in \mathbb{Z}$. *Then, if* T_γ *is bounded from* $L^{p_1(\cdot)}(\omega^{p_1(\cdot)})$ *to* $L^{p_2(\cdot)}(\omega^{p_2(\cdot)})$, T_γ *is also bounded from* $M\dot{K}^{\alpha,\lambda}_{q_1,p_1(\cdot)}(\omega^{p_1(\cdot)})$ *to* $M\dot{K}^{\alpha,\lambda}_{q_2,p_2(\cdot)}(\omega^{p_2(\cdot)})$.

COROLLARY 3.2. *Let* $\lambda, p_1(\cdot), p_2(\cdot), q_1, q_2, \delta_1, \delta_2, \gamma, \omega$ *and* α *be as in Theorem 3.2, if a sublinear operator* T_γ *satifies the size conditions*

$$|T_\gamma f(x)| \leqslant C \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\gamma}} dy, \;\; x \notin suppf, \tag{3.11}$$

for any integrable and compactly supported functions f *and* T_γ *is bounded from* $L^{p_1(\cdot)}(\omega^{p_1(\cdot)})$ *to* $L^{p_2(\cdot)}(\omega^{p_2(\cdot)})$, *then* T_γ *is also bounded from* $M\dot{K}^{\alpha,\lambda}_{q_1,p_1(\cdot)}(\omega^{p_1(\cdot)})$ *to* $M\dot{K}^{\alpha,\lambda}_{q_2,p_2(\cdot)}(\omega^{p_2(\cdot)})$.

Proof of Theorem 3.2. Let $f \in M\dot{K}^{\alpha,\lambda}_{q_1,p_1(\cdot)}(\omega^{p_1(\cdot)})$. As in the proof of Theorem

3.1, we can write

$$\begin{aligned}
\|T_\gamma(f)\|^{q_1}_{M\dot{K}^{\alpha,\lambda}_{q_2,p_2(\cdot)}(\omega^{p_2(\cdot)})} &= \sup_{L\in\mathbb{Z}} 2^{-L\lambda q_1}\left(\sum_{k=-\infty}^{L} 2^{\alpha q_2 k}\|T_\gamma(f)\chi_k\|^{q_2}_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})}\right)^{q_1/q_2} \\
&\leqslant C\sup_{L\in\mathbb{Z}} 2^{-L\lambda q_1}\sum_{k=-\infty}^{L} 2^{\alpha q_1 k}\|T_\gamma(f)\chi_k\|^{q_1}_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})} \\
&\leqslant C\sup_{L\in\mathbb{Z}} 2^{-L\lambda q_1}\sum_{k=-\infty}^{L} 2^{\alpha q_1 k}\left(\sum_{j=-\infty}^{k-2}\|T_\gamma(f_j)\chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})}\right)^{q_1} \\
&\quad + C\sup_{L\in\mathbb{Z}} 2^{-L\lambda q_1}\sum_{k=-\infty}^{L} 2^{\alpha q_1 k}\left(\sum_{j=k-1}^{k+1}\|T_\gamma(f_j)\chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})}\right)^{q_1} \\
&\quad + C\sup_{L\in\mathbb{Z}} 2^{-L\lambda q_1}\sum_{k=-\infty}^{L} 2^{\alpha q_1 k}\left(\sum_{j=k+2}^{\infty}\|T_\gamma(f_j)\chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})}\right)^{q_1} \\
&=: J_1 + J_2 + J_3.
\end{aligned}$$

For J_1. Noting that $j \leqslant k-2$ and $x \in R_k$, by (3.9) and Lemma 2.1, we get

$$\begin{aligned}
|T_\gamma(f_j)(x)| &\leqslant C|x|^{\gamma-n}\|f_j\|_{L^1(\mathbb{R}^n)} \\
&\leqslant C2^{k(\gamma-n)}\|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}\|\chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'}.
\end{aligned} \tag{3.12}$$

Based on the fact that

$$\|\chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'} \leqslant C2^{j(n-\gamma)}\|\chi_{B_j}\|^{-1}_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})},$$

see [10, p.11]. From (3.12), Lemma 2 and Lemma 4, it follows that

$$\begin{aligned}
&\|T_\gamma(f_j)\chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})} \\
&\leqslant C2^{k(\gamma-n)}\|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}\|\chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'}\|\chi_{B_k}\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})} \\
&\leqslant C2^{k\gamma}\|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}\|\chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'}\|\chi_{B_k}\|^{-1}_{(L^{p_2(\cdot)}(\omega^{p_2(\cdot)}))'} \\
&\leqslant C2^{k\gamma}\|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}\|\chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'}\|\chi_{B_j}\|^{-1}_{(L^{p_2(\cdot)}(\omega^{p_2(\cdot)}))'}\frac{\|\chi_{B_j}\|_{(L^{p_2(\cdot)}(\omega^{p_2(\cdot)}))'}}{\|\chi_{B_k}\|_{(L^{p_2(\cdot)}(\omega^{p_2(\cdot)}))'}} \\
&\leqslant C2^{(k-j)\gamma}2^{(j-k)n\delta_2}\|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}2^{jn}\|\chi_{B_j}\|^{-1}_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})}\|\chi_{B_j}\|^{-1}_{(L^{p_2(\cdot)}(\omega^{p_2(\cdot)}))'} \\
&\leqslant C2^{(k-j)(\gamma-n\delta_2)}\|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}.
\end{aligned} \tag{3.13}$$

A simple caculation shows that

$$\begin{aligned}
\|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})} &= 2^{-j\alpha}\left(2^{j\alpha q}\|f_j\|^q_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}\right)^{1/q} \\
&\leqslant C2^{-j\alpha}\left(\sum_{l=-\infty}^{j} 2^{l\alpha q}\|f_l\|^q_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}\right)^{1/q} \\
&= C2^{j(\lambda-\alpha)}\left(2^{-j\lambda}\left(\sum_{l=-\infty}^{j} 2^{l\alpha q}\|f_l\|^q_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}\right)^{1/q}\right) \\
&\leqslant C2^{j(\lambda-\alpha)}\|f\|_{M\dot{K}^{\alpha,\lambda}_{q_1,p_1(\cdot)}(\omega^{p_1(\cdot)})}.
\end{aligned} \tag{3.14}$$

Hence, in view of $\alpha < n\delta_2 - \beta + \lambda$, combining (3.13) and (3.14), we get

$$\begin{aligned} J_1 &\leqslant C \sup_{L\in\mathbb{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^{L} 2^{\alpha q_1 k} \Big(\sum_{j=-\infty}^{k-2} 2^{(k-j)(\gamma - n\delta_2)} \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})} \Big)^{q_1} \\ &\leqslant C \|f\|^{q_1}_{M\dot{K}^{\alpha,\lambda}_{q_1,p_1(\cdot)}(\omega^{p_1(\cdot)})} \sup_{L\in\mathbb{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^{L} 2^{\alpha q_1 k} \Big(\sum_{j=-\infty}^{k-2} 2^{(k-j)(\gamma - n\delta_2)} 2^{j(\lambda-\alpha)} \Big)^{q_1} \\ &\leqslant C \|f\|^{q_1}_{M\dot{K}^{\alpha,\lambda}_{q_1,p_1(\cdot)}(\omega^{p_1(\cdot)})} \sup_{L\in\mathbb{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^{L} 2^{\lambda q_1 k} \Big(\sum_{j=-\infty}^{k-2} 2^{(k-j)(\gamma - n\delta_2+\alpha-\lambda)} \Big)^{q_1} \\ &\leqslant C \|f\|^{q_1}_{M\dot{K}^{\alpha,\lambda}_{q_1,p_1(\cdot)}(\omega^{p_1(\cdot)})} \sup_{L\in\mathbb{Z}} 2^{-L\lambda q_1} \Big(\sum_{k=-\infty}^{L} 2^{\lambda q_1 k} \Big) \\ &\leqslant C \|f\|^{q_1}_{M\dot{K}^{\alpha,\lambda}_{q_1,p_1(\cdot)}(\omega^{p_1(\cdot)})}. \end{aligned}$$

For J_2, by the boundedness of T_γ from $L^{p_1(\cdot)}(\omega^{p_1(\cdot)})$ to $L^{p_2(\cdot)}(\omega^{p_2(\cdot)})$, one can easily obtain

$$J_2 \leqslant C \sup_{L\in\mathbb{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^{L} 2^{\alpha q_1 k} \|f\chi_k\|^{q_1}_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})} \leqslant C \|f\|^{q_1}_{M\dot{K}^{\alpha,\lambda}_{q_1,p_1(\cdot)}(\omega^{p_1(\cdot)})}.$$

For J_3. Noting that $j \geqslant k+2$ and $x \in R_k$, Lemma 2.1 yields

$$\begin{aligned} |T_\gamma(f_j)(x)| &\leqslant C 2^{j(\gamma-n)} \|f_j\|_{L^1(\mathbb{R}^n)} \\ &\leqslant C 2^{j(\gamma-n)} \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})} \|\chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'}. \end{aligned} \tag{3.15}$$

Consequently, we have

$$\begin{aligned} &\|T_\gamma(f_j)\chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})} \\ &\leqslant C 2^{j(\gamma-n)} \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})} \|\chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'} \|\chi_{B_k}\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})} \\ &\leqslant C 2^{j(\gamma-n)} \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})} \|\chi_{B_j}\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'} \|\chi_{B_j}\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})} \frac{\|\chi_{B_k}\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})}}{\|\chi_{B_j}\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})}} \\ &\leqslant C 2^{j(\gamma-n)} 2^{(k-j)n\delta_1} \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})} \|\chi_{B_j}\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'} \|\chi_{B_j}\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})}. \end{aligned} \tag{3.16}$$

Since $\omega^{p_2(\cdot)} \in A_1 \subset A_{1+p_2(\cdot)/p_1'(\cdot)}$, so $\omega^{p_2(\cdot)} \in A(p_1(\cdot), p_2(\cdot))$, then we obtain

$$\begin{aligned} &\|\chi_{B_j}\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'} \|\chi_{B_j}\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})} \\ &\leqslant C \|\omega^{-1}\chi_{B_j}\|_{L^{p_1'(\cdot)}} \|\omega\chi_{B_j}\|_{L^{p_2(\cdot)}} \\ &\leqslant C 2^{j(n-\gamma)}. \end{aligned} \tag{3.17}$$

This combined with (3.16) shows that

$$\|T_\gamma(f_j)\chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})} \leqslant C 2^{(k-j)n\delta_1} \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}. \tag{3.18}$$

From (3.14) and (3.18), since $\alpha > -n\delta_1 + \lambda$, we get

$$\begin{aligned}
J_3 &\leqslant C \sup_{L\in\mathbb{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^{L} 2^{\alpha q_1 k} \Big(\sum_{j=k+2}^{\infty} 2^{(k-j)n\delta_1} \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})} \Big)^{q_1} \\
&\leqslant C \|f\|^{q_1}_{M\dot{K}^{\alpha,\lambda}_{q_1,p_1(\cdot)}(\omega^{p_1(\cdot)})} \sup_{L\in\mathbb{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^{L} 2^{\alpha q_1 k} \Big(\sum_{j=k+2}^{\infty} 2^{(k-j)n\delta_1} 2^{j(\lambda-\alpha)} \Big)^{q_1} \\
&\leqslant C \|f\|^{q_1}_{M\dot{K}^{\alpha,\lambda}_{q_1,p_1(\cdot)}(\omega^{p_1(\cdot)})} \sup_{L\in\mathbb{Z}} 2^{-L\lambda q_1} \sum_{k=-\infty}^{L} 2^{\lambda q_1 k} \Big(\sum_{j=k+2}^{\infty} 2^{(k-j)(n\delta_1+\alpha-\lambda)} \Big)^{q_1} \\
&\leqslant C \|f\|^{q_1}_{M\dot{K}^{\alpha,\lambda}_{q_1,p_1(\cdot)}(\omega^{p_1(\cdot)})} \sup_{L\in\mathbb{Z}} 2^{-L\lambda q_1} \Big(\sum_{k=-\infty}^{L} 2^{\lambda q_1 k} \Big) \\
&\leqslant C \|f\|^{q_1}_{M\dot{K}^{\alpha,\lambda}_{q_1,p_1(\cdot)}(\omega^{p_1(\cdot)})}.
\end{aligned}$$

Thus, the proof of Theorem 3.2 is complete. □

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